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# EXISTENCE OF NONTRIVIAL SOLUTIONS FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract.** In this paper, the existence and multiplicity of nontrivial solutions for higher-order ordinary differential equations are discussed based on the critical point theory. Two examples are provided to support our main results.
**Keywords.** Higher order differential equations; Nontrivial solutions; Critical point theory.

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## 1. INTRODUCTION

This paper is concerned with the existence of solutions for the system of 2m-2n order ordinary differential equations

$$\begin{cases} (-1)^m y^{(2m)}(x) = ay + bz + f_1(x,y,z), \quad 0 < x < 1, \\ (-1)^n z^{(2n)}(x) = by + cz + f_2(x,y,z), \quad 0 < x < 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \quad 0 \le i \le m-1, \\ z^{(2j)}(0) = z^{(2j)}(1) = 0, \quad 0 \le j \le n-1, \end{cases} \tag{1.1}$$

where $a, b$ ad $c$ are real constants, $m, n \in N^+$ and $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty) (i = 1,2)$ are continuous.

In recent years, the existence, multiplicity and nonexistence of positive solutions for higher order differential equations have been extensively studied using different methods, such as, fixed point methods in cones, method of upper and lower solutions, nonlinear alternatives of Leray-Schauder and variational methods. We refer the readers to [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. To the best of our knowledge, using fixed point theorems and the method of upper and lower solutions, the existence, multiplicity and nonexistence results of positive solutions for the nonlinear differential systems with

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eigenvalue parameters were established in [9, 10]

$$\begin{cases} (-1)^p u^{(2p)} = \lambda a(t) f(u(t), v(t)), \quad t \in [a,b], \\ (-1)^q v^{(2q)} = \mu b(t) g(u(t), v(t)), \quad t \in [a,b], \\ u^{(2i)}(a) = u^{(2i)}(b) = 0, \quad 0 \le i \le p-1, \\ v^{(2j)}(a) = v^{(2j)}(b) = 0, \quad 0 \le j \le q-1. \end{cases}$$

In [11], Ji and Tang considered the following coupled system of second and fourth order elliptic equations

$$\begin{cases} \Delta^2 y + k(y-z)^+ = f_1(x,y,z), \quad \text{in } \Omega, \\ -\Delta z - k(y-z)^+ = f_2(x,y,z), \quad \text{in } \Omega, \\ \Delta y = y = 0, \quad \text{on } \partial\Omega, \\ z = 0, \quad \text{on } \partial\Omega. \end{cases}$$

Using the linking theorem, they obtained two existence results of nontrivial solutions for the above system.

Inspired and motivated by these results, the purpose of this paper is to establish some new existence and multiplicity of nontrivial solutions for (1.1) using the critical point theory under some suitable assumptions. The rest of this paper is organized as follows. In Section 2, we give some preliminary and some suitable assumptions of $f_i$. In Section 3, the last section, we give the main results and two examples to illustrate our them.

## 2. Preliminaries

Let $\Omega = [0,1]$, and let $\mathbf{H} = L^2(\Omega) \times L^2(\Omega)$ be the Hilbert space equipped with the inner product

$$(u,v)_{\mathbf{H}} = \int_\Omega (u_1 v_1 + u_2 v_2) dx$$

and the deduced norm

$$\|u\|^2_{\mathbf{H}} = \|u_1\|^2_{L^2} + \|u_2\|^2_{L^2},$$

for $u = (u_1,u_2), v = (v_1,v_2) \in \mathbf{H}$. Let $\mathbf{V} = H_0^{(m)}(\Omega) \times H_0^{(n)}(\Omega)$ be the Hilbert space equipped with the inner product

$$(u,v)_{\mathbf{V}} = \int_\Omega (u_1^{(m)} v_1^{(m)} + u_2^{(n)} v_2^{(n)}) dx$$

and the deduced norm

$$\|u\|^2_{\mathbf{V}} = \int_\Omega |u_1^{(m)}|^2 dx + |u_2^{(n)}|^2 dx,$$

for $u = (u_1,u_2), v = (v_1,v_2) \in \mathbf{V}$. It is obvious that the embedding from $\mathbf{V}$ into $\mathbf{H}$ is continuous and dense. Indeed, it is compact. Let $V^*$ denote the dual space of $V$. It is not hard to see that the eigenvalue problem

$$\begin{cases} (-1)^p y^{(2n)}(x) = \lambda y, \quad x \in (0,1), \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \quad 0 \le i \le n-1 \end{cases}$$

has infinitely many eigenvalues $\lambda = k^{2n}$, $k = 1,2,\cdots\cdots$. Then it is clear that $\lambda = 1$ is the smallest eigenvalue of

$$\begin{cases} (-1)^m y^{(2m)}(x) = \lambda y, & x \in (0,1), \\ (-1)^n z^{(2n)}(x) = \lambda z, & x \in (0,1), \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \le i \le m-1, \\ z^{(2j)}(0) = z^{(2j)}(1) = 0, & 0 \le j \le n-1. \end{cases}$$

Furthermore, the corresponding Poincare inequality is

$$\|u\|_{\mathbf{H}}^2 \le \|u\|_{\mathbf{V}}^2, \tag{2.1}$$

for all $u = (y,z) \in \mathbf{V}$.

For problem (1.1), the following assumptions will be used

$(F_1)$ $f(x,u) = (f_1, f_2) = \nabla F(x,u)$ is a Caratheodory function, and there exist $p > 2, c > 0$ such that

$$|f_1(x,u)| + |f_2(x,u)| \le c(1 + |u|^{p-1}),$$

for all $u = (y,z) \in R^2$ and almost every $x \in \Omega$;

$(F_2)$ there exist $\theta \in (0, \frac{1}{2})$ and $M > 0$ such that

$$0 < F(x,u) \le \theta u \cdot \nabla F(x,u),$$

for$|u| \ge M$ and almost every $x \in \Omega$;

$(F_3)$ $\liminf_{|u|\to 0} F(x,u)/|u|^2 = 0$, uniformly for a.e. $x \in \Omega$;

$(F_4)$ $\liminf_{|u|\to \infty} F(x,u)/|u|^2 = \infty$, uniformly for a.e. $x \in \Omega$.

Define the functional $J : \mathbf{V} \to \mathbf{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega (|y^{(m)}|^2 + |z^{(n)}|^2)dx - \frac{1}{2}\int_\Omega \Big(Au, u\Big)dx - \int_\Omega F(x,u)dx,$$

where $u = (y,z) \in \mathbf{V}$ , $\nabla F(x,u) = f(x,u) = (f_1, f_2)$, and $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with the norm

$$\|A\| := \max\{|\mu| : \mu \textit{ is an eigenvalue of } A\}.$$

Under $(F_1)$, it is easy to show that functional $J(u)$ is well defined, and $J(u) \in C^1(\mathbf{V}, \mathbf{R})$. Moreover,

$$(J'(u), v) = \int_\Omega (y^{(m)} v_1^{(m)} + z^{(n)} v_2^{(n)})dx - \int_\Omega \Big(Au, v\Big)dx - \int_\Omega \nabla F(x,u) \cdot v dx,$$

for all $u = (y,z), v = (v_1, v_2) \in \mathbf{V}$. Therefore, a vector $u \in \mathbf{V}$ is a weak solution of (1.1) if and only if it is a critical point of energy functional $J(u)$.

For the sake of convenience, we give the notation of (PS)-condition and two essential theorems.

**Definition 2.1.** Let $\mathbf{E}$ be a real Banach space. Afunctional $I \in C^1(\mathbf{E}, R)$ is said to satisfy the (PS)-condition, if, whenever $\{u_n\} \subset \mathbf{E}$ is such that $|I(u_n)| \le c$, uniformly in $n$, while $\|I'(u_n)\| \to 0$ as $n \to \infty$, then $\{u_n\}$ possesses a convergent subsequence.

**Theorem A.** [13] *Let* $\mathbf{E}$ *be a real Banach space and* $I \in C^1(\mathbf{E}, R)$ *satisfy the (PS)-condition. Suppose that* *(i) There exist $\rho > 0$ and $\alpha > 0$ such that*

$$I\mid_{\partial B_\rho} \geq I(0) + \alpha,$$

*where $B_\rho = \{u \in \mathbf{E} : \|u\| \leq \rho\}$;*

*(ii) There is an $e \in \mathbf{E}$ and $\|e\| > \rho$ such that*

$$I(e) \leq I(0).$$

*Then $I(u)$ has a critical value c, which can be characterized as*

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[0,1]} I(u),$$

*where*

$$\Gamma = \{\gamma \in C([0,1], \mathbf{E}) : \gamma(0) = 0, \gamma(1) = e\}.$$

**Theorem B.** [13] *Let $\mathbf{E}$ be an infinite dimensional real Banach space and let $\varphi \in C^1(\mathbf{E}, \mathbf{R^1})$ be even, satisfy the (PS)-condition, and $\varphi(0) = 0$. Suppose that $\mathbf{E} = \mathbf{V} \oplus \mathbf{X}$, where $\mathbf{V}$ is finite dimensional, and $\varphi$ satisfies*

*(i) there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in \mathbf{X}$ with $\|u\| = \rho$;*

*(ii) for any finite dimensional subspace $\mathbf{W} \subset \mathbf{E}$, there is $R = R(W)$ such that $\varphi(u) \leq 0$ for all $u \in \mathbf{W}$ with $\|u\| \geq R$.*

*Then $f$ possesses an unbounded sequence of critical values.*

## 3. Main results

**Theorem 3.1.** *Assume that $(F_1)$-$(F_3)$ hold. If $\|A\| < 1$, then problem (1.1) has at least one nontrivial solution.*

*Proof.* From the Sobolev embedding theorem (see in [12], p. 167, Theorem 7:22), we know that the embedding of $\mathbf{V} \to L^p \times L^p$ is compact for $p \geq 1$. Combining with $(F_2)$, the map $K : \mathbf{V} \to V^*$ is compact. Therefore, we only need to show that any (PS)-sequence for $J$ is bounded in $\mathbf{V}$.

Let $\{u_n\} \subset \mathbf{H}$ be a $(PS)$-sequence. In particular, $\{u_n\}$ satisfies

$$|J(u_n)| \leq c, \quad \textit{and} \quad \langle J'(u_n), u_n\rangle \to 0 \;\; \textit{as} \;\; n \to \infty.$$

From $(F_2)$ and (2.1), we have

$$\begin{aligned}
c + \|u_n\|_{\mathbf{V}} &\geq J(u_n) - \theta\Big(J'(u_n), u_n\Big) \\
&= \frac{1}{2}\|u_n\|^2_{\mathbf{V}} - \frac{1}{2}\int_\Omega \Big(Au_n, u_n\Big)dx - \int_\Omega F(x, u_n)dx \\
&\qquad -\theta\Big[\|u_n\|^2_{\mathbf{V}} - \int_\Omega \Big(Au_n, u_n\Big)dx - \int_\Omega \nabla F(x, u_n)\cdot u_n dx\Big] \\
&= (\frac{1}{2} - \theta)\|u_n\|^2_{\mathbf{V}} - (\frac{1}{2} - \theta)\int_\Omega \Big(Au_n, u_n\Big)dx + \int_\Omega \Big(\theta\nabla F(x, u_n)\cdot u_n - F(x, u_n)\Big)dx \\
&\geq (\frac{1}{2} - \theta)\|u_n\|^2_{\mathbf{V}} - (\frac{1}{2} - \theta)\int_\Omega \Big(Au_n, u_n\Big)dx \\
&\geq (\frac{1}{2} - \theta)[\|u_n\|^2_{\mathbf{V}} - \|A\|\|u_n\|^2_{\mathbf{H}}] \\
&\geq (\frac{1}{2} - \theta)(1 - \|A\|)\|u_n\|^2_{\mathbf{V}},
\end{aligned}$$

for sufficiently large $n$. Since $\theta < \frac{1}{2}$, $\|A\| < 1$, it means that $\{u_n = (y_n, z_n)\} \subset \mathbf{V}$ is bounded.

Secondly, from $(F_3)$ it is clear that $J(0) = 0$. Furthermore, from $(F_1)$ and $(F_3)$, it follows that there exist $0 < \varepsilon < \frac{1}{2}(1 - \|A\|)$ and $C(\varepsilon) > 0$ such that

$$|F(x,u)| \le \varepsilon |u|^2 + C(\varepsilon)|u|^p$$

for all $u \in \mathbf{R^2}$ and almost every $x \in \Omega$. By the Poincare inequality (2.1), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{\mathbf{V}}^2 - \frac{1}{2}\int_\Omega \Big(Au,u\Big)dx - \int_\Omega F(x,u)dx \\ &\ge \frac{1}{2}\|u\|_{\mathbf{V}}^2 - \frac{1}{2}\|A\| \cdot \|u\|_{\mathbf{H}}^2 - \varepsilon\|u\|_{\mathbf{H}}^2 - C(\varepsilon)\|u\|_{\mathbf{H}}^p \\ &\ge (\frac{1}{2} - \frac{1}{2}\|A\| - \varepsilon)\|u\|_{\mathbf{V}}^2 - C_1\|u\|_{\mathbf{V}}^p, \end{aligned}$$

for $u = (y,z) \in \mathbf{V}$ . So, (i) of **TheoremA** is proved if $\|u\| = \rho > 0$ is sufficiently small.

Finally, by condition $(F_2)$, there exists a $\gamma_0(x) > 0, x \in \Omega$ such that

$$F(x,u) \ge \gamma_0(x)|u|^{\frac{1}{\theta}}$$

for $|u| \ge M$, which is given as $(F_2)$.

Hence, if $u = (y,z) \in \mathbf{V}$ does not vanish identically, then for some constants $c_1(u)$, $c_2(u)$, and $\theta \in (0, \frac{1}{2})$, we have

$$\begin{aligned} J(\lambda u) &= \frac{1}{2}\lambda^2\|u\|_{\mathbf{V}}^2 - \frac{\lambda^2}{2}\int_\Omega \Big(Au,u\Big)dx - \int_\Omega F(x,\lambda u)dx \\ &\le \Big(\frac{1}{2}\|u\|_{\mathbf{V}}^2 - \frac{1}{2}\int_\Omega \Big(Au,u\Big)dx\Big)\lambda^2 - \int_\Omega \gamma_0(x)\lambda^{\frac{1}{\theta}}|u|^{\frac{1}{\theta}}dx \\ &= c_1(u)\lambda^2 - c_2(u)\lambda^{\frac{1}{\theta}} \to -\infty, \end{aligned}$$

when $\lambda \to +\infty$. Then $J(u)$ satisfies the assumptions (ii) of **Theorem A**.

From the above discussions, $J(u)$ has at least one critical point in $\mathbf{V}$, namely the problem (1.1) has at least one positive solution. □

**Theorem 3.2.** *Assume that $(F_1)$-$(F_4)$ hold. Let $F(x,u)$ be even in $u$, i.e., $F(x,u) = F(x,-u)$ for all $x \in \Omega$ and $u \in \mathbf{R^2}$. In addition, the matrix $A$ has two real eigenvalues $\mu_1$, $\mu_2$ satisfying $0 < \mu_1 < \mu_2 < 1$. Then problem (1.1) has infinitely many solutions.*

*Proof.* We only need to verify that $J(u)$ satisfies all the conditions of **Theorem B**.

First, since $F(x,u) = F(x,-u)$ for all $x \in \Omega$, $u \in \mathbf{R^2}$, functional $J$ is also even. From the proof of Theorem 3.1, we know that $J$ satisfies (PS)-condition and $J(0) = 0$. So the condition (i) of **Theorem B** is also satisfied.

Second, we will prove that $J$ satisfies the condition (ii) of **Theorem B**. If this is not true, there exists a finite dimensional subspace $\mathbf{W}$ of $H_0^m(\Omega) \times H_0^n(\Omega)$ and a sequence $\{u_n\} \subset \mathbf{W}$ such that $J(u_n) > 0$, $n = 1,2,\cdots$, with $\|u_n\|_{\mathbf{V}} \to \infty$ as $n \to \infty$. Without loss of generality, we may suppose that $\|u_n\|_{\mathbf{V}} > 0$, $n = 1,2,\cdots$. Let $t_n = \|u_n\|_{\mathbf{V}}, u_n^* = \frac{u_n}{t_n}, n = 1,2,\cdots$. Then $\|u_n^*\|_{\mathbf{V}} = 1, n = 1,2,\cdots$, and $t_n \to \infty$ as $n \to \infty$.

On one hand, by the definition $\liminf\limits_{|u|\to\infty} F(x,u)/|u|^2 = \infty$, uniformly for a.e. $x \in \Omega$, there exists a $\tau > 0$ such that

$$F(x,u) \ge a|u|^2, \quad |u| \ge \tau, \quad x \in \Omega, \tag{3.1}$$

where $a$ satisfies

$$(a+\mu_1)\|u_n^*\|_{\mathbf{H}}^2 \geq \xi > 1.$$

On the other hand, since that $F(x,u)-a|u|^2$ is continuous for $x\in\Omega, |u|\leq\tau$, there exists a $N>0$ such that

$$F(x,u)\geq a|u|^2-N, \quad |u|\leq\tau, \quad x\in\Omega. \tag{3.2}$$

Then from (3.1) and (3.2) it follows that

$$F(x,u)\geq a|u|^2-N, \quad |u|\geq 0, \quad x\in\Omega.$$

Finally, we have

$$\begin{aligned}
J(u_n)=J(t_nu_n^*) &= \frac{1}{2}t_n^2\|u_n^*\|_{\mathbf{V}}^2-\frac{1}{2}t_n^2\int_\Omega\Big(Au_n^*,u_n^*\Big)dx-\int_\Omega F(x,t_nu_n^*)dx\\
&\leq \Big(\frac{1}{2}-\frac{1}{2}\int_\Omega\Big(Au_n^*,u_n^*\Big)dx\Big)t_n^2-\int_\Omega at_n^2|u_n^*|^2dx+N\\
&= \Big(\frac{1}{2}-\frac{1}{2}\int_\Omega\Big(Au_n^*,u_n^*\Big)dx\Big)t_n^2-at_n^2\|u_n^*\|_{\mathbf{H}}^2+N\\
&\leq \Big[\frac{1}{2}-\frac{1}{2}(a+\mu_1)\|u_n^*\|_{\mathbf{H}}^2\Big]t_n^2+N\\
&\leq -\eta t_n^2+N,
\end{aligned}$$

where $\eta=\frac{1}{2}(\xi-1)>0$. In view of $t_n\to\infty$, as $n\to\infty$, we have that $J(u_n)\to-\infty$ as $n\to\infty$, which contradicts with the assumption $J(u_n)>0, \quad n=1,2,\cdots$. Therefore according to **Theorem B**, $J$ has infinitely many critical points, namely, problem (1.1) has infinitely many solutions. □

Finally, we give two examples to illustrate our results.

**Example 3.3.** We consider the existence of positive solutions for the following fourth-sixth order systems:

$$\begin{cases}
y^{(4)}(x)=\frac{1}{2}y+\frac{1}{4}z+24y^3+16yz^2-2yz, \quad 0<x<1,\\
-z^{(6)}(x)=\frac{1}{4}y+\frac{1}{2}z+16y^2z-y^2-3z^2, \quad 0<x<1,\\
y(0)=y(1)=y''(0)=y''(1)=0,\\
z(0)=z(1)=z''(0)=z''(1)=z^{(4)}(0)=z^{(4)}(1)=0,
\end{cases} \tag{3.3}$$

where $A=\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}$. By some simple computation, we easily get that $\|A\|=\frac{3}{4}$.

$$f(x,u)=(f_1(x,u),f_2(x,u))=(24y^3+16yz^2-2yz,16y^2z-y^2-3z^2)=\nabla F(x,u),$$

$$F(x,u)=(8y^2-z)(y^2+z^2), u=(y,z)\in R^2, 8y^2>z.$$

Moreover, we can choose $p=11, c=56$, $\theta=\frac{5}{12}$ such that

$$|f_1(x,u)|+|f_2(x,u)\leq c(1+|u|^2)=56(1+(y^2+z^2)^5),$$

$$F(x,u)=(8y^2-z)(y^2+z^2)\leq\frac{5}{12}u\cdot\nabla F(x,u)=\frac{5}{4}(8y^2-z)(y^2+z^2)+8y^2z^2.$$

It is obvious that

$$\liminf_{|u|\to 0}F(x,u)/|u|^2=\liminf_{|u|\to 0}\frac{(8y^2-z)(y^2+z^2)}{y^2+z^2}=0.$$

Therefore, all the conditions in Theorem 3.1 are satisfied, (3.3) has at least one nontrivial solution.

**Example 3.4.** We consider the existence of positive solutions for the following fourth order and sixth order systems:

$$\begin{cases} y^{(4)}(x) = \frac{3}{8}y + \frac{1}{8}z + 3y^2z + z^3, \quad 0 < x < 1, \\ -z^{(6)}(x) = \frac{1}{8}y + \frac{3}{8}z + y^3 + 3yz^2, \quad 0 < x < 1, \\ y(0) = y(1) = y''(0) = y''(1) = 0, \\ z(0) = z(1) = z''(0) = z''(1) = z^{(4)}(0) = z^{(4)}(1) = 0, \end{cases} \tag{3.4}$$

where $A = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{pmatrix}$. By some simple computation, we can easily get that the matrix A has two real eigenvalues $\mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{4}$ respectively.

$$f(x,u) = (f_1(x,u), f_2(x,u)) = (z^3 + 3y^2z, y^3 + 3yz^2) = \nabla F(x,u), F(x,u) = yz(y^2 + z^2).$$

Furthermore, we can choose $p = 4, c = 1, \theta = \frac{3}{8}$ such that

$$|f_1(x,u)| + |f_2(x,u)| = (y+z)^3 \leq 1 + |u|^2 = 1 + (y^2 + z^2)^3,$$

$$F(x,u) = yz(y^2 + z^2) \leq \frac{3}{8} u \cdot \nabla F(x,u) = \frac{3}{2} yz(y^2 + z^2).$$

It is obvious that

$$\liminf_{|u|\to 0} F(x,u)/|u|^2 = \liminf_{|u|\to 0} \frac{yz(y^2+z^2)}{y^2+z^2} = 0, \quad \liminf_{|u|\to\infty} F(x,u)/|u|^2 = \liminf_{|u|\to\infty} \frac{yz(y^2+z^2)}{y^2+z^2} = \infty.$$

Therefore all the conditions in Theorem 3.2 are satisfied and (3.4) has infinitely many solutions.

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