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VISCOSITY METHODS FOR NONEXPANSIVE AND MONOTONE MAPPINGS IN HILBERT SPACES

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Abstract. The purpose of this paper is to study variational inclusions, equilibrium problems and fixed point problems via a viscosity iterative algorithm. Strong convergence of the viscosity iterative algorithm is obtained in a Hilbert space.

Keywords. Inclusion problem; Nonexpansive mapping; Equilibrium problem, Iteration method; Variational inequality.

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1. Introduction-Preliminaries

The study on convex feasibility problems is an important branch of nonlinear analysis optimization theory. Numerous problems in physics, transportation, signal process and economics are reduced to find a solution to a convex feasibility problem, which cover inclusion problems, equilibrium problems, fixed point problems, variational inequalities and so on. The motivation for this subject is mainly due to its possible applications to mathematical modeling of concrete complex problems; see [1]-[5] and the references therein. Solving such problems, we have to obtain some solution which is simultaneously a solution of two or more subproblems or a solution of one subproblem on solution sets of another subproblem; see [6]-[14] and the references therein.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H and let $Proj_C$ be the metric projection from H onto C.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

Recall that A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-inverse-strongly monotone.

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Recall that the celebrated classical variational inequality is to find an $x \in C$ such that

$$\langle y - x, Ax \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, we use $VI(C,A)$ to denote the solution set of (1.1). It is known that $x \in C$ is a solution of variational inequality (1.1) iff x is a solution of the fixed point equation $Proj_C(I - rA)x = x$, where $r > 0$ is a constant and I is the identity mapping on H. Recently, fixed point algorithms have been employed to study solutions of of variational inequality (1.1); see [6, 11, 15, 16] and the references therein.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

In this paper, we use $EP(F)$ to denote the solution set of equilibrium problem (1.2). The equilibrium problem provides us with a natural, unified, and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. This theory has witnessed an explosive growth in theoretical advances and applications. Recently, numerical methods have been introduced and studied for solutions of equilibrium problem (1.2); see [3, 6, 9, 10, 17, 18] and the references therein.

To study the equilibrium problem, we may assume that F satisfies the following conditions:

(A1) $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A2) $F(x,x) = 0$ for all $x \in C$;
(A3) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous;
(A4) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y).$$

Let $T : C \to C$ be a mapping. In this paper, we use $Fix(T)$ to denote the fixed point set of T. Recall that T is said to be contractive iff there exists a constant $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

For such a case, T is also said to be α-contractive. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is known that the fixed point set of nonexpansive mappings is nonempty provided that the subset C is bounded, convex and closed. Finding an optimal point in common fixed/zero point sets of a family of nonlinear mappings is very important, which occurs frequently in various areas of mathematical sciences and engineering; see [19]-[24] and the references therein.

Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and let $\{\gamma_i\}$ be a nonnegative real sequence with $0 \le \gamma_i < 1$, $\forall i \ge 1$. For $n \ge 1$, define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&V_{n,n+1} = I,\\
&V_{n,n} = (1-\gamma_n)I + \gamma_n S_n V_{n,n+1},\\
&V_{n,n-1} = (1-\gamma_{n-1})I + \gamma_{n-1} S_{n-1} V_{n,n},\\
&\vdots\\
&V_{n,k} = (1-\gamma_k)I + \gamma_k S_k V_{n,k+1},\\
&V_{n,k-1} = (1-\gamma_{k-1})I + \gamma_{k-1} S_{k-1} V_{n,k},\\
&\vdots\\
&V_{n,2} = (1-\gamma_2)I + \gamma_2 S_2 V_{n,3},\\
&W_n = V_{n,1} = (1-\gamma_1)I + \gamma_1 S_1 V_{n,2}.
\end{aligned} \tag{1.3}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$, see [20] and the references therein.

Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be monotone iff, for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x-y, f-g\rangle > 0$. M is a maximal mapping iff the graph $Graph(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \ge 0$, for all $(y, g) \in Graph(M)$ implies $f \in Rx$.

For a maximal monotone operator M on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(M)$, where $Dom(M)$ denotes the domain of M. It is known that J_r is firmly nonexpansive, that is, $\|J_r x - J_r y\|^2 \le \langle J_r x - J_r y, x-y\rangle$, $\forall x, y \in H$, and $Fix(J_r) = M^{-1}(0)$, where $Fix(J_r) := \{x \in Dom(M) : x = J_r x\}$, and $M^{-1}(0) := \{x \in H : 0 \in Mx\}$.

Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as signal processing, image recovery, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators. Splitting methods for linear equations were introduced by Peaceman and Rachford [1] and Douglas and Rachford [2]. Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg [25] and Lions and Mercier [26]. The central problem is to iteratively find a zero of the sum of two monotone operators A and M in a Hilbert space H, namely, a solution to the inclusion problem $0 \in (A+M)x$. Many problems can be formulated as a problem of the inclusion problem, such as, stationary solutions to initial value problems of evolution equations. In this paper, we investigate an equilibrium problem, a inclusion problem and a common fixed point problem via a viscosity approximation method. Strong convergence theorems of common solutions to the problems are established in the framework of Hilbert spaces.

Lemma 1.1 ([20]). *Let H be a Hilbert space and let C be a nonempty convex closed subset of H. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \le l < 1$, where l is some real number, $\forall i \ge 1$. Then*

(1) *W_n is nonexpansive and $Fix(W_n) = \cap_{i=1}^{\infty} Fix(S_i)$, for each $n \ge 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} V_{n,k}$ exists.*

(3) *$W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} V_{n,1}x, \quad x \in C$$

is a nonexpansive mapping satisfying $Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Lemma 1.2 ([11]). *Let H be a Hilbert space and let C be a nonempty convex closed subset of H. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \le l < 1$, $\forall i \ge 1$. If K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$.*

From now on, we always assume that $0 < \gamma_i \le l < 1$, $\forall i \ge 1$.

Lemma 1.3 ([27]). *Let H be a Hilbert space and let C be a nonempty convex closed subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $s > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{s}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Define a mapping $Res_s : H \to C$ as follows:

$$Res_s x = \{z \in C : F(z,y) + \frac{1}{s}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C\}, \quad x \in H.$$

Then

(a) *Res_s is single-valued firmly nonexpansive, i.e.,*

$$\|Res_s x - Res_s y\|^2 \le \langle Res_s x - Res_s y, x - y\rangle, \quad \forall x, y \in H,$$

(b) *$EP(F) = Fix(Res_s)$ is closed and convex.*

Lemma 1.4 ([28]). *Let H be a Hilbert space and and let M be a maximal monotone operator. For $r > 0$, $s > 0$ and $x \in H$, we have*

$$J_r y = J_s\big(\frac{s}{r} y + \frac{r-s}{r} J_r y\big),$$

where $J_r = (I + rM)^{-1}$ and $J_s = (I + sM)^{-1}$.

Lemma 1.5 ([24]). *Let H be a Hilbert space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H. Let $x_{n+1} = (1 - a_n)y_n + a_n x_n$, where $\{a_n\}$ is a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < 1$. If*

$$\limsup_{n\to\infty} \big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \le 0$$

then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.6 ([29]). *Let H be a Hilbert space. Let $A : C \to H$ be a mapping and let $M : H \rightrightarrows H$ be a maximal monotone operator. Then $Fix((I + rM)^{-1}(I - rA)) = (M + A)^{-1}(0)$, where r is some positive real number.*

Lemma 1.7 ([30]). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that $\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$. Then $\lim_{n\to\infty} \alpha_n = 0$.*

2. MAIN RESULTS

Theorem 2.1. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $f: C \to C$ be a κ-contraction. Let $A: C \to H$ be an α-inverse-strongly monotone mapping and let $M: H \rightrightarrows H$ be a maximal monotone operator such that $Dom(M) \subset C$. Let S_i be a nonexpansive mapping for each $i \geq 1$ and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $\{x_n\}$ be a sequence generated in the process:*

$$\begin{cases} x_1 \in C, \text{ chosen arbitrarily}, \\ u_n = J_{r_n}(y_n - r_n A y_n), \\ x_{n+1} = \alpha_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n J_{r_n}(u_n - r_n A u_n), \quad \forall n \geq 1, \end{cases}$$

where y_n is in C such that

$$F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

W_n is defined by (1.3), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{r_n\}$ and $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions: $\lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < r \leq r_n \leq r' < 2\alpha$, $0 < s \leq s_n \leq s' < 2\beta$, where r, r', s and s' are four real numbers. If $\cap_{i=1}^{\infty} F(S_i) \cap (A+M)^{-1}(0) \cap EP(F) \neq \emptyset$. Then $\{x_n\}$ converges strongly to $\bar{x} = Proj_{\cap_{i=1}^{\infty} F(S_i) \cap (A+M)^{-1}(0) \cap EP(F)} f(\bar{x})$.

Proof. Fix $x^* \in \cap_{i=1}^{\infty} F(S_i) \cap (A+M)^{-1}(0) \cap EP(F)$. It follows from Lemma 1.3 that $R_{s_n} x^* = x^*$. We also have $x^* = J_{r_n}(I - r_n A)x^*$ and $x^* = S_i x^*$, for each $i \geq 1$. For $\forall x, y \in C$, we have

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ &= \|x - y\|^2 - 2r_n \langle x - y, Ax - Ay \rangle + r_n^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2. \end{aligned}$$

Taking into account that $0 < r \leq r_n \leq r' < 2\alpha$, we see that $I - r_n A$ is nonexpansive. It follows that

$$\begin{aligned} \|u_n - x^*\| &\leq \|J_{r_n}(I - r_n A) R_{s_n} x_n - x^*\| \\ &\leq \|(I - r_n A) R_{s_n} x_n - x^*\| \\ &\leq \|R_{s_n} x_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned}$$

So,

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n \|f(W_n x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|W_n u_n - x^*\| \\ &\leq \alpha_n \kappa \|W_n x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)\|u_n - x^*\| \\ &\leq \alpha_n \kappa \|W_n x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n (1 - \kappa) \frac{\|f(x^*) - x^*\|}{(1 - \kappa)} + (1 - \alpha_n(1 - \kappa))\|x_n - x^*\|, \end{aligned}$$

which yields that $\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{\|f(x^*)-x^*\|}{1-\kappa}\}$. This shows that $\{x_n\}$ is bounded. From Lemma 1.4, we find that

$$\begin{aligned}
&\|J_{r_{n+1}}(y_n - r_nAy_n) - J_{r_n}(y_n - r_nAy_n)\| \\
&= \|J_{r_n}(\frac{r_n}{r_{n+1}}(y_n - r_nAy_n) + (1 - \frac{r_n}{r_{n+1}})J_{r_{n+1}}(y_n - r_nAy_n)) - J_{r_n}(y_n - r_nAy_n)\| \\
&\le |1 - \frac{r_n}{r_{n+1}}|\|J_{r_{n+1}}(y_n - r_nAy_n)) - (y_n - r_nAy_n)\|
\end{aligned} \tag{2.1}$$

Since J_{r_n} is firmly nonexpansive, one finds from (2.1) that

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\le \|J_{r_{n+1}}(y_{n+1} - r_{n+1}Ay_{n+1}) - J_{r_{n+1}}(y_n - r_nAy_n)\| \\
&\quad + \|J_{r_{n+1}}(y_n - r_nAy_n) - J_{r_n}(y_n - r_nAy_n)\| \\
&\le \|y_{n+1} - y_n\| + |r_n - r_{n+1}|\|Ay_n\| + \|J_{r_{n+1}}(y_n - r_nAy_n) - J_{r_n}(y_n - r_nAy_n)\| \\
&\le \|y_{n+1} - y_n\| + |r_n - r_{n+1}|\Theta_1,
\end{aligned} \tag{2.2}$$

where Θ_1 is an appropriate constant. Putting $\xi_n = J_{r_n}(u_n - r_nAu_n)$, we find from (2.2) that

$$\|\xi_{n+1} - \xi_n\| \le \|y_{n+1} - y_n\| + |r_n - r_{n+1}|\Theta_2, \tag{2.3}$$

where Θ_2 is an appropriate constant. From Lemma 1.3, we find that

$$s_{n+1}F(y_{n+1}, y) + \langle y - y_{n+1}, y_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C,$$

and

$$s_nF(y_n, y) + \langle y - y_n, y_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

Putting $y = y_n$ and $y = y_{n+1}$ into the above two inequalities, we find that

$$\langle y_{n+1} - y_n, \frac{y_n - x_n}{s_n} - \frac{y_{n+1} - x_{n+1}}{s_{n+1}}\rangle \ge 0.$$

This implies that

$$\begin{aligned}
&\|y_{n+1} - y_n\|\Big(\|x_{n+1} - x_n\| + |1 - \frac{s_n}{s_{n+1}}|\|y_{n+1} - x_{n+1})\|\Big) \\
&\ge \langle y_{n+1} - y_n, x_{n+1} - x_n + (1 - \frac{s_n}{s_{n+1}})(y_{n+1} - x_{n+1})\rangle \\
&\ge \|y_{n+1} - y_n\|^2.
\end{aligned}$$

It follows that

$$\|x_{n+1} - x_n\| + |s_{n+1} - s_n|\Theta_3 \ge \|y_{n+1} - y_n\|, \tag{2.4}$$

where Θ_3 is an appropriate constant. Combining (2.3) with (2.4), we arrive at

$$\|\xi_{n+1} - \xi_n\| \le \|x_{n+1} - x_n\| + (|s_{n+1} - s_n| + |r_n - r_{n+1}|)\Theta_4, \tag{2.5}$$

where Θ_4 is an appropriate constant. Note that

$$\begin{aligned}
&\|W_{n+1}\xi_{n+1} - W_n\xi_n\| \\
&\le \|W_{n+1}\xi_{n+1} - W\xi_{n+1}\| + \|W\xi_n - W_n\xi_n\| + \|W\xi_{n+1} - W\xi_n\| \\
&\le \sup_{x\in K}\{\|W_{n+1}x - Wx\| + \|Wx - W_nx\|\} + \|\xi_{n+1} - \xi_n\|,
\end{aligned} \tag{2.6}$$

where K is the bounded subset of C. Letting $x_{n+1}=(1-\beta_n)\pi_n+\beta_n x_n$, we see that

$$\begin{aligned}\pi_{n+1}-\pi_n &= \frac{\alpha_{n+1}f(W_{n+1}x_{n+1})+(1-\beta_{n+1}-\alpha_{n+1})W_{n+1}\xi_{n+1}}{1-\beta_{n+1}}\\&\quad-\frac{\alpha_n f(W_nx_n)+(1-\beta_n-\alpha_n)W_n\xi_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}(f(W_{n+1}x_{n+1})-W_{n+1}\xi_{n+1})}{1-\beta_{n+1}}+W_{n+1}\xi_{n+1}\\&\quad-\frac{\alpha_n(f(W_nx_n)-W_n\xi_n)}{1-\beta_n}-W_n\xi_n.\end{aligned}$$

It follows from (2.6) that

$$\begin{aligned}&\|\pi_{n+1}-\pi_n\|\\&\le\frac{\alpha_n}{1-\beta_n}\|f(W_nx_n)-W_n\xi_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(W_{n+1}x_{n+1})-W_{n+1}\xi_{n+1}\|\\&\quad+\|W_{n+1}\xi_{n+1}-W_n\xi_n\|\\&\le\frac{\alpha_n}{1-\beta_n}\|f(W_nx_n)-W_n\xi_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(W_{n+1}x_{n+1})-W_{n+1}\xi_{n+1}\|\\&\quad+\sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+\|\xi_{n+1}-\xi_n\|.\end{aligned}\tag{2.7}$$

From (2.5) and (2.7), we arrive at

$$\begin{aligned}&\|\pi_{n+1}-\pi_n\|-\|x_{n+1}-x_n\|\\&\le\frac{\alpha_n}{1-\beta_n}\|f(W_nx_n)-W_n\xi_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(W_{n+1}x_{n+1})-W_{n+1}\xi_{n+1}\|\\&\quad+\sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+(|s_{n+1}-s_n|+|r_n-r_{n+1}|)\Theta_4.\end{aligned}$$

This yields that

$$\limsup_{n\to\infty}\big(\|\pi_{n+1}-\pi_n\|-\|x_{n+1}-x_n\|\big)\le 0.$$

From Lemma 1.5, we find that $\lim_{n\to\infty}\|x_n-\pi_n\|=0$, which yields from the definition of π_n that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{2.8}$$

Now, we are in a position to show $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-z\rangle\le 0$, where

$$\bar{x}=Proj_{\cap_{i=1}^{\infty}F(S_i)\cap(A+M)^{-1}(0)\cap EP(F)}f(\bar{x}).$$

To prove this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle=\lim_{i\to\infty}\langle f(\bar{x})-\bar{x},x_{n_i}-\bar{x}\rangle.\tag{2.9}$$

Since $\{x_{n_i}\}$ is bounded, without loss of generality, we may assume that $x_{n_i}\rightharpoonup q$. For any $x^*\in\cap_{i=1}^{\infty}F(S_i)\cap(A+M)^{-1}(0)\cap EP(F)$, we see that

$$\begin{aligned}\|y_n-x^*\|^2&\le\|Res_{s_n}x_n-Res_{s_n}x^*\|^2\\&\le\langle Res_{s_n}x_n-Res_{s_n}x^*,x_n-x^*\rangle\\&\le\frac{\|y_n-x^*\|^2+\|x_n-x^*\|^2-\|x_n-y_n\|^2}{2},\end{aligned}$$

which yields that

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2.$$

On the other hand, one has

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|W_n\xi_n - x^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|x_n - y_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|x_n - y_n\|^2 &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).\end{aligned}$$

Since

$$0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1,$$

and $\lim_{n\to\infty}\alpha_n = 0$, we arrive at

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.10}$$

Note that

$$\begin{aligned}\|u_n - x^*\|^2 &\le \|(I - r_nA)y_n - (I - r_nA)x^*\|^2\\ &= \|y_n - x^*\|^2 - 2r_n\langle y_n - x^*, Ay_n - Ax^*\rangle + r_n^2\|Ay_n - Ax^*\|^2\\ &\le \|x_n - x^*\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ax^*\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|W_n\xi_n - x^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_nr_n(2\alpha - r_n)\|Ay_n - Ax^*\|^2.\end{aligned}$$

Hence, one has

$$\begin{aligned}&\gamma_nr_n(2\alpha - r_n)\|Ay_n - Ax^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\ &\le \alpha_n\|f(W_nx_n) - x^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).\end{aligned}$$

In view of the restrictions on $\{\alpha_n\}$ and $\{\gamma_n\}$, one finds that

$$\lim_{n\to\infty}\|Ay_n - Ax^*\| = 0. \tag{2.11}$$

On the other hand, one also has

$$\begin{aligned}&\|u_n - x^*\|^2\\ &\le \langle (I - r_nA)y_n - (I - r_nA)x^*, u_n - x^*\rangle\\ &= \frac{1}{2}\big(\|(I - r_nA)y_n - (I - r_nA)x^*\|^2 - \|y_n - u_n - r_n(Ay_n - Ax^*)\|^2 + \|u_n - x^*\|^2\big)\\ &\le \frac{1}{2}\big(\|y_n - x^*\|^2 - \|y_n - u_n\|^2 + 2r_n\|y_n - u_n\|\|Ay_n - Ax^*\| + \|u_n - x^*\|^2\big),\end{aligned}$$

which yields that

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|y_n - u_n\|^2 + 2r_n\|y_n - u_n\|\|Ay_n - Ax^*\|.$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(W_nx_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|W_n\xi_n-x^*\|^2\\ &\le \alpha_n\|f(W_nx_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|u_n-x^*\|^2\\ &\le \alpha_n\|f(W_nx_n)-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\|y_n-u_n\|^2\\ &\quad +2r_n\gamma_n\|y_n-u_n\|\|Ay_n-Ax^*\|.\end{aligned}$$

Hence,

$$\begin{aligned}\gamma_n\|y_n-u_n\|^2 &\le \alpha_n\|f(W_nx_n)-x^*\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\ &\quad +2r_n\gamma_n\|y_n-u_n\|\|Ay_n-Ax^*\|.\end{aligned}$$

This implies from (2.11) that $\lim_{n\to\infty}\|y_n-u_n\|=0$. In a similar way, one finds that $\lim_{n\to\infty}\|\xi_n-u_n\|=0$. Using (2.8), one has $\lim_{n\to\infty}\|W_n\xi_n-x_n\|=0$. This implies that $\lim_{n\to\infty}\|W_n\xi_n-\xi_n\|=0$. Notice that

$$\frac{y_n-u_n}{r_n}-Ay_n\in Mu_n.$$

Let μ and ν be in H such that $\mu\in M\nu$. Taking into account that M is monotone, we find that

$$\langle\frac{y_n-u_n}{r_n}-Ay_n-\mu,u_n-\nu\rangle\ge 0.$$

This implies that

$$\langle -Aq-\mu,q-\nu\rangle\ge 0.$$

This implies that $-Aq\in Mq$, that is, $q\in(A+M)^{-1}(0)$.

Next, we prove that $q\in\cap_{i=1}^{\infty}F(S_i)$. We assume $q\notin\cap_{i=1}^{\infty}F(S_i)$, i.e., $Wq\ne q$. Taking into account that $\xi_{n_i}\rightharpoonup q$ and the space satisfies Opial's condition, one has

$$\begin{aligned}\liminf_{i\to\infty}\|\xi_{n_i}-q\| &< \liminf_{i\to\infty}\|\xi_{n_i}-Wq\|\\ &\le \liminf_{i\to\infty}\{\|\xi_{n_i}-W\xi_{n_i}\|+\|W\xi_{n_i}-Wq\|\}\\ &\le \liminf_{i\to\infty}\{\|\xi_{n_i}-W\xi_{n_i}\|+\|\xi_{n_i}-q\|\}.\end{aligned}$$

From Lemma 1.2, we see that $\lim_{n\to\infty}\|W\xi_n-\xi_n\|=0$. This derives a contradiction. Thus, we have $q\in\cap_{i=1}^{\infty}F(S_i)$.

Now, we are in a position to show that $q\in EP(F)$. From $y_n=Res_{s_n}x_n$, we see that

$$\langle y-y_n,y_n-x_n\rangle\ge s_nF(y,u_n),\quad \forall y\in C.$$

So, $0\ge F(y,q)$. For $0<t\le 1$ and $y\in C$, we set

$$y_t=(1-t)q+ty.$$

It follows that that $y_t\in C$. Hence, $0\ge F(y_t,q)$. Taking into account the fact that

$$tF(y_t,y)\ge tF(y_t,y)+(1-t)F(y_t,q)\ge F(y_t,y_t)=0,$$

we arrive at $F(q,y)\ge 0$, $\forall y\in C$. This shows that $q\in EP(F)$. So,

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle\le 0.$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\leq \beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| + \gamma_n\|W_n\xi_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\ &\quad + \alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle + \alpha_n\langle f(W_nx_n)-f(\bar{x}), x_{n+1}-\bar{x}\rangle \\ &\leq \frac{1-\alpha_n(1-\kappa)}{2}\|x_n-\bar{x}\|^2 + \frac{1}{2}\|x_{n+1}-\bar{x}\|^2 + \alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1}-\bar{x}\|^2 \leq \big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle.$$

By virtue of Lemma 1.7, one obtains that $\lim_{n\to\infty}\|x_n-\bar{x}\| = 0$. This completes the proof. □

If S_i is an identity mapping for each $i \geq 1$, we immediately find the following result.

Corollary 2.2. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $f: C \to C$ be a κ-contraction. Let $A: C \to H$ be an α-inverse-strongly monotone mapping and let $M: H \rightrightarrows H$ be a maximal monotone operator such that $Dom(M) \subset C$. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in C$ and*

$$\begin{cases} u_n = J_{r_n}(y_n - r_nAy_n), \\ x_{n+1} = \alpha_nf(x_n) + \beta_nx_n + \gamma_nJ_{r_n}(u_n - r_nAu_n), \quad \forall n \geq 1, \end{cases}$$

where y_n is in C such that $s_nF(y_n,y) + \langle y-y_n, y_n-x_n\rangle \geq 0$, $\forall y \in C$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n = 1$ and $\{r_n\}$ and $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions: $\lim_{n\to\infty}|s_n-s_{n+1}| = \lim_{n\to\infty}|r_n-r_{n+1}| = 0$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < r \leq r_n \leq r' < 2\alpha$, $0 < s \leq s_n \leq s' < 2\beta$, where r, r', s and s' are four real numbers. If $(A+M)^{-1}(0)\cap EP(F) \neq \emptyset$. Then $\{x_n\}$ converges strongly to $\bar{x} = Proj_{(A+M)^{-1}(0)\cap EP(F)}f(\bar{x})$.

Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator. It is clearly that $J_rx = Proj_Cx$, $\forall x \in H$. From Corollary 2.2, the following result is not hard to derive.

Corollary 2.3. *Let H be a real Hilbert space and let C be a nonempty convex closed subset of H. Let $f: C \to C$ be a κ-contraction. Let $A: C \to H$ be an α-inverse-strongly monotone mapping and let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in C$ and*

$$\begin{cases} u_n = Proj_C(y_n - r_nAy_n), \\ x_{n+1} = \alpha_nf(x_n) + \beta_nx_n + \gamma_nProj_C(u_n - r_nAu_n), \quad \forall n \geq 1, \end{cases}$$

where y_n is in C such that $s_nF(y_n,y) + \langle y-y_n, y_n-x_n\rangle \geq 0$, $\forall y \in C$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n = 1$ and $\{r_n\}$ and $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions: $\lim_{n\to\infty}|s_n-s_{n+1}| = \lim_{n\to\infty}|r_n-r_{n+1}| = 0$,

$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < r \leq r_n \leq r' < 2\alpha$, $0 < s \leq s_n \leq s' < 2\beta$, *where* r, r', s *and* s' *are four real numbers. If* $VI(C,A)\cap EP(F) \neq \emptyset$*. Then* $\{x_n\}$ *converges strongly to* $\bar{x} = Proj_{VI(C,A)\cap EP(F)}f(\bar{x})$.

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