



A HYBRID DESCENT ITERATIVE ALGORITHM FOR A SPLIT INCLUSION PROBLEM

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Abstract. The purpose of this paper is to introduce a hybrid descent iterative algorithm for solving a split variational inclusion problem and a fixed point problem of a strict pseudocontraction mapping. We establish a strong convergence theorem of common solutions of the two problems in the framework of Hilbert spaces without any compact assumptions on any mapping.

Keywords. Iterative algorithm; Monotone operator; Split inclusion problem; Variational inequality.

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1. INTRODUCTION

Throughout the paper unless otherwise stated, let H_1 and H_2 be two real Hilbert spaces endowed with inner products and induced norms denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, while H refers to as any of these spaces. Let D be a nonempty closed and convex subset of H and Let $Proj_D^H$ be the metric projection onto D . A very common problem in diverse areas of mathematics and physical science consists of trying to find a "solution" satisfying certain "constraints". This problem is referred to as a convex feasibility problem, which is to find a common element in a family of nonempty closed and convex subsets of a Hilbert space.

Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2 . In 1994, Censor and Elfving [1] first introduced the following split feasibility problem for modelling inverse problems formulated as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \quad (1.1)$$

where $A : H_1 \rightarrow H_2$ is a bounded linear mapping. This problem can also be viewed as a convex feasibility problem. There are a number of significant applications of the split feasibility problem in intensity-modulated radiation therapy, signal processing, image reconstruction and so on. An efficient algorithm

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for solving (1.1) is the Byrne's CQ algorithm. For any $x_0 \in H_1$, the CQ algorithm generates an iterative sequence as

$$x_{n+1} = Proj_C^{H_1}(I + \gamma A^*(Proj_Q^{H_2} - I)A)x_n,$$

where $0 < \gamma < 2/\|A\|^2$. It is known that the CQ algorithm converges weakly to a solution of problem (1.1) if such a solution exists. To guarantee the strong convergence of the sequence, a number of regularization methods have been investigated to solve the split feasibility problem and their related convex optimization problems recently; see [2]-[10] and the references therein.

Let M and N be two maximal monotone operators on H_1 and H_2 , respectively. In this paper, we study the following split variational inclusion problem: Find $x^* \in H_1$ such that

$$0 \in M(x^*), \quad (1.2)$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in N(y^*). \quad (1.3)$$

In this paper, we use $SFP(M, N)$ to denote the solution set of the split variational inclusion problem. Recently, the above split variational inclusion problem has been introduced and studied based on Mann-like methods by many authors; see [11, 12, 13, 14, 15] and the references therein. In this paper, we investigate the split variational inclusion problem via a hybrid descent iterative algorithm in Hilbert spaces. Strong convergence theorems are established without any compact assumptions on mappings. The main results presented in this paper partly improve the results in [6, 8, 11, 12]. The organization of this paper is as follows. In Section 2, we provide the necessary mathematical preliminaries. The last section is devoted to the strong convergence analysis of the hybrid descent iterative algorithm. Some subresults are derived as corollaries of the main results.

2. PRELIMINARIES

Let T be a mapping on D . The fixed-point set of T is denoted by $Fix(T)$. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D.$$

T is said to be quasi-nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\|x - Ty\| \leq \|x - y\|, \quad \forall x \in Fix(T), y \in D.$$

It is known that every nonexpansive mapping satisfies the following properties

$$\langle Tx - Ty, (y - Ty) - (x - Tx) \rangle \leq \frac{1}{2} \|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in D.$$

In particular, every quasi-nonexpansive mapping satisfies the following properties

$$\langle x - Ty, (y - Ty) \rangle \leq \frac{1}{2} \|y - Ty\|^2, \quad \forall x \in Fix(T), y \in D. \quad (2.1)$$

Recall that T is said to be firmly nonexpansive iff

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in D.$$

Recall that T is said to be κ -strictly pseudocontractive iff there is a real number $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D.$$

The class of κ -strictly pseudocontractive mappings was first introduced and studied by Browder and Petryshy [16] in the framework of Hilbert spaces. Since then, many authors have studied fixed points of κ -strictly pseudocontractive mappings via different methods and techniques; see [17, 18, 19, 20, 21] and the references therein.

Recall that a mapping $M : H \rightarrow 2^H$ is said to be monotone iff, for all $x, y \in H_1$, $u \in Mx$ and $v \in My$

$$\langle x - y, u - v \rangle \geq 0.$$

It is said to be maximal iff the $Graph(M)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal iff for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in Graph(M)$ implies that $u \in M(x)$. The resolvent mapping, $J_r^M : H \rightarrow H$ associated with M , is defined by

$$J_r^M x = (I + rM)^{-1}(x), \forall x \in H,$$

where $r > 0$ is some real number and I stands for identity operator on H . It is known that J_r^M is single-valued, nonexpansive and firmly nonexpansive. The resolve operator plays an important role in many convex optimization problem; see [3, 8, 9, 13, 15] and the references therein.

Lemma 2.1. [16] *Let D be a nonempty convex and closed subset of a Hilbert space H . Let T be a κ -strict pseudocontraction and let $\{\beta_n\}$ be a sequence in $(0, 1)$. Define a mapping S by $Sx = (1 - \beta_n)Tx + \beta_n x$, $\forall x \in D$. If $\beta \in [\kappa, 1)$, then S is nonexpansive and $Fix(S) = Fix(T)$.*

Lemma 2.2. [22] *Let H be a real Hilbert space and let M be a maximal operator. For $\lambda > 0$ and $\mu > 0$ we have*

$$(I + \mu A)^{-1} \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) (I + \lambda M)^{-1} x \right) = (I + \lambda M)^{-1} x, \forall x \in H.$$

Lemma 2.3. [23] *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n \in [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. Let $\{\lambda_n\}$ be a sequence of nonnegative real numbers such that $\lambda_{n+1} \leq \alpha_n \beta_n + (1 - \alpha_n) \lambda_n$. Then $\lim_{n \rightarrow \infty} \lambda_n = 0$.*

Lemma 2.4. [24] *Let H be a Hilbert space and let F is a η -strongly monotone, \mathcal{L} -Lipschitz continuous mapping on H . Define a mapping $T^\alpha : H \rightarrow H$ by $T^\alpha x = (I - \mu \alpha F)x$, $\forall x \in H_1$, where α is a real number in $(0, 1)$. If $0 < \mu \in (0, \frac{2\eta}{\mathcal{L}^2})$, then T^α is a contraction, that is, $\|T^\alpha x - T^\alpha y\| \leq (1 - \alpha \tau) \|x - y\|$, $\forall x, y \in H$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu \mathcal{L}^2)} \in (0, 1]$.*

Lemma 2.5. [16] *Let D be a nonempty convex and closed subset of a Hilbert space H . Let T be a κ -strict pseudocontraction with fixed points. Then $I - T$, where I is the identity mapping, is demiclosed on any point in D .*

3. MAIN RESULTS

Theorem 3.1. *Let C be a nonempty closed and convex subset of a Hilbert space H_1 and let Q be a nonempty closed and convex subset of a real Hilbert space H_2 . Let T be a κ -strictly pseudocontractive mapping on H_1 with fixed points. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and let A^* be the adjoint operator of A . Let $F : H_1 \rightarrow H_1$ be a \mathcal{L} -Lipschitz continuous and τ -strongly monotone mapping. Let M be a maximal monotone mapping on H_1 and let N be a maximal monotone mapping on H_2 . Assume that*

$\text{Fix}(T) \cap \text{SFP}(M, N) \neq \emptyset$. Let γ be a positive real number. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ and let $\{r_n\}$ be a positive sequence. Let $\{x_n\}$ be a sequence generated by the following process: $x_1 \in H_1$ and

$$\begin{cases} x_{n+1} = T_{\beta_n} y_n - \mu \alpha_n F T_{\beta_n} y_n, \\ y_n = J_{r_n}^M (x_n + \gamma A^* (J_{r_n}^N - I) A x_n), \quad n \geq 1, \end{cases}$$

where $T_{\beta_n} = (1 - \beta_n)T + \beta_n I$. Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\gamma \in (0, \frac{1}{\|A\|^2})$, μ is a real number in $(0, \frac{2\tau}{\tau^2})$, $\{r_n\}$ is a real number sequence such that $\liminf_{n \rightarrow \infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\{\beta_n\}$ is a real number sequence in $[\kappa, 1)$ such that $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then $\{x_n\}$ converge strongly to $\bar{x} \in \text{SFP}(M, N) \cap \text{Fix}(T)$, which is the unique solution of the following variational inequality

$$\langle F\bar{x}, \bar{x} - y \rangle \leq 0, \quad \forall y \in \text{SFP}(M, N) \cap \text{Fix}(T). \quad (3.1)$$

Proof. Since the set of common solutions is nonempty, we can fix a point in $\text{SFP}(M, N) \cap \text{Fix}(T)$, say, x . It follows that $Ax = J_{r_n}^N Ax$, $x = J_{r_n}^M x$ and $x = Tx$. Note that

$$\begin{aligned} \|y_n - x\|^2 &= \|J_{r_n}^M x - J_{r_n}^M (x_n + \gamma A^* (J_{r_n}^N - I) A x_n)\|^2 \\ &\leq \|x - x_n - \gamma A^* (J_{r_n}^N - I) A x_n\|^2 \\ &\leq \|x - x_n\|^2 + 2\gamma \langle A^* (J_{r_n}^N - I) A x_n, x_n - x \rangle + \gamma^2 \|A\|^2 \|(J_{r_n}^N - I) A x_n\|^2 \\ &\leq \|x - x_n\|^2 + 2\gamma \langle A(x_n - x) + (J_{r_n}^N - I) A x_n, (J_{r_n}^N - I) A x_n \rangle \\ &\quad - 2\gamma \|(J_{r_n}^N - I) A x_n\|^2 + \gamma^2 \|A\|^2 \|(J_{r_n}^N - I) A x_n\|^2. \end{aligned}$$

Since both $J_{r_n}^N$ and $J_{r_n}^M$ are nonexpansive, we find from (2.1) that

$$2\gamma \langle A(x_n - x) + (J_{r_n}^N - I) A x_n, (J_{r_n}^N - I) A x_n \rangle \leq \gamma \|(J_{r_n}^N - I) A x_n\|^2$$

Hence, one has

$$\|y_n - x\|^2 \leq \gamma(\gamma \|A\|^2 - 1) \|(J_{r_n}^N - I) A x_n\|^2 + \|x - x_n\|^2. \quad (3.2)$$

From the restriction on constant γ , one reaches that $\|y_n - x\| \leq \|x - x_n\|$. Using Lemma 2.1, one finds that T_{β_n} is nonexpansive and $\text{Fix}(T) = \text{Fix}(T_{\beta_n})$ for each $n \geq 1$. So,

$$\|T_{\beta_n} y_n - x\| \leq \|y_n - x\| \leq \|x - x_n\|.$$

This in turn implies that

$$\begin{aligned} \|x_{n+1} - x\| &\leq \mu \alpha_n \|Fx\| + \|(I - \mu \alpha_n F) T_{\beta_n} y_n - (I - \mu \alpha_n F)x\| \\ &\leq \mu \alpha_n \|Fx\| + (1 - \tau \alpha_n) \|T_{\beta_n} y_n - x\| \\ &\leq \mu \alpha_n \|Fx\| + (1 - \tau \alpha_n) \|x_n - x\| \\ &\leq (1 - \tau \alpha_n) \|x_n - x\| + \tau \alpha_n \frac{\mu \|Fx\|}{\tau} \\ &\leq \max\left\{ \frac{\mu \|Fx\|}{\tau}, \|x_n - x\| \right\} \\ &\leq \dots \\ &\leq \max\left\{ \frac{\mu \|Fx\|}{\tau}, \|x_1 - x\| \right\}. \end{aligned}$$

This proves that $\{x_n\}$ is bounded. It is easy to see that $\{y_n\}$ is also bounded. Note that

$$\begin{aligned}
\|y_n - y_{n+1}\| &\leq \|J_{r_n}^M(x_n + \gamma A^*(J_{r_n}^N - I)Ax_n) - J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&\leq \|J_{r_n}^M(x_n + \gamma A^*(J_{r_n}^N - I)Ax_n) - J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&\quad + \|J_{r_n}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&\leq \|(x_n + \gamma A^*(J_{r_n}^N - I)Ax_n) - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&\quad + \|J_{r_n}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\|.
\end{aligned} \tag{3.3}$$

Using Lemma 2.2, we see that

$$\begin{aligned}
&\|J_{r_n}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&= \|J_{r_n}^M\left(\frac{r_n}{r_{n+1}}(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) + \left(1 - \frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\right) \\
&\quad - J_{r_n}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&\leq \left\|\left(\frac{r_n}{r_{n+1}}(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) + \left(1 - \frac{r_n}{r_{n+1}}\right)J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\right) \right. \\
&\quad \left. - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\right\| \\
&\leq \frac{|r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\|.
\end{aligned} \tag{3.4}$$

Using Lemma 2.2 again, we also have

$$\begin{aligned}
&\|(x_n + \gamma A^*(J_{r_n}^N - I)Ax_n) - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&= \|x_n - x_{n+1} + \gamma A^*((J_{r_n}^N - I)Ax_n - (J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
&\leq \|x_n - x_{n+1} + \gamma A^*(Ax_{n+1} - Ax_n)\| + \|\gamma A^*(J_{r_n}^N Ax_n - J_{r_{n+1}}^N Ax_{n+1})\| \\
&\leq (1 - \gamma \|A\|^2) \|x_n - x_{n+1}\| + \gamma \|A\| \|J_{r_n}^N Ax_n - J_{r_{n+1}}^N \left(\frac{r_n}{r_{n+1}} Ax_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right) J_{r_{n+1}}^N Ax_{n+1}\right)\| \\
&\leq (1 - \gamma \|A\|^2) \|x_n - x_{n+1}\| + \gamma \|A\| \left(\frac{|r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^N (Ax_{n+1} - Ax_n)\| + \|Ax_{n+1} - Ax_n\|\right) \\
&\leq \|x_n - x_{n+1}\| + \frac{\gamma \|A\| |r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^N (Ax_{n+1} - Ax_n)\|.
\end{aligned} \tag{3.5}$$

From (3.3), (3.4) and (3.5), we see that

$$\begin{aligned}
&\|y_n - y_{n+1}\| \\
&\leq \|x_n - x_{n+1}\| + \frac{\gamma \|A\| |r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^N (Ax_{n+1} - Ax_n)\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^M(x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\|.
\end{aligned} \tag{3.6}$$

Hence, one has

$$\begin{aligned}
& \|T_{\beta_n}y_n - T_{\beta_{n+1}}y_{n+1}\| \\
& \leq \|T_{\beta_n}y_n - T_{\beta_n}y_{n+1}\| + \|T_{\beta_n}y_{n+1} - T_{\beta_{n+1}}y_{n+1}\| \\
& \leq \|y_n - y_{n+1}\| + \|T_{\beta_n}y_{n+1} - T_{\beta_{n+1}}y_{n+1}\| \\
& \leq \|y_n - y_{n+1}\| + |\beta_{n+1} - \beta_n| \|Ty_{n+1} - y_{n+1}\| \\
& \leq \|x_n - x_{n+1}\| + \frac{\gamma\|A\|\|r_{n+1} - r_n\|}{r_{n+1}} \|J_{r_{n+1}}^N (Ax_{n+1} - Ax_{n+1})\| \\
& \quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^M (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
& \quad + |\beta_{n+1} - \beta_n| \|Ty_{n+1} - y_{n+1}\|.
\end{aligned} \tag{3.7}$$

On the other hand, one has

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| & \leq \|(I - \mu\alpha_{n+1}F)T_{\beta_{n+1}}y_{n+1} - (I - \mu\alpha_{n+1}F)T_{\beta_n}y_n\| \\
& \quad + \|(I - \mu\alpha_{n+1}F)T_{\beta_n}y_n - (I - \mu\alpha_nF)T_{\beta_n}y_n\| \\
& \leq (1 - \tau\alpha_{n+1})\|T_{\beta_{n+1}}y_{n+1} - T_{\beta_n}y_n\| + \mu|\alpha_{n+1} - \alpha_n|\|FT_{\beta_n}y_n\|.
\end{aligned} \tag{3.8}$$

Combing (3.7) with (3.8), one arrives at

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| & \leq \mu|\alpha_{n+1} - \alpha_n|\|FT_{\beta_n}y_n\| + (1 - \tau\alpha_{n+1})\|x_n - x_{n+1}\| \\
& \quad + \frac{\gamma\|A\|\|r_{n+1} - r_n\|}{r_{n+1}} \|J_{r_{n+1}}^N (Ax_{n+1} - Ax_{n+1})\| \\
& \quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|J_{r_{n+1}}^M (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1}) - (x_{n+1} + \gamma A^*(J_{r_{n+1}}^N - I)Ax_{n+1})\| \\
& \quad + |\beta_{n+1} - \beta_n| \|Ty_{n+1} - y_{n+1}\|.
\end{aligned} \tag{3.9}$$

An application of Lemma 2.3 to (3.9) yields that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Since $J_{r_n}^N$ is firmly nonexpansive, we obtain from (2.1) that

$$\begin{aligned}
& \|y_n - x\|^2 \\
& \leq \langle y_n - x, x_n + \gamma A^*(J_{r_n}^N - I)Ax_n - x \rangle \\
& = \frac{1}{2} \{ \|y_n - x\|^2 - \|x_n + \gamma A^*(J_{r_n}^N - I)Ax_n - y_n\|^2 \\
& \quad + \|x_n - x\|^2 + 2\gamma \langle x_n - x, A^*(J_{r_n}^N - I)Ax_n \rangle + \gamma^2 \|A^*(J_{r_n}^N - I)Ax_n\|^2 \} \\
& \leq \frac{1}{2} \{ \|y_n - x\|^2 - (\|x_n - y_n\|^2 + 2\gamma \langle x_n - y_n, A^*(J_{r_n}^N - I)Ax_n \rangle + \|\gamma A^*(J_{r_n}^N - I)Ax_n\|^2) \\
& \quad + \|x_n - x\|^2 + 2\gamma (\langle J_{r_n}^N Ax_n - Ax, (J_{r_n}^N - I)Ax_n \rangle - \|(J_{r_n}^N - I)Ax_n\|^2) + \gamma^2 \|A\|^2 \|(J_{r_n}^N - I)Ax_n\|^2 \} \\
& \leq \frac{1}{2} \{ \|x_n - x\|^2 + \gamma(\gamma\|A\|^2 - 1) \|(J_{r_n}^N - I)Ax_n\|^2 + \|y_n - x\|^2 - \|x_n - y_n\|^2 \\
& \quad + 2\gamma \|A(x_n - y_n)\| \|(J_{r_n}^N - I)Ax_n\| - \|\gamma A^*(J_{r_n}^N - I)Ax_n\|^2 \} \\
& \leq \frac{1}{2} \{ \|y_n - x\|^2 + \|x_n - x\|^2 + 2\gamma \|A\| \|x_n - y_n\| \|(J_{r_n}^N - I)Ax_n\| - \|x_n - y_n\|^2 \}.
\end{aligned}$$

It follows that

$$\|y_n - x\|^2 \leq \|x_n - x\|^2 + 2\gamma\|A\|\|x_n - y_n\|\|(J_{r_n}^N - I)Ax_n\| - \|x_n - y_n\|^2. \quad (3.11)$$

This implies that

$$\begin{aligned} \|x_{n+1} - x\|^2 &= \|T_{\beta_n}y_n - x - \mu\alpha_n FT_{\beta_n}y_n\|^2 \\ &\leq \|T_{\beta_n}y_n - x\|^2 - 2\mu\alpha_n \langle FT_{\beta_n}y_n, x_{n+1} - x \rangle \\ &\leq \|y_n - x\|^2 + 2\mu\alpha_n \|FT_{\beta_n}y_n\| \|x_{n+1} - x\| \\ &\leq \|x_n - x\|^2 + 2\gamma\|A\|\|x_n - y_n\|\|(J_{r_n}^N - I)Ax_n\| - \|x_n - y_n\|^2 + 2\mu\alpha_n \|FT_{\beta_n}y_n\| \|x_{n+1} - x\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_n - y_n\|^2 &\leq \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + 2\gamma\|A\|\|x_n - y_n\|\|(J_{r_n}^N - I)Ax_n\| + 2\mu\alpha_n \|FT_{\beta_n}y_n\| \|x_{n+1} - x\| \\ &\leq (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| + 2\gamma\|A\|\|x_n - y_n\|\|(J_{r_n}^N - I)Ax_n\| \\ &\quad + 2\mu\alpha_n \|FT_{\beta_n}y_n\| \|x_{n+1} - x\|. \end{aligned} \quad (3.12)$$

From Lemmas 2.1 and 2.4, we find from (3.2) that

$$\begin{aligned} \|x_{n+1} - x\|^2 &= \|(I - \mu\alpha_n F)T_{\beta_n}y_n - (I - \mu\alpha_n F)x\|^2 + \mu^2\alpha_n^2 \|Fx\|^2 \\ &\quad - 2\mu\alpha_n \langle (I - \mu\alpha_n F)T_{\beta_n}y_n - (I - \mu\alpha_n F)x, Fx \rangle \\ &= \|(I - \mu\alpha_n F)T_{\beta_n}y_n - (I - \mu\alpha_n F)x\|^2 - \mu^2\alpha_n^2 \|Fx\|^2 \\ &\quad - 2\mu\alpha_n \langle (I - \mu\alpha_n F)T_{\beta_n}y_n - x, Fx \rangle \\ &\leq (1 - \tau\alpha_n)^2 \|T_{\beta_n}y_n - T_{\beta_n}x\|^2 - 2\mu\alpha_n \langle (I - \mu\alpha_n F)T_{\beta_n}y_n - x, Fx \rangle \\ &\leq (1 - \tau\alpha_n)^2 \|y_n - x\|^2 - 2\mu\alpha_n \langle (I - \mu\alpha_n F)T_{\beta_n}y_n - x, Fx \rangle \\ &\leq (1 - \tau\alpha_n)^2 (\|x_n - x\|^2 - \gamma_n(1 - \gamma\|A\|^2)\|(J_{r_n}^N - I)Ax_n\|^2) \\ &\quad - 2\mu\alpha_n \langle x_{n+1} - x, Fx \rangle \\ &\leq \|x_n - x\|^2 - 2\mu\alpha_n \|x_{n+1} - x\| \|Fx\| - (1 - \tau\alpha_n)^2 \gamma(1 - \gamma\|A\|^2)\|(J_{r_n}^N - I)Ax_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\gamma(1 - \gamma\|A\|^2)(1 - \tau\alpha_n)^2 \|(J_{r_n}^N - I)Ax_n\|^2 \\ &\leq \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + 2\mu\alpha_n \|x_{n+1} - x\| \|Fx\| \\ &\leq \|x_n - x_{n+1}\|(\|x_n - x\| + \|x_{n+1} - x\|) + 2\mu\alpha_n \|x_{n+1} - x\| \|Fx\|. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|J_{r_n}^N Ax_n - Ax_n\| = 0. \quad (3.13)$$

Using (3.12) and (3.13), one arrives at

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.14)$$

Since $T_{\beta_n}y_n - y_n = (1 - \beta_n)(Ty_n - y_n)$, one obtains

$$\lim_{n \rightarrow \infty} \|Ty_n - y_n\| = 0. \quad (3.15)$$

Since F is strongly monotone and Lipschitz continuous, we next use \bar{x} to denote the unique solution of variational inequality (3.1). Note that

$$\langle F\bar{x}, \bar{x} - x_{n+1} \rangle = \langle F\bar{x}, \bar{x} - x_n \rangle + \langle F\bar{x}, x_n - x_{n+1} \rangle. \quad (3.16)$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$.

Next, we show that x^* is the common solution. From (3.15), we see from Lemma 2.5 that x^* is a fixed point of T . On the other hand, one has

$$\frac{x_n - y_n + \gamma A^*(J_{r_n}^N A x_n - A x_n)}{\gamma} \in M y_n$$

Note that $y_{n_k} \rightarrow x^*$ as $k \rightarrow \infty$. It follows from (3.13) and (3.14) that $0 \in Bx^*$. Since A is a linear bounded operator, we find that $Ax_{n_k} \rightarrow Ax^*$ as $k \rightarrow \infty$. Using Lemma 2.2, we find that

$$\begin{aligned} \|J_{r_n}^N A x_n - J_r^N A x_n\| &= \|J_r^N \left(\frac{r}{r_n} A x_n + \left(1 - \frac{r}{r_n}\right) J_{r_n}^N A x_n \right) - J_r^N A x_n\| \\ &\leq \left\| \frac{r}{r_n} A x_n + \left(1 - \frac{r}{r_n}\right) J_{r_n}^N A x_n - A x_n \right\| \\ &\leq \frac{|r_n - r|}{r} \|J_{r_n}^N A x_n - A x_n\|, \end{aligned} \quad (3.17)$$

where r is some positive real number. From (3.13), one obtains that $\|J_{r_n}^N A x_n - J_r^N A x_n\| \rightarrow \infty$ as $n \rightarrow \infty$. In view of

$$\|J_r^N A x_n - A x_n\| \leq \|J_r^N A x_n - J_{r_n}^N A x_n\| + \|J_{r_n}^N A x_n - A x_n\|,$$

we find that $\|J_r^N A x_n - A x_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Since J_r^N is nonexpansive, we find that $Ax^* \in \text{Fix}(J_r^N)$, that is, $0 \in N A x^*$. This proves that $x \in SFP(M, N) \cap \text{Fix}(T)$. This shows from (3.16) that

$$\limsup_{n \rightarrow \infty} \langle F\bar{x}, \bar{x} - x_{n+1} \rangle \leq 0.$$

Finally, we show that $\{x_n\}$ converges strongly to \bar{x} .

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= \|(I - \mu \alpha_n F) T_{\beta_n} y_n - (I - \mu \alpha_n F) \bar{x} - \mu \alpha_n F \bar{x}\|^2 \\ &\leq \|(I - \mu \alpha_n F) T_{\beta_n} y_n - (I - \mu \alpha_n F) \bar{x}\|^2 - 2\mu \alpha_n \langle F\bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq (1 - \tau \alpha_n)^2 \|T_{\beta_n} y_n - \bar{x}\|^2 - 2\mu \alpha_n \langle F\bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq (1 - \tau \alpha_n)^2 \|y_n - \bar{x}\|^2 - 2\mu \alpha_n \langle F\bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq (1 - \tau \alpha_n)^2 \|x_n - \bar{x}\|^2 - 2\mu \alpha_n \langle F\bar{x}, x_{n+1} - \bar{x} \rangle \\ &\leq (1 - 2\tau \alpha_n) \|x_n - \bar{x}\|^2 + 2\alpha_n \mu \langle F\bar{x}, \bar{x} - x_{n+1} \rangle + \tau^2 \alpha_n^2 \|x_n - \bar{x}\|^2. \end{aligned}$$

Using Lemma 2.3, we get that $\|x_n - \bar{x}\| \rightarrow 0$. This completes the proof. \square

From Theorem 3.1, the following results are not hard to derived easily.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a Hilbert space H_1 and let Q be a nonempty closed and convex subset of a real Hilbert space H_2 . Let T be a nonexpansive mapping on H_1 with fixed points. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and let A^* be the adjoint operator of A . Let $F : H_1 \rightarrow H_1$ be a \mathcal{L} -Lipschitz continuous and τ -strongly monotone mapping. Let M be a maximal monotone mapping on H_1 and let N be a maximal monotone mapping on H_2 . Assume that*

$\text{Fix}(T) \cap \text{SFP}(M, N) \neq \emptyset$. Let γ be a positive real number. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ and let $\{r_n\}$ be a positive sequence. Let $\{x_n\}$ be a sequence generated by the following process: $x_1 \in H_1$ and

$$\begin{cases} x_{n+1} = Ty_n - \mu\alpha_n FTy_n, \\ y_n = J_{r_n}^M(x_n + \gamma A^*(J_{r_n}^N - I)Ax_n), \end{cases}$$

Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\gamma \in (0, \frac{1}{\|A\|^2})$, μ is a real number in $(0, \frac{2\tau}{\mathcal{L}^2})$, $\{r_n\}$ is a real number sequence such that $\liminf_{n \rightarrow \infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converge strongly to $\bar{x} \in \text{SFP}(M, N) \cap \text{Fix}(T)$, which is the unique solution of the following variational inequality

$$\langle F\bar{x}, \bar{x} - y \rangle \leq 0, \quad \forall y \in \text{SFP}(M, N) \cap \text{Fix}(T).$$

Corollary 3.3. Let C be a nonempty closed and convex subset of a Hilbert space H_1 and let Q be a nonempty closed and convex subset of a real Hilbert space H_2 . Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and let A^* be the adjoint operator of A . Let $F : H_1 \rightarrow H_1$ be a \mathcal{L} -Lipschitz continuous and τ -strongly monotone mapping. Let M be a maximal monotone mapping on H_1 and let N be a maximal monotone mapping on H_2 . Assume that $\text{SFP}(M, N) \neq \emptyset$. Let γ be a positive real number. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ and let $\{r_n\}$ be a positive sequence. Let $\{x_n\}$ be a sequence generated by the following process: $x_1 \in H_1$, $x_{n+1} = y_n - \mu\alpha_n Fy_n$, where $y_n = J_{r_n}^M(x_n + \gamma A^*(J_{r_n}^N - I)Ax_n)$. Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\gamma \in (0, \frac{1}{\|A\|^2})$, μ is a real number in $(0, \frac{2\tau}{\mathcal{L}^2})$, $\{r_n\}$ is a real number sequence such that $\liminf_{n \rightarrow \infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converge strongly to $\bar{x} \in \text{SFP}(M, N)$, which is the unique solution of the following variational inequality $\langle F\bar{x}, \bar{x} - y \rangle \leq 0$, $\forall y \in \text{SFP}(M, N)$.

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