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GLOBAL STABILITY OF AN INFECTION AND VACCINATION AGE-STRUCTURED MODEL WITH GENERAL NONLINEAR INCIDENCE

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Abstract. In this paper, a model structured in vaccination and infection ages with a general class of nonlinear incidence is introduced and investigated. The resulting model is a system of two age-structured PDEs and an ODE with a non local term. We give a necessary and sufficient condition for global asymptotic stability of the free-equilibrium related to the basic reproduction number. Further, by constructing a new Lyapunov functional, we show the global asymptotic stability of the endemic equilibrium whenever it exists. Finally, a discussion about controlling the spread of the disease is provided.

Keywords. Epidemiology; Nonlinear incidence; Compact attractor; Lyapunov functional; Global stability.

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1. Introduction

There is a vast and a growing literature on the analysis of the dynamics of the well-known Susceptible-Vaccinated-Infected-Removed (SVIR) ODE model, see [1, 2, 3, 4] and the references therein. The huge number of existing papers on the subject can easily threw the non-specialist reader off balance, so we begin by highlighting our main contribution. Indeed, an important aspect in disease modeling is the incidence rate, that is, newly infected individuals per unit of time. A bilinear form of incidence rate (action mass type) such as, SI or SJ, where $J = \int_0^\infty \beta(a)i(t,a)da$, ($\beta(a)$ being the transmission rate with respect to the age of infection and i being the density of infection individuals) is frequently used. This is often considered to characterize the fact that the contact number between susceptible and infective is proportional to the product of both sub populations, see [5, 6, 7, 8, 9]. Nevertheless, several authors suggested that the disease transmission process may have a non linear incidence to ensure a good description of the disease dynamics [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Our main contribution consists in considering a nonlinear incidence function in its most general form that is

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$f(S,J)$, which generalizes most of usual forms of incidence rates. This generalization has an obvious biological interest, but it also has a purely mathematical interest, as it requires the introduction of some new ideas to construct a suitable Lyapunov function and by the sequel the study of the global dynamics of our model.

The evolution of the disease is linked to a threshold quantity R_0 (basic reproduction number) being smaller or larger than one. These models do not make distinction between individuals. Thus, its assume constant per capita rates in a living class. However, an individual varies from others according to some particular aspects, such as, the infection age, that corresponds to the time since the infection began. Moreover, when vaccination is considered, the duration of immunity of an individual vaccine, (which is the definition of age of vaccination) vary also between individuals. So, the introduction of these types of age into the model may provide more realistic prediction for persistence and extinction of a disease under investigation. In the context of age infection model, Thieme and Castillo-Chavez [26] developed and analyzed a model of the infection age-dependent infectivity. This work has been followed by many age structured models [8, 9, 21, 27, 28, 29, 30, 31]. On the other hand, many works have been devoted to study the age vaccination model, see [5, 8, 9, 21, 24, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42] and the references therein.

In this paper, we will consider a nonlinear incidence function in its very general form $f(S,J)$ and under suitable hypotheses on f and present the global stability of the free-disease equilibrium provided that $R_0^{\psi} \leq 1$ and of the endemic equilibrium whenever it exists, under the condition $R_0^{\psi} > 1$.

2. Model formulation

In this paper, we deal with the following model

$$\begin{cases} S'(t) = A - (\mu+\psi)S(t) - f\big(S(t),J(t)\big) + \int_0^\infty \alpha(a)v(t,a)da, \quad t>0, \\ v_t(t,b) + v_b(t,b) = -(\mu+\alpha(b))v(t,b), \quad t>0, \ b>0, \\ i_t(t,a) + i_a(t,a) = -(\mu+\gamma(a))i(t,a), \quad t>0, \ a>0, \\ J(t) = \int_0^\infty \beta(a)i(t,a)da, \ t>0, \end{cases} \tag{2.1}$$

associated to the following boundary and initial conditions

$$\begin{cases} v(t,0) = \psi S(t), \ t>0, \\ i(t,0) = f\big(S(t),J(t)\big), \ t>0, \\ S(0) = S_0 \geq 0, \\ v(0,.) = v_0(.) \in L^1_+(\mathbb{R}^+), \\ i(0,.) = i_0(.) \in L^1_+(\mathbb{R}^+). \end{cases} \tag{2.2}$$

$S(t)$ represents the number of susceptible individuals at time t.

$v(t,b)$ represents the density of vaccinated individuals at time t, and at age b, which is the duration of immunity of a vaccine for an individual.

$i(t,a)$ represents the density of infected individuals at time t ant at age a, being the infection age.

$\beta(a)$ and $\gamma(a)$ are respectively, the transmission coefficient, the recovery rate of infected individuals with age a.

$\alpha(b)$ is the vaccine wane rate with age b.

The parameters A, μ, ψ represent respectively the entering flux into the susceptible class (S), the mortality rate of the total population and the vaccination rate of susceptible individuals.

Our incidence rate $f(S,J)$ generalizes a lot of common forms of incidence rate such as, βSJ, $\frac{\beta SJ}{S+J}$, $\frac{\beta SJ}{1+aS+bJ+cSJ}$.

Recently, Yang, Martcheva and Wang [24] and Duan, Yuan and Lie [32] considered the structured SVIR model with age of vaccination. Both works considered the case where J is governed by an ODE (roughly speaking $\beta(a)$ is constant), in addition, the first one considered only a mass action type incidence ($f(S,I) = SI$) and the second one analyzed the particular case $f(S,I) = SG(I)$, which make our present model and results more general than the cited works and by the sequel much more complicated to study.

Up to our knowledge the forms of incidence rate as $\frac{\beta SJ}{1+aS+bJ}$, (Beddington-DeAngelis functional response), $\frac{\beta SJ}{1+aS+bJ+abSJ}$ (Crowley-Martin functional response) have not been analyzed before. We emphasize that the global asymptotic stability of the equilibria of problem (2.1)-(2.2) is established by the use of a more complex Lyapunov function than the one used in [34].

We present now our basic assumptions and notations that will be used later.

The nonlinear incidence function is assumed to fulfill the following classical conditions:

(H1) f is strictly increasing for both components S and J with $f(0,J) = f(S,0) = 0$ for all $S,J \geq 0$,

(H2) The function $\dfrac{\partial f}{\partial J}(.,0)$ is positive continuous,

(H3) For all $S \geq 0$ the function $f(S,J)$ is concave with respect to J,

(H4) The function f is locally Lipschitz continuous in S and J, with a Lipschitz constant $L > 0$, i.e. for every $K > 0$ there exists some $L := L_K > 0$ such that

$$|f(S_2,J_2) - f(S_1,J_1)| \leq L(|S_2 - S_1| + |J_2 - J_1|),$$

whenever $0 \leq S_2, S_1, J_2, J_1 \leq K$.

All parameters are positive, we also assume that $\beta \in C_{BU}(\mathbb{R}^+, \mathbb{R}^+)$, where $C_{BU}(\mathbb{R}^+, \mathbb{R}^+)$ is the set of all bounded and uniformly continuous functions from $\mathbb{R}^+$ into $\mathbb{R}^+$. The functions α, γ are assumed to belong to the set $L^\infty_+(\mathbb{R}^+)$, the nonnegative cone of $L^\infty(\mathbb{R}^+)$.

The disease splits the population (N) into susceptible, infective and removed individuals, that is, $N(t) = S(t) + V(t) + I(t) + R(t)$ with $I(t) = \int_0^\infty i(t,a)da$ and $V(t) = \int_0^\infty v(t,b)db$. The removed class is modeled as

$$R'(t) = \int_0^\infty \gamma(a) i(t,a)da - \mu R(t). \tag{2.3}$$

The total population $N(t)$ satisfies:

$$N'(t) = A - \mu N(t),$$

the solution $N(t)$ of the last equation converges to $\frac{A}{\mu}$, when t goes to ∞. Hence, the equation of R in (2.3) can be omitted.

Throughout this paper, we denote

$$\begin{cases} \Gamma_1(b) = e^{-\int_0^b \alpha(\theta)}d\theta, \\ \Gamma_2(a) = e^{-\int_0^a \gamma(\theta)}d\theta, \\ \bar{\alpha} = \int_0^\infty \alpha(b)\Gamma_1(b)e^{-\mu b}db, \\ \bar{\beta} = \int_0^\infty \beta(a)\Gamma_2(a)e^{-\mu a}da, \end{cases} \tag{2.4}$$

and

$$\bar{N}_\psi = \frac{A}{\mu + (1-\bar{\alpha})\psi}, \quad v^0(b) = e^{-\mu b}\Gamma_1(b)\psi\bar{N}_\psi. \tag{2.5}$$

Next, we set $\Phi(t,.) = (v(t,.), i(t,.))$, $\Phi_0(.) = (v_0(.), i_0(.))$ and

$$|\Phi(t,.)| = |v(t,.)| + |i(t,.)| \text{ and } ||\Phi||_1 = \int_0^\infty |\Phi(a)|da.$$

Using the same idea as in [34], we show existence, uniqueness and positivity of the solution of (2.1)-(2.2).

Theorem 2.1. *Assume that **(H4)** holds and let us suppose* $(S_0, \Phi_0) \in \mathbb{R}^+ \times (L^1_+(\mathbb{R}^+))^2$. *Then there exists a unique non-negative solution* $(S, \Phi) \in C^1(\mathbb{R}^+) \times C(\mathbb{R}^+; (L^1(\mathbb{R}^+))^2)$ *of problem (2.1)-(2.2). Moreover, we have the following estimates,*

$$S(t) + \int_0^\infty |\Phi(t,a)|da \leq \max\{S_0 + ||\Phi_0||_1, \frac{A}{\mu}\} := \tilde{M}, \quad J(t) \leq ||\overline{\_}||_\infty \tilde{M},$$

for all $t \geq 0$, *and the upper bounds satisfy,*

$$\limsup_{t\to\infty} \left(S(t) + \int_0^\infty |\Phi(t,a)|da\right) \leq \frac{A}{\mu}, \quad \limsup_{t\to\infty} J(t) \leq ||\overline{\_}||_\infty \frac{A}{\mu}.$$

Finally,

$$\liminf_{t\to\infty} S(t) \geq \Lambda, \tag{2.6}$$

$$\bar{\alpha}\psi\Lambda \leq \liminf_{t\to\infty} \int_0^\infty \alpha(b)v(t,b)db \leq ||\alpha||_\infty \frac{A}{\mu}. \tag{2.7}$$

with $\Lambda := \frac{A}{\mu + \psi + L}$, *and L is a Lipschitz constant defined in **(H4)**.*

Notice that, for (2.1)-(2.2), R_0^ψ of secondary infections produced by a single infected individual is defined by

$$R_0^\psi = \bar{\beta}\frac{\partial f}{\partial J}(\bar{N}_\psi, 0).$$

For more details about the reproduction rate, we refer the reader to [43].

The rest of this paper is organized as follows. The next section focuses on the existence of compact attractors. Next, we will prove that the disease-free equilibrium is globally asymptotically stable whenever $R_0^{\psi} \leq 1$. Furthermore, we will investigate the global dynamic of the endemic equilibrium, whenever it exists. Finally, we present a discussion on a possible way to control the epidemic disease.

3. Preliminaries

3.1. **Volterra formulations and Global compact attractors.**

We begin by writing (2.1) in the form of a Volterra type equation, namely,

$$v(t,b) = \begin{cases} e^{-\mu b}\mathcal{F}_1(b)\psi S(t-b), & t > b \geq 0, \\[2ex] e^{-\mu t}\dfrac{\mathcal{F}_1(b)}{\mathcal{F}_1(b-t)}v_0(b-t), & b > t \geq 0. \end{cases} \tag{3.1}$$

$$i(t,a) = \begin{cases} e^{-\mu a}\mathcal{F}_2(a) f(S(t-a),J(t-a)), & t > a \geq 0, \\[2ex] e^{-\mu t}\dfrac{\mathcal{F}_2(a)}{\mathcal{F}_2(a-t)}i_0(a-t), & a > t \geq 0 \end{cases} \tag{3.2}$$

and

$$\begin{cases} S'(t) = A - (\mu+\psi)S(t) - f\big(S(t),J(t)\big) + U(t), \\[2ex] S(0) = S_0 \end{cases} \tag{3.3}$$

with

$$\begin{cases} J(t) = \int_0^\infty \underline{\;}(a)i(t,a)da, \\[2ex] U(t) = \int_0^\infty \alpha(b)v(t,b)db. \end{cases} \tag{3.4}$$

Combining (3.2) and (3.3) and by a simple computation, we obtain

$$\begin{cases} S'(t) = A - (\mu+\psi)S(t) - f\big(S(t),J(t)\big) + U(t), \\[2ex] J(t) = J_1(t) + J_2(t), \end{cases} \tag{3.5}$$

where

$$J_1(t) = \int_0^t \beta(a) f(S(t-a),J(t-a))\mathcal{F}_2(a)e^{-\mu a}da, \tag{3.6}$$

$$J_2(t) = e^{-\mu t}\int_0^\infty \beta(a+t)i_0(a)\frac{\mathcal{F}_2(a+t)}{\mathcal{F}_2(a)}da. \tag{3.7}$$

Now, it is not difficult to show the existence of a continuous semiflow

$$\Phi(t,(S_0,v_0(.),i_0(.))) = (S(t),v(t,.),i(t,.)), \tag{3.8}$$

with (S,v,i) is solution of the autonomous problem (3.2)-(3.4).

We choose $X = \mathbb{R}^+ \times L^1(\mathbb{R}^+) \times L^1(\mathbb{R}^+)$ endowed with the natural norm

$$||(S,v,i)||_X = |S| + ||v||_1 + ||i||_1.$$

The following theorem states the existence of a compact attractor of all bounded sets of X, (the concept of global attractors is presented in, e.g., [44, 45, 46]).

Theorem 3.1. *Suppose (**H4**) holds. If $\digamma_2 \in L^1(\mathbb{R}^+)$, then Φ has a compact attractor $\mathbf{A}$ of bounded sets of X.*

Proof. By Theorem 2.1, the semiflow Φ is point-dissipative and eventually bounded on bounded sets on X. Hence, from Theorem 2.33 in [45], we only need to show the asymptotic smoothness of Φ to complete the proof. In order to prove this property, we apply Theorem 2.46 in [45]. We define

$$\Theta_1(t,(S_0,v_0(.),i_0(.))) = (0, v_1(t,.), w_1(t,.)),$$

as

$$v_1(t,b) = \begin{cases} 0, \quad t > b, \\ \\ e^{-\mu t}\dfrac{\digamma_1(b)}{\digamma_1(b-t)} v_0(b-t), \quad b > t, \end{cases} \tag{3.9}$$

$$w_1(t,a) = \begin{cases} f\big(S(t-a), J_2(t-a)\big) e^{-\mu a}\digamma_2(a), \quad t > a, \\ \\ i_0(a-t) e^{-\mu t}\dfrac{\digamma_2(a)}{\digamma_2(a-t)}, \quad a > t, \end{cases} \tag{3.10}$$

and $\Theta_2(t,(S_0,v_0(.),i_0(.))) = (S(t), v_2(t,.), w_2(t,.))$ with

$$v_2(t,b) = \begin{cases} e^{-\mu b}\digamma_1(b)\psi S(t-b), \quad t > b, \\ \\ 0, \quad b > t, \end{cases} \tag{3.11}$$

$$w_2(t,a) = \begin{cases} \big(f(S(t-a), J(t-a)) - f(S(t-a), J_2(t-a))\big) e^{-\mu a}\digamma_2(a), \quad t > a, \\ \\ 0, \quad a > t. \end{cases} \tag{3.12}$$

From (3.2), we have

$$\Phi(t,(S_0,v_0(.),i_0(.))) = \Theta_1(t,(S_0,v_0(.),i_0(.))) + \Theta_2(t,(S_0,v_0(.),i_0(.))).$$

Now, by using the same arguments as in [34], we can prove the asymptotic smoothness of Φ. This complete the proof. □

3.2. **Total trajectories.** We describe now the total trajectories of system (2.1), that are solutions of (2.1) defined for all $t \in \mathbb{R}$. These extended solutions play an important role in the global asymptotic stability of equilibria.

We consider $\bar{\phi}$ a total $\Phi-$trajectory, $\bar{\phi}(t) = (S(t), v(t,.), i(t,.))$. By a straightforward calculation (see, also [34, 45]), we obtain, for all $s \in \mathbb{R}$,

$$\begin{cases} S'(s) = A - (\mu+\psi)S(s) - f(S(s),J(s)) + U(s), \\ U(s) = \int_0^\infty \alpha(b) e^{-\mu b} \digamma_1(b) \psi S(s-b) db, \\ v(s,b) = e^{-\mu b} \digamma_1(b) \psi S(s-b), \quad b \geq 0, \\ J(s) = \int_0^\infty \beta(a) e^{-\mu a} \digamma_2(a) f(S(s-a), J(s-a)) da, \\ i(s,a) = e^{-\mu a} \digamma_2(a) f(S(s-a), J(s-a)), \quad a \geq 0. \end{cases} \tag{3.13}$$

The next lemma presents some useful estimates of the total trajectory, related to the compact attractor **A**.

Lemma 3.2. *Assume that **(H4)** holds. Then, for all $(S_0, v_0, i_0) \in \mathbf{A}$,*

$$S_0 + \int_0^\infty v_0(b) db + \int_0^\infty i_0(a) da \leq \frac{A}{\mu}, \quad and$$

$$\int_0^\infty \beta(a) i_0(a) da \leq \frac{A}{\mu} ||\beta||_\infty, \quad S_0 \geq \frac{A}{\mu+\psi+L},$$

$$\frac{\psi \bar{\alpha} A}{\mu+\psi+L} \leq \int_0^\infty \alpha(b) v_0(b) db \leq \frac{A}{\mu} ||\alpha||_\infty, \;\; i_0(a) \leq f(\tfrac{A}{\mu}, ||\beta||_\infty \tfrac{A}{\mu}) e^{-\mu a} \digamma_2(a),$$

$$\frac{\psi A e^{-\mu b} \digamma_1(b)}{\mu+\psi+L} \leq v_0(b) \leq \frac{\psi A}{\mu} e^{-\mu b} \digamma_1(b), \quad a \geq 0, \, b \geq 0,$$

*where L is the Lipschitz constant defined in **(H4)**.*

Proof. First, since $\digamma_2(a) \leq 1$ for all $a \geq 0$, we have

$$I(t) := \int_0^\infty i(t,a) da \leq \int_0^\infty e^{-\mu a} f(S(t-a), J(t-a)) da =: \tilde{I}(t).$$

After a change of variable, the function $\tilde{I}(t)$ becomes

$$\tilde{I}(t) = \int_{-\infty}^t e^{-\mu(t-s)} f(S(s), J(s)) ds,$$

and satisfies the following equation

$$\tilde{I}'(t) = f(S(t), J(t)) - \mu \tilde{I}(t), \quad t \in \mathbb{R}. \tag{3.14}$$

From (3.13), we have

$$\begin{aligned} V(t) &:= \int_0^\infty v(t,b) db, \\ &= \int_{-\infty}^t e^{-\mu(t-s)} \digamma_1(t-s) \psi S(s) ds. \end{aligned}$$

The derivation of this last equation leads to

$$V'(t) = \psi S(t) - \mu V(t) - U(t). \tag{3.15}$$

From (3.14) and (3.15), we find from the equation of S in (3.13) that

$$S'(t)+V'(t)+\tilde{I}'(t)=A-\mu(S(t)+V(t)+\tilde{I}(t)).$$

For $t>r$, and by a direct computation, we have

$$S(t)+V(t)+\tilde{I}(t)=(S(r)+V(r)+\tilde{I}(r))e^{-\mu(t-r)}+A\int_r^t e^{-\mu(t-s)}ds.$$

By letting $r\to-\infty$, we obtain

$$S(t)+V(t)+\tilde{I}(t)=\frac{A}{\mu},\quad t\in\mathbb{R}.$$

Since $I\le\tilde{I}$, we have

$$S(t)+V(t)+I(t)\le\frac{A}{\mu}\ \text{ and }\ J(t)\le||\beta||_\infty\frac{A}{\mu}\ ,\ U(t)\le||\alpha||_\infty\frac{A}{\mu},$$

$$i(t,a)\le f(\frac{A}{\mu},||\beta||_\infty\frac{A}{\mu})e^{-\mu a}\digamma_2(a),\quad a\ge 0,\ \ \forall t\in\mathbb{R},$$

and

$$v(t,b)\le\psi\frac{A}{\mu}e^{-\mu b}\digamma_1(b),\quad b\ge 0,\ \ \forall t\in\mathbb{R}.$$

Now we deal with S in (3.13). By the boundedness of J and **(H4)**, we have

$$\begin{aligned}
S'(t)\ &\ge\ A-(\mu+\psi)S(t)-f(S(t),||\beta||_\infty\frac{A}{\mu}),\\
&\ge\ A-(\mu+\psi)S(t)-LS(t),\\
&\ge\ A-(\mu+\psi+L)S(t).
\end{aligned}$$

Finally, by a straightforward computation, we have

$$S(t)\ge\frac{A}{\mu+\psi+L}\quad\forall t\in\mathbb{R}.\tag{3.16}$$

Similarly as above, we can show the rest of estimations. This completes the proof. □

4. The global asymptotic stability of the disease-free equilibrium

This section is devoted to the global asymptotic stability of disease-free equilibrium.

We observe that system (2.1) always has a disease-free equilibrium $(\bar{N}_\psi,v^0(.),0)$ with $\bar{N}_\psi$ and $v^0(.)$ are defined in (2.5).

Theorem 4.1. *Let $\digamma_2\in L^1(\mathbb{R}^+)$, and suppose that **(H1)-(H4)** hold. Then, the disease free equilibrium $(\bar{N}_\psi,v^0(.),0)$ is globally asymptotically stable whenever $R_0^\psi\le 1$.*

Proof. Let us define the function ϕ as

$$\phi(a)=\frac{\partial f}{\partial J}(\bar{N}_\psi,0)\int_a^\infty\beta(\xi)e^{-\mu(\xi-a)}\frac{\digamma_2(\xi)}{\digamma_2(a)}d\xi,$$

which is solution of the following problem

$$\begin{cases}\phi'(a)=(\mu+\gamma(a))\phi(a)-\beta(a)\dfrac{\partial f}{\partial J}(\bar{N}_\psi,0),\quad a>0,\\[2ex] \phi(0)=R_0^\psi.\end{cases}\tag{4.1}$$

We first define the functions,

$$G(y)=\bar{\alpha}\big(y-\int_{\bar{N}_\psi}^{y}\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(\eta,J)}d\eta-\bar{N}_\psi\big),$$

and

$$\theta(b)=\psi\int_b^\infty\frac{\alpha_1(\xi)}{\bar{\alpha}}d\xi,$$

with

$$\alpha_1(b)=\alpha(b)\digamma_1(b)e^{-\mu b},$$

and $\bar{\alpha}$ is defined in (2.4). For $x:=(S_0,v_0(.),i_0(.))\in\mathbf{A}$, we consider as the Lyapunov functional $V(x)=V_1(x)+V_2(x)+V_3(x)$, where

$$V_1(x)=S_0-\int_{\bar{N}}^{S_0}\lim_{J\to 0^+}\frac{f(\bar{N},J)}{f(\eta,J)}d\eta-\bar{N},$$

$$V_2(x)=\int_0^\infty\phi(a)i_0(a)da,$$

and

$$V_3(x)=\int_0^\infty\theta(b)G\Big(\frac{e^{\mu b}v_0(b)}{\psi\digamma_1(b)}\Big)db.$$

Let $\chi:\mathbb{R}\to\mathbf{A}$ be a total $\Phi-$trajectory, $\chi(t)=(S(t),v(t,.),i(t,.))$, $S(0)=S_0$, $v(0,a)=v_0(a)$ and $i(0,a)=i_0(a)$, with $(S(t),v(t,a),i(t,a))$ is solution of problem (3.13)

$$\frac{d}{dt}V_1(\chi(t)) = \big(1-\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)}\big)\big(A-(\mu+\psi)S(t)-f(S(t),J(t))+\psi\textstyle\int_0^\infty\alpha_1(b)S(t-b)db\big).$$

Using (2.5) ($A=(\mu+(1-\alpha)\psi)\bar{N}_\psi$), we get

$$\begin{aligned}\frac{d}{dt}V_1(\chi(t)) &= (\mu+(1-\bar{\alpha})\psi)(1-\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)})\big(\bar{N}-S(t)\big)-\psi\bar{\alpha}S(t)\big(1-\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)}\big)\\ &- f(S(t),J(t))\big(1-\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)}\big)+\psi\big(1-\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)}\big)\textstyle\int_0^\infty\alpha_1(b)S(t-b)db.\end{aligned}$$

Next,

$$V_2(\chi(t)) = \textstyle\int_0^\infty\phi(a)i(t,a)da.$$

From the expression of i in (3.13), we have

$$V_2(\chi(t)) = \textstyle\int_0^\infty\phi_1(a)\xi(t-a)da,$$

with

$$\xi(t)=f(S(t),J(t)),\ \text{ and }\ \phi_1(a)=\phi(a)e^{-\mu a}\digamma_2(a). \tag{4.2}$$

Following the same arguments as in the proof of Lemma 9.18 in ([45]), we can show that V_2 is absolutely continuous and

$$\frac{d}{dt}V_2(\chi(t))=\phi_1(0)\xi(t)+\textstyle\int_0^\infty\phi_1'(a)\xi(t-a)da.$$

In view of (4.2), we arrive at

$$\frac{d}{dt}V_2(\chi(t))=\phi(0)f(S(t),J(t))+\textstyle\int_0^\infty\big(\phi'(a)-\mu\phi(a)-\gamma(a)\phi(a)\big)e^{-\mu a}\digamma_2(a)f(S(t-a),J(t-a))da,$$

which together with (3.13) and (4.1) implies that

$$\begin{aligned}\frac{d}{dt}V_2(\chi(t)) &= \phi(0)f(S(t),J(t))+\int_0^\infty \phi'(a)i(t,a)da-\int_0^\infty(\mu+\gamma(a))i(t,a)\phi(a)da,\\ &= R_0^\psi f(S(t),J(t))-J(t)\frac{\partial f}{\partial J}(\bar{N}_\psi,0).\end{aligned}$$

On the other hand, we find from the expression of v in (3.13) that

$$V_3(\chi(t))=\int_0^\infty \theta(b)H(S(t-b))db.$$

So, also as in the proof of Lemma 9.18 in [45], we can show that V_3 is absolutely continuous and

$$\begin{aligned}\frac{d}{dt}V_3(\chi(t)) &= \theta(0)G(S(t))+\int_0^\infty \theta'(b)G(S(t-b))db,\\ &= \psi\int_0^\infty\big(G(S(t))-G(S(t-b))\big)\frac{\alpha_1(b)}{\bar{\alpha}}db\end{aligned}$$

Adding $\frac{dV_1}{dt}$, $\frac{dV_2}{dt}$ and $\frac{dV_3}{dt}$, and remarking that

$$G'(y)=\bar{\alpha}\big(1-\lim_{J\to0^+}\frac{f(\bar{N}_\psi,J)}{f(y,J)}\big),$$

we find

$$\begin{aligned}\frac{d}{dt}V(\chi(t)) &= (\mu+\psi(1-\bar{\alpha}))(\bar{N}_\psi-S(t))\big(1-\lim_{J\to0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)}\big)\\ &+ \psi\int_0^\infty\big(G(S(t))-G(S(t-b))+G'(S(t))(S(t-b)-S(t))\big)\frac{\alpha_1(b)}{\bar{\alpha}}db\\ &+ \big(R_0^\psi-1\big)f(S(t),J(t))+f(S(t),J(t))\lim_{J\to0^+}\frac{f(\bar{N}_\psi,J)}{f(S(t),J)}-J(t)\frac{\partial f(\bar{N}_\psi,0)}{\partial J}.\end{aligned}$$

By using the fact that H is a convex function, $\bar{\alpha}<1$, and $R_0^\psi\le1$, we get that the first three terms of this above equation are negative. We claim that the fourth term is also negative. Indeed, the concavity of the function f with respect to J ensures that

$$f(S,J)\le J\frac{\partial f}{\partial J}(S,0).$$

Hence

$$\begin{aligned}f(S,J)\lim_{J\to0^+}\frac{f(\bar{N}_\psi,J)}{f(S,J)}-J\frac{\partial f}{\partial J}(\bar{N}_\psi,0) &= f(S,J)\frac{\frac{\partial f}{\partial J}(\bar{N}_\psi,0)}{\frac{\partial f}{\partial J}(S,0)}-J\frac{\partial f}{\partial J}(\bar{N}_\psi,0),\\ &= \frac{\frac{\partial f}{\partial J}(\bar{N}_\psi,0)}{\frac{\partial f}{\partial J}(S,0)}\big(f(S,J)-J\frac{\partial f}{\partial J}(S,0)\big)\le0.\end{aligned}$$

Notice that $\frac{d}{dt}V(\chi(t))=0$ implies that $S(t)=\bar{N}_\psi$. Let Q be the largest invariant set, for which $\frac{d}{dt}V(\chi(t))=0$. Then in Q we must have $S(t)=\bar{N}_\psi$ for all $t\in\mathbb{R}$. We substitute this into the equation of S in (3.13) to get $J(t)=0$ for all $t\in\mathbb{R}$. Thus, from the equation of i in (3.13), we obtain $i(t,.)=0$ for all $t\in\mathbb{R}$. In addition, according to equation of v in (3.13), we also have $v(t,.)=v^0(.)$ for all $t\in\mathbb{R}$. Then the largest invariant set with the property that $\frac{d}{dt}V(\chi(t))=0$ is $(\bar{N}_\psi,v^0(.),0)$ (LaSalle's Invariant Principle).

Now, since $\mathbf{A}$ is compact, the $\omega(x)$ and $\alpha(x)$ are non-empty, compact, invariant and attract $\chi(t)$ as $t\to\pm\infty$, respectively, we know that V is constant on the $\omega(x)$ and $\alpha(x)$, and thus $\omega(x)=\alpha(x)=\{(\bar{N}_\psi,v^0(.),0)\}$. Consequently, $\lim_{t\longrightarrow\pm\infty}\chi(t)=(\bar{N}_\psi,v^0(.),0)$ and

$$\lim_{t\longrightarrow-\infty}V(\chi(t))=\lim_{t\longrightarrow+\infty}V(\chi(t))=V(\bar{N}_\psi,v^0(.),0).$$

Since $V(\chi(t))$ is a decreasing function of t, we obtain $V(\chi(t))=V(\bar{N}_\psi,v^0(.),0)$ for all $t\in\mathbb{R}$. It follows that $\chi(t)=(\bar{N}_\psi,v^0(.),0)$ for all $t\in\mathbb{R}$. In particular,

$$(S_0,v_0(.),i_0(.))=(S(0),v(0,.),i(0,.))=(\bar{N}_\psi,v^0(.),0).$$

Therefore the attractor $\mathbf{A}$ is the singleton set formed by the disease free equilibrium $(\bar{N}_\psi,v^0(.),0)$. By Theorem 2.39 in [45], the disease-free equilibrium is globally asymptotically stable. □

5. Existence of endemic equilibrium states and the Uniform persistence

In this section, we ensures the existence of a positive equilibrium states and then, we establish the strongly uniform persistence of the solution to problem (2.1).

Lemma 5.1. *Let* $\lim_{J\to 0^+}\frac{f(\bar{N}_\psi,J)}{f(S,J)}>1$ *for* $S\in[0,\bar{N}_\psi)$. *Then, if* $R_0^\psi>1$, *system (2.1) has positive equilibrium states.*

Proof. An endemic equilibrium is a fixed point of the semiflow Φ,

$$\Phi(t,(S^*,\upsilon^*,i^*))=(S^*,\upsilon^*,i^*),\ \text{ with }\ i^*\neq 0,\ \ \forall t\geq 0.$$

From (3.1)-(3.2) and (3.3)-(3.4), we obtain

$$\begin{cases} v^*(b)=e^{-\mu b}\digamma_1(b)\psi S^*, \\ \\ i^*(a)=e^{-\mu a}\digamma_2(a)f(S^*,J^*), \end{cases} \tag{5.1}$$

and

$$\begin{cases} A-(\mu+\psi)S^*-f(S^*,J^*)+\int_0^\infty\alpha(b)\upsilon^*(b)db=0, \\ \\ J^*=\int_0^\infty\beta(a)i^*(a)da. \end{cases} \tag{5.2}$$

Combining the equations (5.2) and (5.1), we get

$$\begin{cases} A=(\mu+\psi(1-\bar{\alpha}))S^*+f(S^*,J^*), \\ \\ \bar{\beta}f(S^*,J^*)=J^*. \end{cases} \tag{5.3}$$

with $\bar{\beta}$, which is defined in (2.4). Following the same argument as [14, 15], we prove the existence of positive equilibrium states. □

We emphasis now on the uniform persistence; see, for instance, [44, 45, 47, 48]. For this purpose, we apply Theorem 5.2 in [45]. We first suppose that

$$\int_0^\infty i_0(a)\Pi(a)da > 0, \tag{5.4}$$

with

$$\Pi(a) = \int_0^\infty e^{-\mu t}\beta(a+t)e^{-\int_a^{a+t}\gamma(\sigma)d\sigma}dt.$$

Remark 5.2. If $\int_0^\infty i_0(a)\Pi(a)da = 0$, then, by Volterra's formulation of the solution, (3.5)-(3.6)-(3.7) and Gronwall's inequality, we can show that $J(t) = 0$ for all $t \geq 0$.

In order to state our main result, we need the following lemma, which is not difficult to prove.

Lemma 5.3. *If the function f is differentiable and concave with respect to J, then the following assertions are verified.*

(1) There exists a positive equilibrium (S^*, J^*) verifying (5.3) such that, for all $S > 0$,

$$\begin{cases} \dfrac{x}{J^*} < \dfrac{f(S,x)}{f(S,J^*)} < 1 \ \ for \ \ 0 < x < J^*, \\ \\ 1 < \dfrac{f(S,x)}{f(S,J^*)} < \dfrac{x}{J^*} \ \ for \ \ x > J^*. \end{cases} \tag{5.5}$$

(2) There exists $\varepsilon > 0$ and there exists $\eta > 0$ such that, for all $S \in [\bar{N}-\varepsilon, \bar{N}+\varepsilon]$,

$$\frac{f(S,J_1)}{J_1} \geq \frac{f(S,J_2)}{J_2}, \tag{5.6}$$

for all $0 < J_1 \leq J_2 \leq \eta$.

We define a persistence function $\rho : X \to \mathbb{R}^+$ by

$$\rho(S_0, i_0, v_0) = \int_0^\infty \beta(a)i_0(a)da.$$

Then

$$\rho(\Phi(t,x)) = \int_0^\infty \beta(a)i(t,a)da := J(t).$$

The following lemma affirm that the hypothesis (H1) in Theorem 5.2. [45] holds. The proof is similar as the one presented in [34].

Lemma 5.4. *Under assumptions (5.4), (5.5), J is positive on $\mathbb{R}$.*

Now we are ready to prove the strong uniform persistence of the disease.

Theorem 5.5. *Let $\digamma_2 \in L^1(\mathbb{R}^+)$ and suppose (5.4), (5.5) and (5.6) hold. Then there exists some $\varepsilon > 0$ such that*

$$\liminf_{t\to\infty} J(t) > \varepsilon,$$

for all nonnegative solutions of (2.1) provided that $R_0^\psi > 1$.

Proof. By Theorem 5.2 in [45], and Lemma 5.4, the solution of problem (2.1)-(2.2) is strongly uniformly persistent if it is weakly uniformly persistent. Suppose that the disease is not uniformly weakly persistent, that is, there exists an arbitrarily small $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} J(t) < \varepsilon.$$

Next, we set $\liminf_{t\to\infty} S(t) = S_\infty$. Using the fluctuation method (see, for instance, [46]), there exists a sequence t_k such that $\lim_{t_k\to\infty} S'(t_k) = 0$ and $\lim_{t_k\to\infty} S(t_k) = S_\infty$.

First, from (3.1), we observe that

$$\lim_{t_k\to\infty} \int_0^\infty \alpha(b)v(t_k,b)db \geq \psi\bar{\alpha}S_\infty.$$

Combining this with the equation of S in (2.1), we have

$$0 \quad \geq \quad A - (\mu + (1-\bar{\alpha})\psi)S_\infty - f(\frac{A}{\mu},\varepsilon).$$

Therefore, we find that

$$S_\infty \geq \bar{N}_\psi - \theta(\varepsilon),$$

where $\theta(\varepsilon) = \dfrac{f(\frac{A}{\mu},\varepsilon)}{\mu + (1-\bar{\alpha})\psi}$, and $\bar{N}_\psi$ is defined in (2.5).

On the other hand, we introduce the following auxiliary problem

$$\begin{cases} \tilde{i}_t(t,a) + \tilde{i}_a(t,a) = -(\mu + \gamma(a))\tilde{i}(t,a), \\ \tilde{i}(t,0) = f\big(\bar{N}_\psi - \theta(\varepsilon), \tilde{J}(t)\big), \\ \tilde{J}(t) = \int_0^\infty \beta(a)\tilde{i}(t,a)da, \\ \tilde{i}(0,a) = i_0(a). \end{cases} \tag{5.7}$$

From the monotonicity of the function f with respect to S and J, we deduce that (for t so large and using a comparison principle)

$$\tilde{i}(t,.) \leq i(t,.), \tag{5.8}$$

Now by employing the same argument as in the proof of Theorem 5.5 in [34], we reach a contradiction. This completes the proof. □

Let X_0 be a subset defined as

$$X_0 = \{(S_0, \upsilon_0(.), i_0(.)) \in X; \int_0^\infty i_0(a)\Pi(a)da = 0\},$$

where Π is defined in (5.4).

From Theorem 5.7 in [45], we have the following result.

Theorem 5.6. *There exists a compact attractor $\mathbf{A_1}$ that attracts all solutions with initial condition belonging to $X \setminus X_0$. Moreover $\mathbf{A_1}$ is $\rho-$ uniformly positive, i.e., there exists some $\delta > 0$ such that*

$$\int_0^\infty \beta(a)i_0(a)da \geq \delta \ \textit{ for all } \ (S_0, \upsilon_0, i_0) \in \mathbf{A_1}. \tag{5.9}$$

6. The global asymptotic stability of the endemic steady state

In this section, we discuss the global stability of the endemic equilibrium $(S^*, v^*(.), i^*(.))$ of system (3.13). Before stating the main result of this section, we need the following estimate, which guarantees that all solutions of (3.13) with initial data satisfying (5.4), are bounded away from 0.

Proposition 6.1. *There exists $\bar{\delta} > 0$ such that, for all $(S_0, v_0(.), i_0(.)) \in \mathbf{A_1}$,*

$$i_0(a) \geq \bar{\delta}\digamma_2(a)e^{-\mu a}, \ \ a \geq 0.$$

Proof. Since $\mathbf{A_1}$ is invariant, there exists a total trajectory $\Psi : \mathbb{R} \to \mathbf{A_1}$, $\Psi(t) = (S(t), v(t,.), i(t,.))$ with $S(0) = S_0$, $v(0,.) = v_0(.)$ and $i(0,a) = i_0(a)$. By (5.9), we have

$$J(t) = \int_0^\infty \beta(a)i(t,a)da \geq \delta,$$

and

$$i(t,a) = f(S(t-a), J(t-a))\digamma_2(a)e^{-\mu a}.$$

According to Lemma 3.2, we have

$$i(t,a) \geq \bar{\delta}\digamma_2(a)e^{-\mu a}, \ \ t \in \mathbb{R}, \ \ a \geq 0,$$

where $\bar{\delta} := f(\dfrac{A}{\mu+\psi+L}, \delta)$. □

Corollary 6.2. *The following estimates hold*

$$\frac{i(t,a)}{i^*(a)} \geq \frac{\bar{\delta}}{f(S^*,J^*)},$$

$$\int_0^\infty \frac{i(t,a)}{i^*(a)}dm_2(a) \geq \frac{\bar{\delta}}{f(S^*,J^*)}, \ \ with \ \ dm_2(a) = \frac{\beta(a)e^{-\mu a}\digamma_2(a)}{\bar{\beta}}da,$$

and

$$f(S(t),J(t)) \geq \bar{\delta},$$

$$\frac{v(t,b)}{v^*(b)} \geq \frac{A}{(\mu+\psi+L)S^*},$$

for all $t \in \mathbb{R}$ and $\bar{\delta} := f(\dfrac{A}{\mu+\psi+L}, \delta)$.

Theorem 6.3. *Under the assumptions of Theorem 5.5, problem (3.13) has a unique positive endemic equilibrium $(S^*, v^*(.), i^*(.))$, which is globally asymptotically stable in $X \setminus X_0$.*

Proof. Let $\Psi : \mathbb{R} \to \mathbf{A_1}$ be a total $\Phi-$trajectory, $\Psi(t) = (S(t), v(t,.), i(t,.))$, $S(0) = S_0$, $v(0,b) = v_0(b)$ and $i(0,a) = i_0(a)$, where $(S(t), v(t,b), i(t,a))$ is solution of problem (3.13). We set

$$dm_1(b) = \alpha(b)e^{-\mu b}\digamma_1(b)db \ \ \text{and} \ \ dm_2(a) = \frac{\beta(a)}{\bar{\beta}}e^{-\mu a}\digamma_2(a)da,$$

where $\bar{\beta}$ is defined in (2.4). We also define

$$\phi_1(b) = \psi S^* \int_b^\infty dm_1(\xi), \ \ \text{and} \ \ \phi_2(a) = f(S^*,J^*)\int_a^\infty dm_2(\xi),$$

and

$$H(y) = y - ln(y) - 1, \ \ \text{and} \ \ G(y) = y - \int_1^y \frac{f(S^*,J^*)}{f(\eta S^*,J^*)}d\eta - 1.$$

Then, for $x := (S_0, v_0(.), i_0(.)) \in \mathbf{A_1}$, we consider the following Lyapunov functional $V(x) = V_1(x) + V_2(x) + V_3(x)$ with

$$V_1(x) = S_0 - S^* - \int_{S^*}^{S_0} \frac{f(S^*,J^*)}{f(\eta,J^*)} d\eta,$$

$$V_2(x) = \int_0^\infty H(\frac{i_0(a)}{i^*(a)})\phi_2(a)da,$$

and

$$V_3(x) = \int_0^\infty G(\frac{v_0(b)}{v^*(b)})\phi_1(b)db.$$

By analyzing the derivative of V_1, and using the definition of the positive steady state in (5.3), we have

$$\begin{aligned}
\frac{d}{dt}V_1(\Psi(t)) &= (1-\frac{f(S^*,J^*)}{f(S(t),J^*)})(A - f(S(t),J(t)) - (\mu+\psi)S(t) + \psi\textstyle\int_0^\infty S(t-b)dm_1(b)) \\
&= (1-\frac{f(S^*,J^*)}{f(S(t),J^*)})(f(S^*,J^*) - \bar{\alpha}\psi S^*) + (\mu+\psi)(S^* - S(t))(1-\frac{f(S^*,J^*)}{f(S(t),J^*)}) \\
&- f(S(t),J(t))(1-\frac{f(S^*,J^*)}{f(S(t),J^*)}) + \psi(1-\frac{f(S^*,J^*)}{f(S(t),J^*)})\textstyle\int_0^\infty S(t-b)dm_1(b).
\end{aligned}$$

Following the same arguments as in the proof of Lemma 9.18 in [45] and noticing that $\int_0^\infty dm_2(a) = 1$ and (3.13), we find

$$\begin{aligned}
\frac{d}{dt}V_2(\Psi(t)) &= H(\frac{f(S(t),J(t))}{f(S^*,J^*)})\phi_2(0) + \textstyle\int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})\phi_2'(a)da \\
&= H(\frac{f(S(t),J(t))}{f(S^*,J^*)})f(S^*,J^*) - f(S^*,J^*)\textstyle\int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})dm_2(a).
\end{aligned}$$

Next, for V_3, we get

$$\begin{aligned}
\frac{d}{dt}V_3(\Psi(t)) &= G(\frac{S(t)}{S^*})\phi_1(0) + \textstyle\int_0^\infty G(\frac{S(t-b)}{S^*})\phi_1'(b)db, \\
&= G(\frac{S(t)}{S^*})\psi\bar{\alpha}S^* - \psi S^*\textstyle\int_0^\infty G(\frac{S(t-b)}{S^*})dm_1(b).
\end{aligned}$$

First, using the definition of H, we remark that

$$\begin{aligned}
H(\frac{f(S(t),J(t))}{f(S^*,J^*)}) &= -\ln(\frac{f(S(t),J(t))}{f(S^*,J^*)}) + \frac{f(S(t),J(t))}{f(S^*,J^*)} - 1 \\
&= -\ln(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - \ln(\frac{f(S(t),J^*)}{f(S^*,J^*)}) + \frac{f(S(t),J(t))}{f(S^*,J^*)} - 1.
\end{aligned}$$

Hence, for $V = V_1 + V_2 + V_3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}V(\Psi(t)) \\
= \quad & (\mu+\psi)(S^* - S(t))(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) + (1 - \frac{f(S^*,J^*)}{f(S(t),J^*)})(f(S^*,J^*) - \bar{\alpha}\psi S^*) \\
- \quad & f(S(t),J(t))(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) - f(S^*,J^*)\ln(\frac{f(S(t),J^*)}{f(S^*,J^*)}) - f(S^*,J^*)\ln(\frac{f(S(t),J(t))}{f(S(t),J^*)}) \\
+ \quad & f(S(t),J(t)) - f(S^*,J^*) + \psi(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) \int_0^\infty S(t-b)dm_1(b) \\
- \quad & f(S^*,J^*) \int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})dm_2(a) \\
+ \quad & G(\frac{S(t)}{S^*})\psi\bar{\alpha}S^* - \psi S^* \int_0^\infty G(\frac{S(t-b)}{S^*})dm_1(b).
\end{aligned}$$

Reorganizing these terms, it follows that

$$\begin{aligned}
&\frac{d}{dt}V(\Psi(t)) \\
= \quad & (\mu+\psi)(S^* - S(t))(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) + f(S^*,J^*)(-\ln(\frac{f(S(t),J^*)}{f(S^*,J^*)}) - \frac{f(S^*,J^*)}{f(S(t),J^*)} + 1) \\
+ \quad & f(S^*,J^*)\big(\frac{f(S(t),J(t))}{f(S(t),J^*)} - \ln(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - 1\big) \\
- \quad & f(S^*,J^*) \int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})dm_2(a) - \bar{\alpha}\psi S^*(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) + G(\frac{S(t)}{S^*})\psi\bar{\alpha}S^* \\
- \quad & \psi S^* \int_0^\infty G(\frac{S(t-b)}{S^*})dm_1(b) + \psi S^* \int_0^\infty (1 - \frac{f(S^*,J^*)}{f(S(t),J^*)})(\frac{S(t-b)}{S^*} - \frac{S(t)}{S^*})dm_1(b) \\
+ \quad & (1 - \frac{f(S^*,J^*)}{f(S(t),J^*)})S(t)\psi\bar{\alpha}.
\end{aligned}$$

Using the definition of H, we arrive at

$$\begin{aligned}
\frac{d}{dt}V(\Psi(t)) \quad = \quad & (\mu+\psi)(S^* - S(t))(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) + f(S^*,J^*)(-\ln(\frac{f(S(t),J^*)}{f(S^*,J^*)}) - \frac{f(S^*,J^*)}{f(S(t),J^*)} + 1) \\
+ \quad & f(S^*,J^*)H\big(\frac{f(S(t),J(t))}{f(S(t),J^*)}\big) - f(S^*,J^*) \int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})dm_2(a) \\
+ \quad & \bar{\alpha}\psi(S(t) - S^*)(1 - \frac{f(S^*,J^*)}{f(S(t),J^*)}) \\
+ \quad & \psi S^* \int_0^\infty (G(\frac{S(t)}{S^*}) - G(\frac{S(t-b)}{S^*}) + (1 - \frac{f(S^*,J^*)}{f(S(t),J^*)})(\frac{S(t-b)}{S^*} - \frac{S(t)}{S^*}))dm_1(b).
\end{aligned}$$

Further,

$$G'(\frac{S(t)}{S^*}) = 1 - \frac{f(S^*,J^*)}{f(S(t),J^*)},$$

from which it follows that

$$\begin{aligned}
\frac{d}{dt}V(\Psi(t)) &= (\mu+\psi(1-\bar{\alpha}))(S^*-S(t))(1-\frac{f(S^*,J^*)}{f(S(t),J^*)}) \\
&+ f(S^*,J^*)\big(-\ln(\frac{f(S(t),J^*)}{f(S^*,J^*)}) - \frac{f(S^*,J^*)}{f(S(t),J^*)} + 1\big) \\
&+ \psi S^* \int_0^\infty \big(G(\frac{S(t)}{S^*}) - G(\frac{S(t-b)}{S^*}) + G'(\frac{S(t)}{S^*})(\frac{S(t-b)}{S^*} - \frac{S(t)}{S^*})\big)dm_1(b) \\
&+ f(S^*,J^*)\big(H(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - \int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})dm_2(a)\big).
\end{aligned}$$

Observe that the function $-ln(x) - \frac{1}{x} + 1 \le 0$, for all $x > 0$, and $\bar{\alpha} < 1$. Then the first two terms of this above equation are negative. Now since G is convex, we have

$$G(b) - G(a) + G'(b)(a-b) \le 0, \text{ for all } a,b \text{ positive.}$$

Thus the third term is also negative. Next, we will claim that the last term is negative. For this, we set

$$\begin{aligned}
D &:= H(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - \int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})dm_2(a), \\
&= H(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - \int_0^\infty H(\frac{f(S(t-a),J(t-a))}{f(S^*,J^*)})\frac{\beta(a)e^{-\mu a}\digamma_2(a)}{\bar{\beta}}da.
\end{aligned}$$

Since H is convex, we find from Jensen inequality [46, 49] that

$$D \le H(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - H\big(\int_0^\infty \frac{f(S(t-a),J(t-a))}{f(S^*,J^*)}\frac{\beta(a)e^{-\mu a}\digamma_2(a)}{\bar{\beta}}da\big).$$

In view of the definition of J and J^* in (3.13), (5.2), we obtain

$$D \le H(\frac{f(S(t),J(t))}{f(S(t),J^*)}) - H(\frac{J(t)}{J^*}).$$

Let us consider a time t such that $\frac{J(t)}{J^*} < 1$. Then, from the hypothesis (5.5), we have

$$\frac{J(t)}{J^*} < \frac{f(S(t),J(t))}{f(S(t),J^*)} < 1.$$

Hence, since $H(1) = 0$ and H is decreasing in $(0,1)$, we have

$$H(\frac{J(t)}{J^*}) > H(\frac{f(S(t),J(t))}{f(S(t),J^*)}).$$

Therefore, $D < 0$. For other values of t, i.e., $\frac{J(t)}{J^*} > 1$. From (5.5), and by taking into account that H is an increasing function over $(1,\infty)$, we arrive at the claim. Consequently,

$$\frac{d}{dt}V(\Psi(t)) \le 0.$$

Further, $\frac{d}{dt}V(\Psi(t)) = 0$ implies that $S(t) = S^*$.

Now we look for the largest invariant set Q for which $\frac{d}{dt}V(\Psi(t)) = 0$. In Q, we must have $S(t) = S^*$ for all $t \in \mathbb{R}$. First, according to expressions of v and v^* in (3.13) and (5.1), it is clear that

$$v(t,.) = v^*(.), \text{ for all } t \in \mathbb{R}.$$

Using the equation of S in (3.13), we obtain

$$A - (\mu + \psi(1-\bar{\alpha}))S^* = f(S^*, J(t)),$$

from which and the first equation of (5.3), we have

$$f(S^*, J(t)) = f(S^*, J^*), \text{ for all } t \in \mathbb{R}.$$

It follows that

$$J(t) = J^*, \text{ for all } t \in \mathbb{R}.$$

Consequently, from the expressions of i and i^* in (3.13) and (5.1), we conclude that

$$i(t,.) = i^*(.), \text{ for all } t \in \mathbb{R}.$$

Following the same arguments as in the proof of Theorem 4.1, we conclude the global asymptotic stability of the endemic equilibrium. Uniqueness is a direct consequence of the fact that $\frac{d}{dt}V(\Psi(t)) = 0$ holds only on the line $S = S^*$. □

7. Numerical simulations

In this section, we present some simulations to illustrate our results. To this end, we choose the saturation response incidence rate $f(S,J) = \frac{\beta IJ}{I+J}$. In the first figure 1, we choose values of the parameters in such a way to have $R_0^\psi < 1$, and one can observe the global stability of the disease free equilibrium. In the second figure 2, parameters are chosen to have $R_0^\psi > 1$, and in that case we can observe the global stability of the endemic equilibrium. In both cases, we take

$$\alpha(b) = \begin{cases} 0.66(b-15)^2 e^{-0.6(b-15)}, & 15 < b \leq 30, \\ 0.0185, & 30 < b \leq 100, \\ 0, & elsewhere. \end{cases}$$

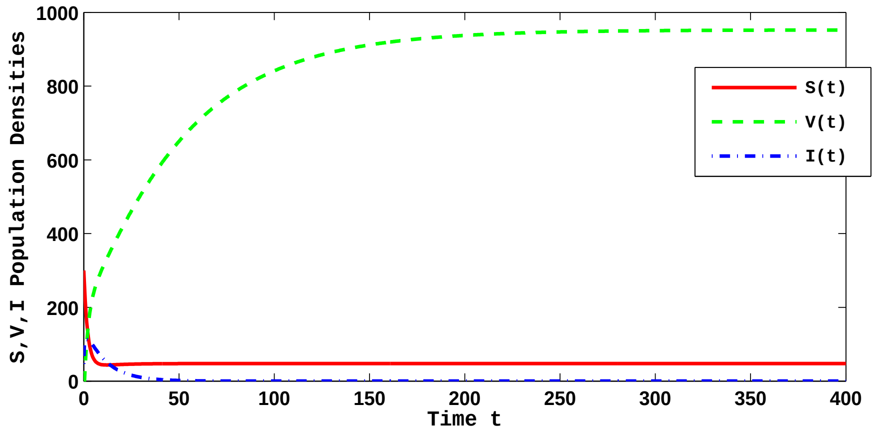

FIGURE 1. Simulation of solutions of system (2.1), for $\psi = 0.4$, $A = 20$, $\mu = 0.02$, $\gamma = 0.1$. The disease-free equilibrium $(\bar{N}, \upsilon^0(.), 0)$ is globally asymptotically stable.

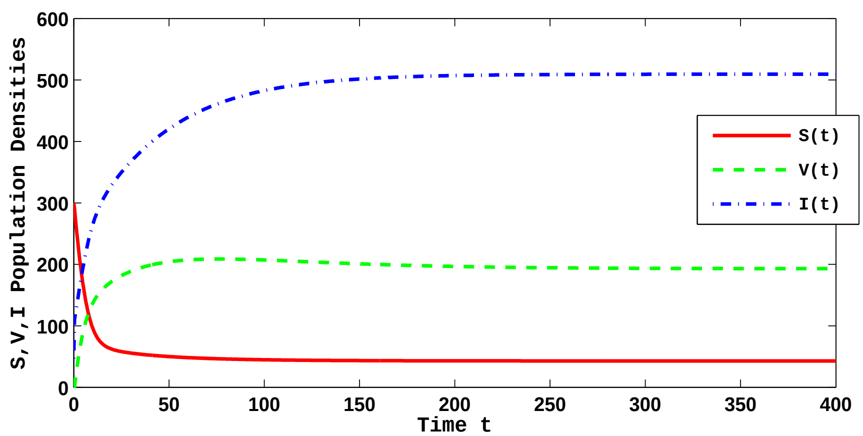

FIGURE 2. Simulation of solutions of system (2.1), for $\psi = 0.09$, $A = 20$, $\mu = 0.02$, $\gamma = 0.01$. The endemic equilibrium $(S^*, \upsilon^*(.), i^*(.))$ is globally asymptotically stable.

8. DISCUSSION

In this paper, we established a SVIR age structured model with general incidence rate. The main objective of this paper is to find, under what conditions on incidences, the global stability of equilibria. The results are linked to the basic reproduction rate which naturally depends of the vaccination rate ψ. So if this rate is less than one, the infection disappears and if this one is strictly greater than one then, the endemic equilibrium appears and becomes globally asymptotically stable. Moreover, one can control the spread of the epidemic disease by letting the basic reproduction number less than one, which can be done by controlling the vaccination rate ψ. In this framework and in the context of vaccination models, an important question arises: if the R_0^0 (which corresponds to the basic reproduction rate of (2.1)-(2.2) without vaccination ($\psi = 0$)) is strictly greater than one, how many persons have to be vaccinated to eradicate the disease? To this end, we begin by supposing that $R_0^0 := \lim_{\psi \to 0} R_0^{\psi} > 1$.

Next the derivative of R_0^ψ with respect to ψ is

$$\frac{dR_0^\psi}{d\psi} = -\frac{\bar{\beta}A(1-\bar{\alpha})}{\mu+(1-\bar{\alpha}\psi)}F'(\bar{N}_\psi),$$

with $F(\bar{N}_\psi) = \dfrac{\partial f}{\partial J}(\bar{N}_\psi, 0)$. Since $f(S,J)$ is an increasing function with respect to S, we have $F'(\bar{N}_\psi) \geq 0$ and thus $\dfrac{dR_0^\psi}{d\psi} \leq 0$. On the other hand, we have

$$\lim_{\psi\to\infty} R_0^\psi = \bar{\beta}\frac{\partial f}{\partial J}(0,0) = 0.$$

So there exists a unique $\psi^* > 0$ such that $R_0^{\psi^*} = 1$. Therefore, we have a positive probability to eradicate the disease provided that we apply a vaccination rate $\psi > \psi^*$.

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