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A NEW ITERATIVE ALGORITHM FOR SOLVING A SYSTEM OF GENERALIZED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, under the framework of real Hilbert spaces, we introduce a new iterative algorithm for finding a common element in the solution set of a generalized equilibrium problem and in the fixed-point sets of a family of nonexpansive mappings. We obtain strong convergence theorems of the common solution problem. An example is provided to support the convergence analysis.

Keywords. Iterative algorithm; Generalized equilibrium problem; Variational inequality; Nonexpansive mapping; Strongly monotone operator.

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1. Introduction

Let $\mathscr{H}$ be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$, and let $\Upsilon : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ be a bifunction.

The equilibrium problem for $\Upsilon : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ is to find $x \in \mathscr{C}$ such that

$$\Upsilon(x, y) \geq 0, \quad \forall y \in \mathscr{C}. \tag{1.1}$$

The solution set of (1.1) is denoted by $EP(\Upsilon)$ in this paper. This equilibrium problem, which was first introduced by Ky Fan [1], and further studied by Blum and Oettli [2], has been extensively investigated based on fixed point methods; see [3, 4, 5, 6, 7, 8] and the references therein.

In 2005, Combettes and Hirstoaga [9] introduced an iterative scheme for finding the best approximation to the initial data when $EP(\Upsilon)$ is nonempty, and proved a strong convergence theorem. The equilibrium problem has emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, ecology, transportation, network, elasticity and optimization; see [10, 11, 12] and the references therein.

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Let $S:\mathscr{C}\to\mathscr{C}$ be a mapping. Throughout this paper, the set of fixed points of mapping S is denoted by

$$Fix(S)=\{x\in\mathscr{C}:x=Sx\}.$$

Recall that $S:\mathscr{C}\to\mathscr{C}$ is said to be nonexpansive if

$$\| Sx-Sy \|\leq\| x-y \|,\quad \forall x,y\in\mathscr{C}.$$

It was proved in [9] that equilibrium problem (1.1) is equivalent to a fixed point problem of nonexpansive mappings. Let $r>0$ be a positive real number. Define a mapping $T_r:\mathscr{H}\rightrightarrows\mathscr{C}$ by

$$T_r(x)=\{z\in\mathscr{C}:\Upsilon(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in\mathscr{C}\}$$

Then T_r is single-valued, and $Fix(T_r)=EP(\Upsilon)$.

Let $A:\mathscr{C}\to\mathscr{H}$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax-Ay,x-y\rangle\geq 0,\quad \forall x,y\in\mathscr{C}.$$

Recall that A is said to be α-inverse strongly monotone if there exists a constant $\alpha>0$ such that

$$\langle Ax-Ay,x-y\rangle\geq\alpha\,\| Ax-Ay \|^2,\ \forall x,y\in\mathscr{C}.$$

A bounded linear operator A on $\mathscr{H}$ is said to be strongly positive if there is a $\delta>0$ such that

$$\langle Ax,x\rangle\geq\delta\|x\|^2,\quad \forall x\in\mathscr{H}.$$

Let $A:\mathscr{C}\to\mathscr{C}$ be an inverse strongly monotone mapping and let $\Upsilon:\mathscr{C}\times\mathscr{C}\to\mathbb{R}$ be a bifunction. In this paper, we consider the following *Generalized Equilibrium Problem* (GEP):

$$\text{Find}\quad \tilde{x}\in\mathscr{C}\quad\text{such that}\quad \Upsilon(\tilde{x},y)+\langle A\tilde{x},y-\tilde{x}\rangle\geq 0,\ \forall y\in\mathscr{C}. \tag{1.2}$$

The set of solutions of this problem is denoted by GEP(Υ,A), i.e.,

$$GEP(\Upsilon,A)=\{x\in\mathscr{C}:\Upsilon(x,y)+\langle Ax,y-x\rangle\geq 0,\ \forall y\in\mathscr{C}\}.$$

It is desirable to have a look at some special cases of generalized equilibrium problem (1.2):

(i): if $A\equiv 0$, then problem (1.2) reduces to the following *Equilibrium Problem*:

$$\text{Find}\quad \tilde{x}\in\mathscr{C}\quad\text{such that}\quad \Upsilon(\tilde{x},y)\geq 0,\ \forall y\in\mathscr{C};$$

(ii): if $\Upsilon\equiv 0$, then problem (1.2) reduces to the following *Classical Variational Inequality*:

$$\text{Find}\quad \tilde{x}\in\mathscr{C}\quad\text{such that}\quad \langle A\tilde{x},y-\tilde{x}\rangle\geq 0,\ \forall y\in\mathscr{C}.$$

As in the case of equilibrium problem, the classical variational inequality problem is equivalent to a fixed point problem. To see this, we should recall the orthogonal projection $P_{\mathscr{C}}:\mathscr{H}\to\mathscr{C}$ that assigns $x\in\mathscr{H}$ to its unique nearest point in $\mathscr{C}$ provided that this latter set is nonempty, closed and convex. Indeed, we have

$$P_{\mathscr{C}}(x)=\arg\min\{\|x-y\|:\ y\in\mathscr{C}\}.$$

It is well-known that

$$y=P_{\mathscr{C}}(x)\iff\langle x-y,y-z\rangle\geq 0,\quad \forall z\in\mathscr{C}.$$

We assume that $\lambda > 0$ and $u \in \mathscr{C}$. Then $u \in VI(\mathscr{C},A)$ if and only if $\lambda\langle Au, z-u\rangle \geq 0$ for each $z \in \mathscr{C}$. This in turn is equivalent to

$$\begin{aligned}\langle -\lambda Au, u-z\rangle \geq 0, \quad \forall z \in \mathscr{C} &\iff \langle u-\lambda Au-u, u-z\rangle \geq 0, \quad \forall z \in \mathscr{C}\\ &\iff \langle (I-\lambda A)u-u, u-z\rangle \geq 0, \quad \forall z \in \mathscr{C}\\ &\iff u = P_{\mathscr{C}}(I-\lambda A)u.\end{aligned}$$

This shows that $u \in \mathscr{C}$ is a solution of the variational inequality if and only if u is a fixed point of $P_{\mathscr{C}}(I-\lambda A)$; where $\lambda > 0$ is a constant, and I is the identity operator. Recently, many authors have studied problem (1.2) and its two special cases via fixed-point methods; see [13, 14, 15, 16, 17, 18] and the references therein.

In [19], Shimoji and Takahashi introduced a W_n-mapping induced by an infinite family of nonexpansive mappings $\{S_i : \mathscr{C} \to \mathscr{C}\}$ and a real number sequence of $\{\gamma_i\}$ with the restriction $0 \leq \gamma_i < 1$, $\forall\, i \geq 1$ (see next section). In 2016, Zhang and Hao [13], based on the W_n-mapping, studied the convergence analysis of the following iterative sequence, which is defined as following:

$$\begin{cases} x_1 \in \mathscr{C}, chosen/arbitrarily,\\ F(y_n,y)+\langle Ax_n, y-y_n\rangle + \dfrac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \geq 0,\ \forall y \in \mathscr{C},\\ x_{n+1} = \beta_n x_n + (1-\beta_n)W_n(\alpha_n f(W_n x_n)+(1-\alpha_n)y_n),\ \forall n \geq 1,\end{cases}$$

where F is a bifunction, f is a contraction, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the open interval $(0,1)$ and $\{r_n\}$ is a sequence of positive numbers. They proved that the sequence $\{x_n\}$ converges strongly to a point

$$x \in \Omega := \bigcap_{i=1}^{\infty} Fix(S_i) \cap GEP(F,A) \neq \emptyset,$$

where $x = P_\Omega f(x)$.

Subsequently, Majee and Nahak [14] studied the convergence analysis of the following iterative sequence $\{x_n\}$, which is defined in the following way: $x_1 \in \mathscr{C}$ arbitrarily, and

$$\begin{cases} u_n = T_{r_n}^{(F_1,h_1)}\left(x_n + \gamma A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\right),\\ y_n = \delta_n x_n + (1-\delta_n)S_N^n S_{N-1}^n \cdots S_1^n u_n,\\ x_{n+1} = \alpha_n \eta f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n \mu D\right)y_n,\ n \geq 1,\end{cases}$$

where D is a strongly positive bounded linear operator on $\mathscr{H}$, F_1, F_2, h_1, h_2 are some bifunctions, $\eta, \mu > 0$, $\{r_n\} \subset (0,\infty)$, and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subset [0,1]$.

In this paper, inspired by [13] and [14], we introduce a new iterative algorithm and prove a strong convergence theorem for generalized equilibrium problem (1.2) and common fixed-point problem of a family of nonexpasnive mappings. Comparing with the algorithm in [13], we here consider a system of generalized equilibrium problems (instead of one bifunction and one α-inverse strongly monotone operator). It also deserves mentioning that the number of nonexpansive mappings in our algorithm is infinite. The results presented in this paper improve and extend some recent results announced in [13, 14, 20, 21, 22].

2. Preliminaries

To study generalized equilibrium problem (1.2), we assume that bifunction $\Upsilon : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ satisfies the following conditions:

(C1): $\Upsilon(x,x) = 0$ for all $x \in \mathscr{C}$,

(C2): Υ is monotone, that is,

$$\Upsilon(x,y) + \Upsilon(y,x) \leq 0, \ \forall x,y \in \mathscr{C},$$

(C3): Υ is upper-hemicontinuous, that is,

$$\limsup_{h \to 0^+} \Upsilon(hz + (1-h)x, y) \leq \Upsilon(x,y), \ \forall x,y,z \in \mathscr{C},$$

(C4): $\Upsilon(x,0)$ is convex and lower semicontinuous for each $x \in \mathscr{C}$.

In order to prove our main results, we need to recall some concepts from the theory of Banach spaces. A Banach space X is said to satisfy the Opial's condition [23] if, for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is known that the above inequality is equivalent to

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is known that all Hilbert spaces and l^p satisfy the Opial's condition, however, L^p does not satisfy the Opial's condition, unless $p = 2$.

Let $\mathscr{H}$ be a real Hilbert space. Then, for all $x, y \in \mathscr{H}$, the following assertions hold:

(1) $\| x+y \|^2 \leq \| y \|^2 + 2\langle x, x+y \rangle$,

(2) $\| \alpha x + (1-\alpha)y \|^2 = \alpha \| x \|^2 + (1-\alpha) \| y \|^2 - \alpha(1-\alpha) \| x-y \|^2$,

(3) $\| x-y \|^2 = \| x \|^2 + \| y \|^2 - 2\langle x, y \rangle$.

Let $\mathscr{C}$ be a nonempty closed and convex subset of a Hilbert space $\mathscr{H}$. Let $T : \mathscr{C} \to \mathscr{C}$ be a mapping. Recall that T is said to be firmly nonexpansive iff

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad x, y \in \mathscr{C}.$$

It is easy to see that every firmly nonexpansive mapping is nonexpansive. As a typical example, the orthogonal projection $P_{\mathscr{C}}$, which satisfies

$$\|P_{\mathscr{C}}x - P_{\mathscr{C}}y\|^2 \leq \langle P_{\mathscr{C}}x - P_{\mathscr{C}}y, x - y \rangle, \quad \forall x, y \in \mathscr{C},$$

is firmly nonexpansive.

Lemma 2.1. [2] *Let $\mathscr{C}$ be a nonempty closed and convex subset of a Hilbert space $\mathscr{H}$. Let $\Upsilon : \mathscr{C} \times \mathscr{C} \to \mathbb{R}$ be a bifunction. Assume that Υ satisfies $(\boldsymbol{C1}) - (\boldsymbol{C4})$. Let $r > 0$ and $x \in \mathscr{H}$. Then, there exists $z \in \mathscr{C}$ such that*

$$\Upsilon(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in \mathscr{C}.$$

Further, define a mapping $T_r : \mathscr{H} \rightrightarrows \mathscr{C}$ by

$$T_r(x) = \{z \in \mathscr{C} : \Upsilon(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in \mathscr{C}\}.$$

Then

(1) T_r *is single-valued.*

(2) T_r *is firmly nonexpansive, that is,*

$$||T_r(x) - T_r(y)||^2 \leq \langle T_r(x) - T_r(y), x - y \rangle, \quad \forall x, y \in \mathscr{H}$$

(3) $Fix(T_r) = EP(\Upsilon)$,

(4) $EP(\Upsilon)$ *is closed and convex.*

Lemma 2.2. [17] *Let* $\mathscr{C}, \mathscr{H}, \Upsilon$ *and* T_r *be as in Lemma 2.1. Then the following holds:*

$$||T_s(x) - T_t(x)||^2 \leq \frac{s-t}{s} \langle T_s(x) - T_t(x), T_s(x) - x \rangle,$$

for all $s, t > 0$ *and* $x \in \mathscr{H}$.

Lemma 2.3. [24] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \ \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(2) $\limsup_{n\to\infty} \dfrac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.4. [19] Let $\mathscr{C}$ be a nonempty closed convex subset of a Hilbert space $\mathscr{H}$. Let $\{S_i : \mathscr{C} \to \mathscr{C}\}$ be an infinite family of nonexpansive mappings and let $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1$, $\forall i \geq 1$. For $n \geq 1$, define a mapping $W_n : \mathscr{C} \to \mathscr{C}$ as follows

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \gamma_n S_n U_{n,n+1} + (1-\gamma_n) I, \\ U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1-\gamma_{n-1}) I, \\ \vdots \\ U_{n,k} = \gamma_k S_k U_{n,k+1} + (1-\gamma_k) I, \\ U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1-\gamma_{k-1}) I, \\ \vdots \\ U_{n,2} = \gamma_2 S_2 U_{n,3} + (1-\gamma_2) I, \\ W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1-\gamma_1) I. \end{cases} \tag{2.1}$$

Lemma 2.5. [19] *Let* $\mathscr{C}$ *be a nonempty closed convex subset of a Hilbert space* $\mathscr{H}$. *Let* $\{S_i : \mathscr{C} \to \mathscr{C}\}$ *be an infinite family of nonexpansive mappings with* $\cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ *and let* $\{\gamma_i\}$ *be a real sequence such that* $0 \leq \gamma_i \leq l < 1, \forall\, i \geq 1$. *Then*

(1) W_n, *defined in the previous section, is nonexpansive and* $Fix(W_n) = \cap_{i=1}^{\infty} Fix(S_i)$, *for each* $n \geq 1$;

(2) for each $x \in \mathscr{C}$ *and for each positive integer k, the limit* $\lim_{n\to\infty} U_{n,k}$ *exists;*

(3) the mapping $W : \mathscr{C} \to \mathscr{C}$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \ \ x \in \mathscr{C},$$

is nonexpansive and satisfies $Fix(W) = \cap_{i=1}^{\infty} Fix(S_i)$.

The mapping W defined above is called the W-mapping generated by $S_1, S_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$.

Lemma 2.6. [20] *Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$, $\{S_i : \mathscr{C} \to \mathscr{C}\}$ be an infinite family of nonexpansive mappings with $\cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and $\{\gamma_i\}$ be a real sequence such that $0 \leq \gamma_i \leq l < 1, \forall\, i \geq 1$. If K is any bounded subset of $\mathscr{C}$, then*

$$\lim_{n\to\infty} \sup_{x\in K} \| Wx - W_n x \| = 0.$$

Throughout this paper, we always assume that $0 \leq \gamma_i \leq l < 1$, where l is some real number, $i \geq 1$.

Lemma 2.7. [25] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathscr{H}$ and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} \left(\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \| \right) \leq 0.$$

Then $\lim_{n\to\infty} \| y_n - x_n \| = 0$.

3. Main Results

Theorem 3.1. *Let $\mathscr{C}$ be a nonempty, closed convex subset of a Hilbert space $\mathscr{H}$ and let $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k$ be bifunctions from $\mathscr{C} \times \mathscr{C}$ to $\mathbb{R}$ which satisfy $(\boldsymbol{C1}) - (\boldsymbol{C4})$. Let $A_1, A_2, \cdots, A_k$ be μ_j-inverse strongly monotone mappings from $\mathscr{C}$ to $\mathscr{H}$ and let $\{S_i : \mathscr{C} \to \mathscr{C}\}$ be an infinite family of nonexpansive mappings. Let $f : \mathscr{C} \to \mathscr{C}$ be a contractive mapping with the constant $k \in (0,1)$. Assume that*

$$\Omega := \left(\bigcap_{i=1}^{\infty} Fix(S_i) \right) \cap \left(\bigcap_{j=1}^{k} GEP(\Upsilon_j, A_j) \right) \neq \emptyset.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in \mathscr{C},\ y \in \mathscr{C}, \\ \Upsilon_1(u_{n,1}, y) + \langle A_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \\ \Upsilon_2(u_{n,2}, y) + \langle A_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \\ \vdots \\ \Upsilon_k(u_{n,k}, y) + \langle A_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \\ \omega_n = \dfrac{1}{k} \sum_{j=1}^{k} u_{n,j}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) W_n(\alpha_n f(W_n x_n) + (1-\alpha_n)\omega_n), \end{cases} \tag{3.1}$$

where $\{W_n\}$ is the sequence defined by (2.1), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a sequence of positive numbers. Assume that the above control sequences satisfy the following conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\mu$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ convergence strongly to a point $x \in \Omega$, where $x = P_\Omega f(x)$.

Proof. We first show that $\{x_n\}$ and $\{\omega_n\}$ are bounded. Let $x^* \in \Omega$. We observe that $I - r_nA_j,\ j = 1,2,\cdots,k$, is a nonexpansive mapping. Indeed, for any $x,y \in \mathscr{C}$, one has

$$\begin{aligned}\|(I-r_nA_j)x-(I-r_nA_j)y\|^2 &= \|(x-y)-r_n(A_jx-A_jy)\|^2\\ &= \|x-y\|^2-2r_n\langle x-y,A_jx-A_jy\rangle+r_n^2\|A_jx-A_jy\|^2\\ &\le \|x-y\|^2-r_n(2\mu_j-r_n)\|A_jx-A_jy\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

So

$$\|u_{n,i}-x^*\| \le \|x_n-x^*\|, \tag{3.2}$$

which in turn implies that

$$\|\omega_n-x^*\| \le \|x_n-x^*\|. \tag{3.3}$$

Putting $z_n = \alpha_n f(W_nx_n)+(1-\alpha_n)\omega_n$, we find from (3.3) that

$$\begin{aligned}\|z_n-x^*\| &\le \alpha_n\|f(W_nx_n)-x^*\|+(1-\alpha_n)\|\omega_n-x^*\|\\ &\le (1-\alpha_n(1-k))\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|.\end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \beta_n\|x_n-x^*\|+(1-\beta_n)\|W_nz_n-x^*\|\\ &\le \beta_n\|x_n-x^*\|+(1-\beta_n)\|z_n-x^*\|\\ &\le (1-\alpha_n(1-\beta_n)(1-k))\|x_n-x^*\|+\alpha_n(1-\beta_n)\|f(x^*)-x^*\|\\ &\vdots\\ &\le \max\left\{\|x_1-x^*\|,\frac{\|f(x^*)-x^*\|}{1-k}\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded, so are $\{\omega_n\}$ and $\{z_n\}$. Without loss of generality, we may assume that there exists a bounded set $K \subset \mathscr{C}$ such that $x_n,y_n,z_n \in K$. Note that $u_{n,j}$ can be written as $u_{n,j} = T_{r_n,j}(x_n - r_nA_jx_n)$. For any $j = 1,2,\cdots,k$, one has

$$\begin{aligned}\|u_{n+1,j}-u_{n,j}\| &\le \|T_{r_{n+1},j}(I-r_{n+1}A_j)x_{n+1}-T_{r_{n+1},j}(I-r_nA_j)x_n\|\\ &\quad+\|T_{r_{n+1},j}(I-r_nA_j)x_n-T_{r_n,j}(I-r_nA_j)x_n\|\\ &\le \|(I-r_{n+1}A_j)x_{n+1}-(I-r_nA_j)x_n\|\\ &\quad+\|T_{r_{n+1},j}(I-r_nA_j)x_n-T_{r_n,j}(I-r_nA_j)x_n\|\\ &\le \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|A_jx_n\|\\ &\quad+\frac{r_{n+1}-r_n}{r_{n+1}}\|T_{r_{n+1},j}(I-r_nA_j)x_n-T_{r_n,j}(I-r_nA_j)x_n\|.\end{aligned}$$

Then

$$\|u_{n+1,j}-u_{n,j}\| \le \|x_{n+1}-x_n\|+2M_j|r_{n+1}-r_n|, \tag{3.5}$$

where

$$M_j = \max\left\{\sup\{\frac{\|T_{r_{n+1},j}(I-r_nA_j)x_n-T_{r_n,j}(I-r_nA_j)x_n\|}{r_{n+1}}\},\sup\{\|A_jx_n\|\}\right\}.$$

It follows that

$$\begin{aligned}\|z_{n+1}-z_n\| \le\ & \alpha_{n+1}\|f(W_{n+1}x_{n+1})-f(W_nx_n)\|+|\alpha_{n+1}-\alpha_n|(\|f(W_nx_n)\|+\|\omega_n\|)\\ &+(1-\alpha_{n+1})\|\omega_{n+1}-\omega_n\|\\ \le\ & \alpha_{n+1}k\|W_{n+1}x_{n+1}-W_nx_n\|+|\alpha_{n+1}-\alpha_n|(\|f(W_nx_n)\|+\|\omega_n\|)\\ &+\|\omega_{n+1}-\omega_n\|. \end{aligned} \tag{3.6}$$

Setting $M=\frac{1}{k}\sum_{j=1}^{k}2M_j\le\infty$, we have

$$\|\omega_{n+1}-\omega_n\|\le\frac{1}{k}\sum_{j=1}^{k}\|u_{n+1,j}-u_{n,j}\|\le\|x_{n+1}-x_n\|+M|r_{n+1}-r_n|, \tag{3.7}$$

which shows that

$$\begin{aligned}\|z_{n+1}-z_n\|\le\ & \alpha_{n+1}k\|W_nx_n-W_nx_n\|+|\alpha_{n+1}-\alpha_n|(\|f(W_nx_n)\|+\|\omega_n\|)\\ &+\Big[\|x_{n+1}-x_n\|+M|r_{n+1}-r_n|\Big]. \end{aligned}\tag{3.8}$$

Note that

$$\begin{aligned}\|W_{n+1}z_{n+1}-W_nz_n\| &= \|W_{n+1}z_{n+1}-Wz_{n+1}+Wz_{n+1}-Wz_n+Wz_n-W_nz_n\|\\ &\le \|W_{n+1}z_{n+1}-Wz_{n+1}\|+\|Wz_{n+1}-Wz_n\|+\|Wz_n-W_nz_n\|\\ &\le \sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+\|z_{n+1}-z_n\|. \end{aligned}\tag{3.9}$$

Combing (3.8) with (3.9) yields that

$$\begin{aligned}\|W_{n+1}z_{n+1}-W_nz_n\|-\|x_{n+1}-x_n\|\le\ & \sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}\\ &+\alpha_{n+1}k\|W_{n+1}x_{n+1}-W_nx_n\|\\ &+|\alpha_{n+1}-\alpha_n|\Big(\|f(W_nx_n)\|+\|\omega_n\|\Big)\\ &+\Big[\|x_{n+1}-x_n\|+M|r_{n+1}-r_n|\Big]. \end{aligned}\tag{3.10}$$

By using Lemma 2.6, we find that

$$\limsup_{n\to\infty}\{\|W_{n+1}z_{n+1}-W_nz_n\|-\|x_{n+1}-x_n\|\}\le 0.$$

It follows from Lemma 2.7 that

$$\lim_{n\to\infty}\|W_nz_n-x_n\|=0. \tag{3.11}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|W_nz_n-x_n\|=0. \tag{3.12}$$

Moreover, for any $j\in\{1,2,\cdots,k\}$, one has

$$\begin{aligned}\|u_{n,j}-x^*\|^2 &\le \|(x_n-x^*)-r_n(A_jx_n-A_jx^*)\|^2\\ &= \|x_n-x^*\|^2-2r_n\langle x_n-x^*,A_jx_n-A_jx^*\rangle+r_n^2\|A_jx_n-A_jx^*\|^2\\ &\le \|x_n-x^*\|^2-r_n(2\mu_j-r_n)\|A_jx_n-A_jx^*\|^2. \end{aligned}$$

It follows that

$$\begin{aligned}\|\omega_n - x^*\|^2 &= \|\sum_{j=1}^{k}\frac{1}{k}(u_{n,j}-x^*)\|^2\\ &\leq \frac{1}{k}\sum_{j=1}^{k}\|u_{n,j}-x^*\|^2\\ &\leq \|x_n-x^*\|^2-\frac{1}{k}\sum_{j=1}^{k}r_n(2\mu_j-r_n)\|A_jx_n-A_jx^*\|^2. \end{aligned}\tag{3.13}$$

By (3.13), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n x_n+(1-\beta_n)W_nz_n-x^*\|^2\\ &\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2\\ &\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\Big[(\alpha_n\|f(W_nx_n)-x^*\|^2+(1-\alpha_n)\|\omega_n-x^*\|^2)\Big]\\ &\leq \beta_n\|x_n-x^*\|^2+(1-\beta_n)\alpha_n\|f(W_nx_n)-x^*\|^2\\ &\quad +(1-\beta_n)(1-\alpha_n)\Big[\|x_n-x^*\|^2-\frac{1}{k}\sum_{j=1}^{k}r_n(2\mu_j-r_n)\|A_jx_n-A_jx^*\|^2\Big]\\ &\leq \|x_n-x^*\|^2+\alpha_n\|f(W_nx_n)-x^*\|^2-(1-\alpha_n)(1-\beta_n)\frac{1}{k}\sum_{j=1}^{k}r_n(2\mu_j-r_n)\|A_jx_n-A_jx^*\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}&(1-\alpha_n)(1-\beta_n)\frac{1}{k}\sum_{j=1}^{k}r_n(2\mu_j-r_n)\|A_jx_n-A_jx^*\|^2\\ &\leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|f(W_nx_n)-x^*\|^2\\ &\leq (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\|f(W_nx_n)-x^*\|^2.\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, we arrive at

$$\lim_{n\to\infty}\|A_jx_n-A_jx^*\|=0,\qquad \forall j=1,2,\cdots,k. \tag{3.14}$$

Note that

$$\begin{aligned}\|u_{n,j}-x^*\|^2 &\leq \langle (I-r_nA_j)x_n-(I-r_nA_j)x^*, u_{n,j}-x^*\rangle\\ &= \frac{1}{2}\Big(\|(I-r_nA_j)x_n-(I-r_nA_j)x^*\|^2+\|u_{n,j}-x^*\|^2\\ &\quad -\|(I-r_nA_j)x_n-(I-r_nA_j)x^*-(u_{n,j}-x^*)\|^2\Big)\\ &\leq \frac{1}{2}\Big(\|x_n-x^*\|^2+\|u_{n,j}-x^*\|^2-\|x_n-u_{n,j}-r_n(A_jx_n-A_j)x^*\|^2\Big)\\ &= \frac{1}{2}\Big(\|x_n-x^*\|^2+\|u_{n,j}-x^*\|^2-\|x_n-u_{n,j}\|^2\\ &\quad +2r_n\langle x_n-u_{n,j},A_jx_n-A_jx^*\rangle-r_n^2\|A_jx_n-A_jx^*\|^2\Big).\end{aligned}$$

This implies that

$$\|u_{n,j}-x^*\|^2 \leq \|x_n-x^*\|^2 - \|x_n-u_{n,j}\|^2 + 2r_n\|x_n-u_{n,j}\|\|A_jx_n-A_jx^*\|. \tag{3.15}$$

It follows that

$$\begin{aligned}
\|\omega_n-x^*\|^2 &= \|\sum_{j=1}^{k}\frac{1}{k}(u_{n,j}-x^*)\|^2 \\
&\leq \frac{1}{k}\sum_{j=1}^{k}\|u_{n,j}-x^*\|^2 \\
&\leq \|x_n-x^*\|^2 - \frac{1}{k}\sum_{j=1}^{k}\|u_{n,j}-x_n\|^2 + \frac{1}{k}\sum_{j=1}^{k}2r_n\|x_n-u_{n,j}\|\|A_jx_n-A_jx^*\|. \qquad (3.16)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\Big(\alpha_n\|f(W_nx_n)-x^*\|^2 + (1-\alpha_n)\|\omega_n-x^*\|^2\Big) \\
&\leq \beta_n\|x_n-x^*\|^2 + \alpha_n(1-\beta_n)\|f(W_nx_n)-x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\Big(\|x_n-x^*\|^2 - \frac{1}{k}\sum_{j=1}^{k}\|u_{n,j}-x_n\|^2 \\
&\quad + \frac{1}{k}\sum_{j=1}^{k}2r_n\|x_n-u_{n,j}\|\|A_jx_n-A_jx^*\|\Big) \\
&\leq \|x_n-x^*\|^2 + \alpha_n\|f(W_nx_n)-x^*\|^2 - (1-\alpha_n)(1-\beta_n)\frac{1}{k}\sum_{j=1}^{k}\|u_{n,j}-x_n\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\frac{1}{k}\sum_{j=1}^{k}2r_n\|x_n-u_{n,j}\|\|A_jx_n-A_jx^*\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-\alpha_n)(1-\beta_n)\frac{1}{k}\sum_{j=1}^{k}\|u_{n,j}-x_n\|^2 &\leq \alpha_n\|f(W_nx_n)-x^*\|^2 + \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\frac{1}{k}\sum_{j=1}^{k}2r_n\|x_n-u_{n,j}\|\|A_jx_n-A_jx^*\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1}-x_n\| \to 0$, we have

$$\lim_{n\to\infty}\|u_{n,j}-x_n\| = 0. \tag{3.17}$$

It is easy to see that

$$\lim_{n\to\infty}\|\omega_n-x_n\| = 0. \tag{3.18}$$

Since $z_n = \alpha_nf(W_nx_n) + (1-\alpha_n)\omega_n$, we find that

$$\lim_{n\to\infty}\|z_n-\omega_n\| = 0. \tag{3.19}$$

Notice that

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|W_nz_n-x_n\|.$$

This together with (3.12) gives that

$$\lim_{n\to\infty} \|W_n z_n - x_n\| = 0. \tag{3.20}$$

Observe that

$$\|W_n z_n - z_n\| \le \|z_n - \omega_n\| + \|\omega_n - x_n\| + \|x_n - W_n z_n\|.$$

From (3.18), (3.19), (3.20), we obtain that

$$\lim_{n\to\infty} \|W_n z_n - z_n\| = 0. \tag{3.21}$$

Since the mapping $P_\Omega f$ is contractive, we denote its unique fixed point by x. Next, we prove that

$$\limsup_{n\to\infty} \langle f(x) - x, z_n - x\rangle \le 0.$$

To see this, we choose a subsequence $\{z_{n_m}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x) - x, z_n - x\rangle = \lim_{m\to\infty} \langle f(x) - x, z_{n_m} - x\rangle.$$

Since $\{z_{n_m}\}$ is bounded, there exists a subsequence $\{z_{n_{m_i}}\}$ of $\{z_{n_m}\}$ which converges weakly to z. Without loss of generality, we may assume that $z_{n_m} \rightharpoonup z$. Indeed, we also have $\omega_{n_m} \rightharpoonup f$. First, we show that $z \in \bigcap_{i=1}^{\infty} Fix(S_i)$. Suppose to the contrary that $Wz \ne z$. Note that

$$\begin{aligned}\|z_n - Wz_n\| &\le \|Wz_n - W_n z_n\| + \|W_n z_n - z_n\| \\ &\le \sup_{x\in K}\{\|Wx - W_n x\|\} + \|W_n z_n - z_n\|.\end{aligned}$$

In view of Lemma 2.6, we obtain from (3.21) that $\lim_{n\to\infty} \|z_n - Wz_n\| = 0$. By using the Opial's condition, we see that

$$\begin{aligned}\liminf_{m\to\infty} \|z_{n_m} - z\| &< \liminf_{m\to\infty} \|z_{n_m} - Wz\| \\ &\le \liminf_{m\to\infty} \Big\{ \|z_{n_m} - Wz_{n_m}\| + \|Wz_{n_m} - Wz\| \Big\} \\ &\le \liminf_{m\to\infty} \Big\{ \|z_{n_m} - Wz_{n_m}\| + \|z_{n_m} - z\| \Big\}.\end{aligned}$$

This implies that

$$\liminf_{m\to\infty} \|z_{n_m} - z\| < \liminf_{m\to\infty} \|z_{n_m} - z\|,$$

which is a contradiction. Hence, we have $z \in \bigcap_{i=1}^{\infty} Fix(S_i)$.

Next, we show that $f \in \bigcap_{j=1}^{k} GEP(\Upsilon_j, A_j)$. Since $\{\omega_n\}$ is bounded, there exists a subsequence $\{\omega_{n_m}\}$ of $\{\omega_n\}$ such that $\omega_{n_m} \rightharpoonup f$. Furthermore, we have

$$\Upsilon_j(u_{n,j}, y) + \langle A_j x_n, y - u_{n,j}\rangle + \frac{1}{r_n}\langle y - u_{n,j}, u_{n,j} - x_n\rangle \ge 0, \quad \forall y \in \mathscr{C}.$$

By using condition (**C2**), we see that

$$\langle A_j x_n, y - u_{n,j}\rangle + \frac{1}{r_n}\langle y - u_{n,j}, u_{n,j} - x_n\rangle \ge \Upsilon_j(y, u_{n,j}), \quad \forall y \in \mathscr{C}.$$

Substituting n by n_m , we get that

$$\langle A_j x_{n_m}, y - u_{n_m,j}\rangle + \langle y - u_{n_m,j}, \frac{u_{n_m,j} - x_{n_m}}{r_{n_m}}\rangle \ge \Upsilon_j(y, u_{n_m,j}), \quad \forall y \in \mathscr{C}. \tag{3.22}$$

For $0 < l \leq 1$ and $y \in \mathscr{C}$, let $y_l = ly + (1-l)z$. Since $y \in \mathscr{C}$ and $z \in \mathscr{C}$, we have $y_l \in \mathscr{C}$. It follows from (3.22) that

$$\begin{aligned}\langle y_l - u_{n_m,j}, A_j y_l\rangle &\geq \langle y_l - u_{n_m,j}, A_j y_l\rangle - \langle A_j x_{n_m}, y_l - u_{n_m,j}\rangle \\ &\quad - \langle y_l - u_{n_m,j}, \frac{u_{n_m,j} - x_{n_m}}{r_{n_m}}\rangle + \Upsilon_j(y_l, u_{n_m,j}) \\ &= \langle y_l - u_{n_m,j}, A_j y_l - A_j u_{n_m,j}\rangle + \langle y_l - u_{n_m,j}, A_j u_{n_m,j} - A_j x_{n_m}\rangle \\ &\quad - \langle y_l - u_{n_m,j}, \frac{u_{n_m,j} - x_{n_m}}{r_{n_m}}\rangle + \Upsilon_j(y_l, u_{n_m,j}).\end{aligned} \tag{3.23}$$

Monotonicity of A_j, condition (**C4**) and (3.17) imply that

$$\langle y_l - u_{n_m,j}, A_j y_l - A_j u_{n_m,j}\rangle \geq 0,$$

and

$$\| A_j u_{n_m,j} - A_j x_{n_m} \| \to 0 \quad as\ m \to \infty.$$

It follows from (**C4**) and (3.23) that

$$\langle y_l - z, A_j y_l\rangle \geq \Upsilon_j(y_l, z). \tag{3.24}$$

Now, (**C1**) and (**C4**) together with (3.24) show that

$$\begin{aligned}0 = \Upsilon_j(y_l, y_l) &\leq l\Upsilon_j(y_l, y) + (1-l)\Upsilon_j(y_l, z) \\ &\leq l\Upsilon_j(y_l, y) + (1-l)\langle y_l - z, A_j y_l\rangle \\ &= l\Upsilon_j(y_l, y) + (1-l)l\langle y - z, A_j y_l\rangle,\end{aligned}$$

which yields $\Upsilon_j(y_l, y) + (1-l)\langle y - z, A_j y_l\rangle \geq 0$. By letting $l \to 0$, we have

$$\Upsilon_j(z, y) + \langle y - z, A_j z\rangle \geq 0.$$

This shows that $f \in GEP(\Upsilon_j, A_j)$ for all $j = 1, 2, \cdots, k$, or $f \in \bigcap_{j=1}^{k} GEP(\Upsilon_j, A_j)$. It follows that

$$\limsup_{n\to\infty} \langle f(x) - x, z_n - x\rangle \leq 0. \tag{3.25}$$

Finally, we show that $x_n \to x$, as $n \to \infty$. Note that

$$\begin{aligned}\| z_n - x \|^2 &= \alpha_n \langle f(W_n x_n) - x, z_n - x\rangle + (1-\alpha_n)\langle \omega_n - x, z_n - x\rangle \\ &\leq (1 - \alpha_n(1-k)) \| x_n - x \| \| z_n - x \| + \alpha_n \langle f(x) - x, z_n - x\rangle \\ &\leq \frac{1 - \alpha_n(1-k)}{2}(\| x_n - x \|^2 + \| z_n - x \|^2) + \alpha_n \langle f(x) - x, z_n - x\rangle.\end{aligned}$$

Hence, we have

$$\|z_n - x\|^2 \leq (1 - \alpha_n(1-k))\|x_n - x\|^2 + 2\alpha_n \langle f(x) - x, z_n - x\rangle.$$

This implies that

$$\begin{aligned}\| x_{n+1} - x \|^2 &= \| \beta_n x_n + (1-\beta_n) W_n z_n - x \|^2 \\ &\leq \beta_n \| x_n - x \|^2 + (1-\beta_n) \| z_n - x \|^2 \\ &\leq (1 - \alpha_n(1-\beta_n)(1-k)) \| x_n - x \|^2 + 2\alpha_n(1-\beta_n)\langle f(x) - x, z_n - x\rangle.\end{aligned}$$

By using Lemma 2.3 and (3.25), we find that $\lim_{n\to\infty} \| x_n - x \| = 0$. This completes proof. □

4. Applications and Numerical Examples

In this section, we give some special cases of 3.1 and numerical results to support the convergence analysis of our algorithm.

For a single nonexpansive mapping, we find from Theorem 3.1 the following result immediately.

Corollary 4.1. *Let $\mathscr{C}$ be a nonempty, closed convex subset of a Hilbert space $\mathscr{H}$ and let $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k$ be bifunctions from $\mathscr{C} \times \mathscr{C}$ to $\mathbb{R}$ which satisfy $(\textbf{C1}) - (\textbf{C4})$. Let $A_1, A_2, \cdots, A_k$ be μ_j-inverse strongly monotone mappings from $\mathscr{C}$ to $\mathscr{H}$ and let $S : \mathscr{C} \to \mathscr{C}$ be a nonexpansive mapping. Let $f : \mathscr{C} \to \mathscr{C}$ be a contractive mapping with the constant $k \in (0,1)$. Assume that*

$$\Omega := \Big(Fix(S)\Big) \cap \Big(\bigcap_{j=1}^{k} GEP(\Upsilon_j, A_j)\Big) \neq \emptyset.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in \mathscr{C},\ y \in \mathscr{C}, \\ \Upsilon_1(u_{n,1}, y) + \langle A_1 x_n, y - u_{n,1}\rangle + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \\ \Upsilon_2(u_{n,2}, y) + \langle A_2 x_n, y - u_{n,2}\rangle + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \\ \vdots \\ \Upsilon_k(u_{n,k}, y) + \langle A_k x_n, y - u_{n,k}\rangle + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \\ \omega_n = \dfrac{1}{k}\sum_{j=1}^{k} u_{n,j}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S(\alpha_n f(Sx_n) + (1-\alpha_n)\omega_n), \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a sequence of positive numbers. Assume that the above control sequences satisfy the following conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\mu$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ convergence strongly to a point $x \in \Omega$, where $x = P_{\Omega} f(x)$.

Let S be the identity mapping. For a system of generalized equilibrium problems, we find from Theorem 3.1 the following result immediately.

Corollary 4.2. *Let $\mathscr{C}$ be a nonempty, closed convex subset of a Hilbert space $\mathscr{H}$ and let $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k$ be bifunctions from $\mathscr{C} \times \mathscr{C}$ to $\mathbb{R}$ which satisfy $(\textbf{C1}) - (\textbf{C4})$. Let $A_1, A_2, \cdots, A_k$ be μ_j-inverse strongly monotone mappings from $\mathscr{C}$ to $\mathscr{H}$ and let $f : \mathscr{C} \to \mathscr{C}$ be a contractive mapping with the constant $k \in (0,1)$. Assume that*

$$\Omega := \Big(\bigcap_{j=1}^{k} GEP(\Upsilon_j, A_j)\Big) \neq \emptyset.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in \mathscr{C},\ y \in \mathscr{C}, \\ \Upsilon_1(u_{n,1},y) + \langle A_1 x_n, y - u_{n,1}\rangle + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \\ \Upsilon_2(u_{n,2},y) + \langle A_2 x_n, y - u_{n,2}\rangle + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \\ \vdots \\ \Upsilon_k(u_{n,k},y) + \langle A_k x_n, y - u_{n,k}\rangle + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \\ \omega_n = \dfrac{1}{k}\sum_{j=1}^{k} u_{n,j}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n f(x_n) + (1-\alpha_n)\omega_n), \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a sequence of positive numbers. Assume that the above control sequences satisfy the following conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\mu$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ convergence strongly to a point $x \in \Omega$, where $x = P_{\Omega} f(x)$.

For a system of equilibrium problems and common fixed point problems of nonexpansive mappings, we find from Theorem 3.1 the following result immediately.

Corollary 4.3. *Let $\mathscr{C}$ be a nonempty, closed convex subset of a Hilbert space $\mathscr{H}$ and let $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k$ be bifunctions from $\mathscr{C}\times\mathscr{C}$ to $\mathbb{R}$ which satisfy $(\boldsymbol{C1}) - (\boldsymbol{C4})$. Let $\{S_i : \mathscr{C} \to \mathscr{C}\}$ be an infinite family of nonexpansive mappings and let $f : \mathscr{C} \to \mathscr{C}$ be a contractive mapping with the constant $k \in (0,1)$. Assume that*

$$\Omega := \left(\bigcap_{i=1}^{\infty} Fix(S_i)\right) \cap \left(\bigcap_{j=1}^{k} GEP(\Upsilon_j, A_j)\right) \neq \emptyset.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in \mathscr{C},\ y \in \mathscr{C}, \\ \Upsilon_1(u_{n,1},y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \\ \Upsilon_2(u_{n,2},y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \\ \vdots \\ \Upsilon_k(u_{n,k},y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \\ \omega_n = \dfrac{1}{k}\sum_{j=1}^{k} u_{n,j}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) W_n(\alpha_n f(W_n x_n) + (1-\alpha_n)\omega_n), \end{cases} \tag{4.3}$$

where $\{W_n\}$ is the sequence defined by (2.1), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a sequence of positive numbers. Assume that the above control sequences satisfy the following conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\mu$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ convergence strongly to a point $x \in \Omega$, where $x = P_{\Omega} f(x)$.

Corollary 4.4. *Let $\mathscr{C}$ be a nonempty, closed convex subset of a Hilbert space $\mathscr{H}$ and let $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k$ be bifunctions from $\mathscr{C}\times\mathscr{C}$ to $\mathbb{R}$ which satisfy $(\boldsymbol{C1}) - (\boldsymbol{C4})$. Let $A_1, A_2, \cdots, A_k$ be μ_j-inverse strongly* *monotone mappings from $\mathscr{C}$ to $\mathscr{H}$ and let $\{S_i : \mathscr{C} \to \mathscr{C}\}$ be an infinite family of nonexpansive mappings. Let $f : \mathscr{C} \to \mathscr{C}$ be a contractive mapping with the constant $k \in (0,1)$. Assume that*

$$\Omega := \left(\bigcap_{i=1}^{\infty} Fix(S_i)\right) \cap \left(\bigcap_{j=1}^{k} VI(\mathscr{C}, A_j)\right) \neq \emptyset.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in \mathscr{C}, \\ u_{n,1} = P_{\mathscr{C}}(x_n - r_n A_1 x_n), \\ u_{n,2} = P_{\mathscr{C}}(x_n - r_n A_2 x_n), \\ \vdots \\ u_{n,k} = P_{\mathscr{C}}(x_n - r_n A_k x_n), \\ \omega_n = \frac{1}{k}\sum_{j=1}^{k} u_{n,j}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) W_n(\alpha_n f(W_n x_n) + (1-\alpha_n)\omega_n), \end{cases} \tag{4.4}$$

where $\{W_n\}$ is the sequence defined by (2.1), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a sequence of positive numbers. Assume that the above control sequences satisfy the following conditions: $0 < a \le \beta_n \le b < 1$, $0 < c \le r_n \le d < 2\mu$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ convergence strongly to a point $x \in \Omega$, where $x = P_\Omega f(x)$.

Proof. Putting $\Upsilon \equiv 0$, we see from Theorem 3.1 that

$$\langle A_j x_n, y - u_{n,j}\rangle + \frac{1}{r_n}\langle y - u_{n,j}, u_{n,j} - x_n\rangle \ge 0, \quad \forall y \in \mathscr{C}.$$

This implies that

$$\langle y - u_{n,j}, x_n - r_n A_j x_n - u_{n,j}\rangle \le 0, \quad \forall y \in \mathscr{C},$$

from which it follows that

$$u_{n,j} = P_{\mathscr{C}}(x_n - r_n A_j x_n).$$

This completes the proof. □

Finally, we provide an example of GEP satisfying the conditions of Theorem 3.1, and give some numerical results to illustrate our algorithm.

Consider $X = \mathbb{R}$ and $\mathscr{C} = [\circ, 1]$. For any $i \in I$, we define the bifunctions Υ_i by

$$\begin{cases} \Upsilon_i : \mathscr{C} \times \mathscr{C} \longrightarrow \mathbb{R} \\ \Upsilon_i(x,y) = (y + ix)(y - x). \end{cases}$$

and $A_i x = ix$. It is easy to see that, for each $i \in I$, Υ_i satisfies the conditions $(\mathbf{C1})-(\mathbf{C4})$, also A_i, for each $i \in I$, is $\frac{1}{i+1}$-strongly monotone mapping. Indeed,

$$\begin{aligned}\langle A_i x - A_i y, x-y\rangle &= \langle ix - iy, x-y\rangle \\ &= i\langle x-y, x-y\rangle \\ &= i\|x-y\|^2 \\ &\geq \frac{1}{i+1}\|x-y\|^2.\end{aligned}$$

Let $\left\{S_n(x) := \dfrac{x}{n+1}\right\}_{n\geq 1}$ be an infinite family of nonexpansive mappings on $\mathscr{C}$. For all $n \geq 1$, define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(\circ, 1)$ by $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{n^2+1}$ and $\gamma_n = \frac{1}{n}$. Take $r_n = \frac{1}{n+1}$ and $f(x) = \frac{x}{3}$. Therefore, all conditions of Theorem 3.1 are satisfied. First, we find the sequence $\{u_{n,i}\}$ which satisfies the following *Generalized Equilibrium Problem*, for all $y \in \mathscr{C}$,

$$\Upsilon_i(u_{n,},y) + \langle A_i x_n, y - u_{n,i}\rangle + \frac{1}{r_n}\langle y - u_{n,i}, u_{n,i} - x_n\rangle \geq 0.$$

For all $n \geq 1$ and $i \in I$, we get

$$\begin{aligned}&\Upsilon_i(u_{n,i},y) + \langle A_i x_n, y - u_{n,i}\rangle + \frac{1}{r_n}\langle y - u_{n,i}, u_{n,i} - x_n\rangle \geq 0 \\ &\iff (y + iu_{n,i})(y-u_{n,i}) + (ix_n)(y-u_{n,i}) + \frac{1}{r_n}(y-u_{n,i})(u_{n,i}-x_n) \geq 0 \\ &\iff r_n y^2 - r_n u_{n,i} y + r_n i u_{n,i} y - r_n i u_{n,i}^2 + r_n i x_n y - r_n i x_n u_{n,i} + y u_{n,i} - y x_n - u_{n,i}^2 + u_{n,i} x_n \geq 0 \\ &\iff r_n y^2 + \big((1 + r_n(i-1)u_{n,i}) - (1 - ir_n)x_n\big)y + \big((1-ir_n)u_{n,i}x_n - (1+ir_n)u_{n,i}^2\big) \geq 0.\end{aligned}$$

Put

$$K_i(y) = r_n y^2 + \big((1 + r_n(i-1)u_{n,i}) - (1 - ir_n)x_n\big)y + \big((1-ir_n)u_{n,i}x_n - (1+ir_n)u_{n,i}^2\big).$$

Since K_i is a quadratic function relative to y, $K_i(y) \geq 0$ for all $y \in \mathscr{C}$, if and only if the coefficient of y^2 is positive and the discriminant $\triangle_i \leq \circ$. But

$$\begin{aligned}\triangle_i &= \big((1 + r_n(i-1)u_{n,i}) - (1 - ir_n)x_n\big)^2 - 4r_n\big((1-ir_n)u_{n,i}x_n - (1+ir_n)u_{n,i}^2\big) \\ &= \big((1 + r_n(i-1)u_{n,i}) - (1 - ir_n)x_n\big)^2.\end{aligned}$$

So, $T_{r_n,i}(x_n) = \frac{1-ir_n}{1+r_n(i+1)}x_n$. It is easy to see that Υ_i for all $i \in I$, satisfies the conditions $(\mathbf{C1})-(\mathbf{C4})$. Next, we sketch the graph of Υ_i for $i = 1, 2, \cdots, 100$ in three-dimensional space (see Figure 1). We see that

$$\left(\bigcap_{i=1}^{100} Fix(S_i)\right) \cap \left(\bigcap_{j=1}^{100} GEP(\Upsilon_j, A_j)\right) = \{\circ\}.$$

In this case, x_n converges to zero. Thus, $\{x_n\}_{n\geq 1}$ is convergent.

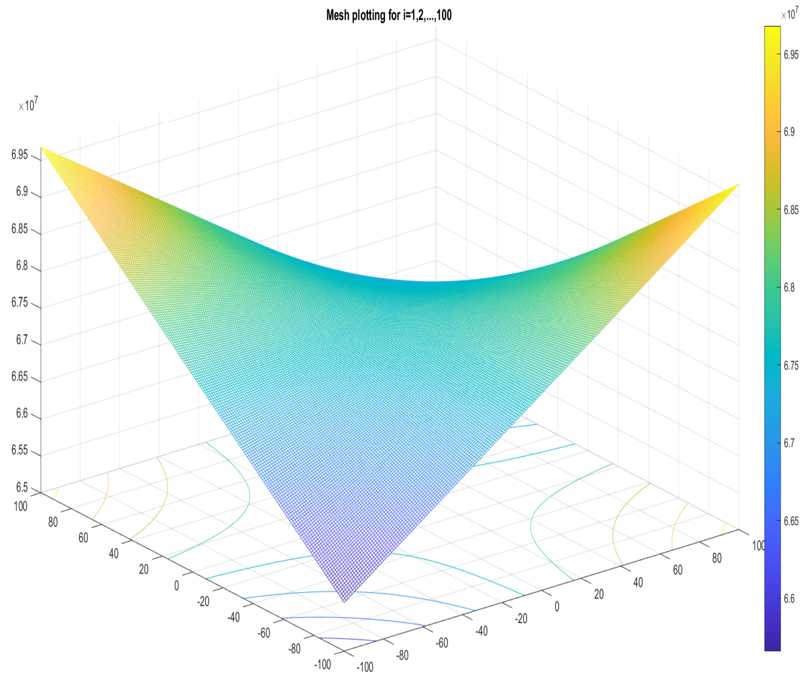

FIGURE 1. Sketched graph of Υ_i for $i = 1, 2, \cdots, 100$.

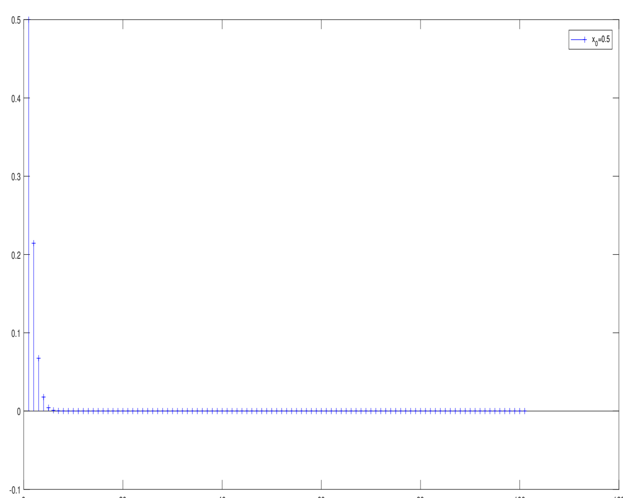

FIGURE 2. Plot of our algorithm.

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