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# SELF-ADAPTIVE ITERATIVE ALGORITHMS FOR SOLVING MULTIPLE-SET SPLIT EQUALITY COMMON FIXED-POINT PROBLEMS OF DEMICONTRACTIVE OPERATORS

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**Abstract.** In this paper, we introduce parallel and cyclic iterative algorithms for solving the multiple-set split equality common fixed-point problem of demicontractive operators. We propose a way of selecting the stepsizes such that the implementation of our algorithms does not need any prior information about operator norms. It thus avoids the difficult task of estimating the operator norms. We also combine the process of cyclic and parallel together and propose two mixed iterative algorithms without prior knowledge of operator norms. The weak convergence theorems of the proposed algorithms are established under some suitable control conditions in a real Hilbert space. Some numerical experiments are given for the proposed iterative algorithms.

**Keywords.** Multiple-set split equality common fixed-point problem; Demicontractive operator; Weak convergence; Iterative algorithm; Hilbert space.

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## 1. INTRODUCTION

Throughout this paper, we always assume that $H$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $I$ denote the identity operator on $H$. Let $T : H \to H$ be an operator. A point $x \in H$ is said to be a fixed point of $T$ provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed-point set of $T$.

Recall that the multiple-sets split feasibility problem (MSFP) which finds application in intensity modulated radiation therapy was proposed in [1] and is formulated as finding a point $x^*$ satisfying the property:

$$x^* \in \bigcap_{i=1}^{p} C_i \ \text{ such that } \ Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

where $p, r \geq 1$ are integers, $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The MSFP (1.1) with

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$p=r=1$ is known as the split feasibility problem (SFP) originally introduced by Censor and Elfving [2], which is defined as follows:

$$x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{1.2}$$

where $C$ and $Q$ are nonempty closed convex subset of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A: \ H_1 \to H_2$ is a bounded linear operator. For solving the SFP (1.2), Censor and Elfving [2] used a multidistance method to obtain iterative algorithms, which involved matrix inverses at each step. In order to avoid usage of the inverse, Byrne [3] proposed an iterative method called the CQ algorithm that involves only the orthogonal projections onto $C$ and $Q$ to solve the SFP (1.2). The CQ algorithm is defined as follows:

$$x_{k+1} = P_C(I-\gamma A^*(I-P_Q)A)x_k, \ k \in N,$$

where $A^*$ denotes the adjoint of $A$ and $\gamma \in (0, \frac{2}{\lambda})$ with $\lambda$ being the spectral radius of the operator $A^*A$. Later, Yang [4] proposed a relaxed CQ algorithm for solving the SFP in which the orthogonal projections $P_C$ and $P_Q$ are replaced by $P_{C_n}$ and $P_{Q_n}$, respectively, i.e., the orthogonal projections onto two halfspaces $C_n$ and $Q_n$. Both the CQ algorithm and the relaxed CQ algorithm used a fixed stepsize related to the largest eigenvalue of $A^*A$, which sometimes affects convergence of the algorithms. Qu and Xiu [5] developed the CQ algorithm and the relaxed CQ algorithm by adopting the Armijo-like searches. Some other related results can be found in [6, 7, 8, 9, 10, 11, 12] and references therein.

Since every closed convex subset of a Hilbert space is the fixed point set of its associating projection, the MSFP (1.1) and the SFP (1.2) are all special cases of the so-called multiple-set split common fixed-point problem (MSCFP), which is formulated as finding a point $x^*$ satisfying the property:

$$x^* \in \bigcap_{i=1}^{p} F(U_i) \ \text{ such that } \ Ax^* \in \bigcap_{j=1}^{r} F(T_j), \tag{1.3}$$

where $p, r \geq 1$ are integers, $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ and $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ are nonlinear operators and $A: \ H_1 \to H_2$ is a bounded linear operator. In particular, if $p=r=1$, then (1.3) reduces to finding a point $x^*$ with the property:

$$x^* \in F(U) \ \text{ such that } \ Ax^* \in F(T), \tag{1.4}$$

which is usually called the split common fixed-point problem (SCFP). The concept of the SCFP in finite-dimensional Hilbert spaces was originally introduced by Censor and Segal [13]. For nonexpansive operators, they proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U(x_k + \gamma A^t(T-I)Ax_k), \ \ k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with $\lambda$ being the largest eigenvalue of $A^tA$ ($A^t$ stands for the matrix transposition). Many methods have been proposed for solving the MSCFP and SCFP or their particular cases; see, for example, [14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein.

For solving the multiple-set common fixed-point problem, some authors proposed the parallel and cyclic iterative algorithms. Censor and Segal [13] introduced the following parallel iterative algorithm to solve MSCFP (1.3) of directed operators:

$$x_{k+1} = x_k - \gamma\Big[\sum_{i=1}^{p} \alpha_i(x_k - U_i(x_k)) + \sum_{j=1}^{r} \beta_j A^*(Ax_k - T_j(Ax_k))\Big],$$

where $\{\alpha_i\}_{i=1}^p$, $\{\beta_j\}_{j=1}^r$ are nonnegative constants, $0<\gamma<2/L$ with $L=\sum_{i=1}^p\alpha_i+\lambda\sum_{j=1}^r\beta_j$ and $\lambda$ being the largest eigenvalue of $A^*A$. Wang and Xu [23] proved the convergence of the following cyclic iterative algorithm for MSCFP (1.3) of directed operators:

$$x_{k+1}=U_{[k]_1}(x_k+\gamma A^*(T_{[k]_2}-I)Ax_k),$$

where $0<\gamma<2/\rho(A^*A)$, $[k]_1:=k(\mod p)$ and $[k]_2:=k(\mod r)$.

For solving MSCFP (1.3) of directed operators, Tang and Liu [24] introduced inner parallel and outer cyclic iterative algorithm:

$$x_{k+1}=U_{[k]_1}(x_k+\gamma_k\Sigma_{j=1}^r\eta_jA^*(T_j-I)Ax_k)$$

and outer parallel and inner cyclic iterative algorithm:

$$x_{k+1}=\Sigma_{j=1}^r\omega_iU_i(x_k+\gamma_kA^*(T_{[k]_2}-I)Ax_k),$$

where $[k]_1:=k(\mod p)$ and $[k]_2:=k(\mod r)$.

In [25], Moudafi proposed an algorithm to solve the SCFP when operators $U$ and $T$ are demicontractive. The class of demicontractive operators is fundamental because many common types of operators arising in optimization belong to this class (see Remark 2.3 below). Moudafi proved that the sequence $\{x_k\}$ generated by the following algorithm converges weakly to the solution of SCFP (1.4):

$$x_{k+1}=(1-\alpha_k)u_k+\alpha_kU(u_k),$$

where $u_k=x_k+\gamma A^*(T-I)Ax_k$, $\gamma\in(0,\frac{1-\mu}{\lambda})$ with $\lambda$ being the spectral radius of the operator $A^*A$ and $\{\alpha_k\}\subseteq(0,1)$. For solving MSCFP (1.3) of the demicontractive mappings, Tang, Peng and Liu [26] proposed the following cyclic algorithm and proved the weak convergence under the suitable conditions:

$$x_{k+1}=(1-\alpha_k)u_k+\alpha_kU_{i(k)}(u_k),$$

where $u_k=x_k+\gamma A^*(T_{j(k)}-I)Ax_k$, $i(k)=k(\mathrm{mod}p)+1$, $j(k)=k(\mathrm{mod}r)+1$, $\gamma\in(0,\frac{1-\mu}{\lambda})$ with $\lambda$ being the spectral radius of the operator $A^*A$ and $\{\alpha_k\}\subseteq(0,1)$.

Recently, Moudafi [27] introduced the split equality common fixed-point problem (SECFP). Let $H_1$, $H_2$, $H_3$ be real Hilbert spaces, let $A:H_1\to H_3$, $B:H_2\to H_3$ be two bounded linear operators, let $U:H_1\to H_1$ and $T:H_2\to H_2$ be two firmly quasi-nonexpansive operators. The SECFP in [27] is to

$$\text{find } x^*\in F(U),\ y^*\in F(T) \text{ such that } Ax^*=By^*. \tag{1.5}$$

If $H_2=H_3$ and $B=I$, then SECFP (1.5) reduces to SCFP (1.4).

For solving SECFP (1.5) of firmly quasi-nonexpansive operators, Moudafi [27] introduced the following alternating algorithm:

$$\begin{cases} x_{k+1}=U(x_k-\gamma_kA^*(Ax_k-By_k)),\\ y_{k+1}=T(y_k+\gamma_kB^*(Ax_{k+1}-By_k)),\end{cases} \tag{1.6}$$

where non-decreasing sequence $\gamma_k\in(\varepsilon,\min(\frac{1}{\lambda_A},\frac{1}{\lambda_B})-\varepsilon)$, $\lambda_A$, $\lambda_B$ stand for the spectral radiuses of $A^*A$ and $B^*B$, respectively.

In [28], Moudafi and Al-Shemas introduced the following simultaneous iterative method to solve SECFP (1.5) of firmly quasi-nonexpansive operators:

$$\begin{cases} x_{k+1}=U(x_k-\gamma_kA^*(Ax_k-By_k)),\\ y_{k+1}=T(y_k+\gamma_kB^*(Ax_k-By_k)),\end{cases} \tag{1.7}$$

where $\gamma_k \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, $\lambda_A$ and $\lambda_B$ stand for the spectral radiuses of $A^*A$ and $B^*B$, respectively.

Let $H_1$, $H_2$, $H_3$ be real Hilbert spaces. let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $U_i : H_1 \to H_1$ $(1 \le i \le p)$ and $T_j : H_2 \to H_2$ $(1 \le j \le r)$ be nonlinear operators. In this paper, inspired and motivated by the works mentioned above, we consider the following multiple-set split equality common fixed-point problem (MSECFP):

$$\text{finding } \; x^* \in \bigcap_{i=1}^{p} F(U_i),\; y^* \in \bigcap_{j=1}^{r} F(T_j), \;\text{ such that }\; Ax^* = By^*. \tag{1.8}$$

This allows asymmetric and partial relations between $x$ and $y$. The interest is to cover many situations, for instance, in decomposition methods for PDEs, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interact only via some components of their decision variables (see [29]). In IMRT, this amounts to envisaging a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [30]). Some iterative algorithms have been proposed to solve the SECFP and MSECFP; see, for example, [31, 32, 33, 34, 35] and the references therein. In this paper, we propose the parallel and cyclic iterative algorithms to solve MSECFP (1.8) of demicontractive operators. We also propose two mixed iterative algorithms which combine the process of cyclic and parallel together. We introduce a way of selecting the stepsizes such that the implementation of our algorithms does not need any prior information about the operator norms of $A$ and $B$, and the convergence is still guaranteed.

The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the self-adaptive iterative algorithms in Section 2. In Section 3, we introduce self-adaptive parallel and cyclic iterative algorithms and the weak convergence theorems of the proposed iterative algorithms is obtained. In Section 4, we propose two mixed self-adaptive iterative algorithms which combine the process of cyclic and parallel together and obtain the weak convergence results. Some numerical experiments are provided to illustrate the efficiency of the proposed iterative algorithms in Section 5, the last section.

## 2. Preliminaries

In this paper, let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces and $\Gamma$ the solution set of MSECFP (1.8). Assume that $A \neq 0$ or $B \neq 0$. We denote by $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ the weak $\omega$-limit set of $\{x_k\}$. We denote the strong convergence and weak convergence by $\to$ and $\rightharpoonup$, respectively.

**Definition 2.1.** An operator $T : H \to H$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - q\| \le \|x - q\|$ for all $x \in H$ and $q \in F(T)$;

(iii) firmly nonexpansive if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$ for all $x, y \in H$;

(iv) firmly quasi-nonexpansive (also called directed operator) if $F(T) \neq \emptyset$ and $\|Tx - q\|^2 \le \|x - q\|^2 - \|x - Tx\|^2$ for all $x \in H$ and $q \in F(T)$;

(v) $k$-strictly pseudocontractive if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$$

for all $x, \; y \in H$;

(vi) $\beta$-demicontractive if $F(T) \neq \emptyset$ and there exists a constant $\beta \in (0,1)$ such that

$$\|Tx-q\|^2 \leq \|x-q\|^2 + \beta\|x-Tx\|^2$$

for all $x \in H$ and $q \in F(T)$.

Two equivalent definitions of demicontractive operator are given by the following lemma.

**Lemma 2.2.** [26] *Let $T: H \to H$ be $\beta$-demicontractive operator. Then the following inequalities are equivalent:*

*(i) $\langle x-Tx,\, x-q\rangle \geq \frac{1-\beta}{2}\|x-Tx\|^2$, $q \in F(T)$, $x \in H$;*

*(ii) $\langle x-Tx,\, q-Tx\rangle \leq \frac{1+\beta}{2}\|x-Tx\|^2$, $q \in F(T)$, $x \in H$.*

**Remark 2.3.** It is easy to see that the class of demicontractive operators contain the class of firmly quasi-nonexpansive operators, the class of quasi-nonexpansive operators and the class of strictly pseudo-contractive operators with a nonempty fixed point.

**Definition 2.4.** An operator $T: H \to H$ is said to be demiclosed at the origin if, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, and $Tx_n \to 0$, then $Tx = 0$.

In real Hilbert spaces, we easily get the following equality:

$$2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x,y \in H. \tag{2.1}$$

**Lemma 2.5.** [36] *Let $H$ be a real Hilbert space. For each $x_1,\ldots,x_m \in H$ and $\alpha_1,\ldots,\alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, we have*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m\|^2 = \Sigma_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Now, we give the following property of the relaxed operator $T_\alpha = (1-\alpha)I + \alpha T$ for demicontractive operators, which will be needed in our main results.

**Lemma 2.6.** [25] *Let $T$ be a $\beta$-demicontractive operator with $F(T) \neq \emptyset$. Set $T_\alpha = (1-\alpha)I + \alpha T$, for $\alpha \in [0, 1-\beta]$. Then, $T_\alpha$ is quasi-nonexpansive and*

$$\|T_\alpha x - q\|^2 \leq \|x-q\|^2 - \alpha(1-\beta-\alpha)\|Tx-x\|^2, \quad x \in H, \quad q \in F(T).$$

**Remark 2.7.** It is easy to check that $F(T) = F(T_\alpha)$. Hence, $F(T)$ is a closed convex subset of $H$, as the fixed-point set of a quasi-nonexpansive operator.

**Lemma 2.8.** [37] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $T: C \to C$ be a $k$-strictly pseudocontractive operator. Then $I-T$ is demiclosed at $0$.*

## 3. Self-adaptive parallel and cyclic iterative algorithms for the MSECFP

In this section, we introduce two self-adaptive parallel and cyclic iterative algorithms for solving MSECFP (1.8) of demicontractive operators, where stepsizes don't depend on the operator norms $\|A\|$ and $\|B\|$. We prove the weak convergence of the proposed algorithms. Let $\{U_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^r$ be a finite family of demicontractive operators, i.e., there exists $\{\beta_i\}_{i=1}^p \subset (0,1)$ and $\{\mu_j\}_{j=1}^r \subset (0,1)$ such that

$$\|U_i x - v\|^2 \leq \|x-v\|^2 + \beta_i\|x - U_i x\|^2, \quad x \in H_1, \quad v \in F(U_i), \quad 1 \leq i \leq p,$$

and

$$\|T_jy-q\|^2 \le \|y-q\|^2+\mu_j\|y-T_jy\|^2,\ \ y\in H_2,\ \ q\in F(T_j),\ \ 1\le j\le r.$$

Let $\beta=\max_{1\le i\le p}\{\beta_i\}$ and $\mu=\max_{1\le j\le r}\{\mu_j\}$. Then

$$\|U_ix-v\|^2 \le \|x-v\|^2+\beta\|x-U_ix\|^2,\ \ x\in H_1,\ \ v\in F(U_i),\ \ 1\le i\le p,$$

and

$$\|T_jy-q\|^2 \le \|y-q\|^2+\mu\|y-T_jy\|^2,\ \ y\in H_2,\ \ q\in F(T_j),\ \ 1\le j\le r.$$

**Algorithm 3.1.** Let $\{\alpha_k^i\}_{k=0}^{\infty}\subset[0,1]$ $(0\le i\le p)$ and $\{\sigma_k^j\}_{k=0}^{\infty}\subset[0,1]$ $(0\le j\le r)$ be sequences such that $\Sigma_{i=0}^p\alpha_k^i=1$ and $\Sigma_{j=0}^r\sigma_k^j=1$ for every $k\ge0$. Take $x_0\in H_1, y_0\in H_2$ and calculate

$$\begin{cases} s_k=x_k-\gamma_kA^*(Ax_k-By_k),\\ x_{k+1}=\alpha_k^0s_k+\alpha_k^1U_1(s_k)+\cdots+\alpha_k^pU_p(s_k),\\ t_k=y_k+\gamma_kB^*(Ax_k-By_k),\\ y_{k+1}=\sigma_k^0t_k+\sigma_k^1T_1(t_k)+\cdots+\sigma_k^rT_r(t_k). \end{cases} \tag{3.1}$$

Assume the stepsize $\gamma_k$ is chosen in such a way that

$$\gamma_k:=\begin{cases} \frac{\rho_k\|Ax_k-By_k\|^2}{\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2}, & Ax_k-By_k\ne0,\\ \gamma, & Ax_k-By_k=0 \end{cases} \tag{3.2}$$

with $0<\liminf_{k\to\infty}\rho_k\le\limsup_{k\to\infty}\rho_k<2$ and $\gamma>0$.

**Lemma 3.2.** *Assume that $\Gamma$ is nonempty. Then $\gamma_k$ defined by (3.2) is well-defined.*

*Proof.* Taking $(x,y)\in\Gamma$, i.e., $x\in\cap_{i=1}^pF(U_i)$, $y\in\cap_{j=1}^rF(T_j)$ and $Ax=By$, we have

$$\langle A^*(Ax_k-By_k),x_k-x\rangle=\langle Ax_k-By_k,Ax_k-Ax\rangle$$

and

$$\langle B^*(Ax_k-By_k),y-y_k\rangle=\langle Ax_k-By_k,By-By_k\rangle.$$

From $Ax=By$, adding the two above equalities, we obtain

$$\begin{aligned}\|Ax_k-By_k\|^2&=\langle A^*(Ax_k-By_k),x_k-x\rangle+\langle B^*(Ax_k-By_k),y-y_k\rangle\\ &\le\|A^*(Ax_k-By_k)\|\cdot\|x_k-x\|+\|B^*(Ax_k-By_k)\|\cdot\|y-y_k\|.\end{aligned} \tag{3.3}$$

For $\|Ax_k-By_k\|>0$, we have $\|A^*(Ax_k-By_k)\|\ne0$ or $\|B^*(Ax_k-By_k)\|\ne0$. So $\gamma_k$ is well-defined. This completes the proof. □

**Theorem 3.3.** *Let $\{U_i,\ \ 1\le i\le p\}:H_1\to H_1$ and $\{T_j,\ \ 1\le j\le r\}:H_2\to H_2$ be $\beta_i$-demicontractive and $\mu_j$-demicontractive operators, respectively. Assume that $U_i-I(1\le i\le p)$, $T_j-I(1\le j\le r)$ are demiclosed at origin and $\Gamma$ is nonempty. Then the sequence $\{(x_k,y_k)\}$ generated by Algorithm 3.1 weakly converges to a solution $(x^*,y^*)$ of (1.8), provided that* $\liminf_{k\to\infty}\alpha_k^0>\beta$, $\liminf_{k\to\infty}\alpha_k^i>0$, $\forall1\le i\le p$ *and* $\liminf_{k\to\infty}\sigma_k^0>\mu$, $\liminf_{k\to\infty}\sigma_k^j>0$, $\forall1\le j\le r$. *Moreover,* $\|Ax_k-By_k\|\to0$, $\|x_k-x_{k+1}\|\to0$ *and* $\|y_k-y_{k+1}\|\to0$ *as* $k\to\infty$.

*Proof.* First, for any $(x^*,y^*)\in\Gamma$, we show that $\lim_{k\to\infty}(\|x_k-x^*\|^2+\|y_k-y^*\|^2)$ exists. Taking $(x^*,y^*)\in\Gamma$, i.e., $x^*\in\bigcap_{i=1}^p F(U_i)$, $y^*\in\bigcap_{j=1}^r F(T_j)$ and $Ax^*=By^*$, we have

$$\begin{aligned}&\|s_k-x^*\|^2\\ =&\|x_k-\gamma_k A^*(Ax_k-By_k)-x^*\|^2\\ =&\|x_k-x^*\|^2-2\gamma_k\langle x_k-x^*,A^*(Ax_k-By_k)\rangle+\gamma_k^2\|A^*(Ax_k-By_k)\|^2.\end{aligned}\tag{3.4}$$

By (2.1), we find that

$$\begin{aligned}&-2\langle x_k-x^*,A^*(Ax_k-By_k)\rangle\\ =&-2\langle Ax_k-Ax^*,Ax_k-By_k\rangle\\ =&-\|Ax_k-Ax^*\|^2-\|Ax_k-By_k\|^2+\|By_k-Ax^*\|^2.\end{aligned}\tag{3.5}$$

From (3.4) and (3.5), we obtain that

$$\begin{aligned}\|s_k-x^*\|^2=&\|x_k-x^*\|^2-\gamma_k\|Ax_k-By_k\|^2-\gamma_k\|Ax_k-Ax^*\|^2\\ &+\gamma_k\|By_k-Ax^*\|^2+\gamma_k^2\|A^*(Ax_k-By_k)\|^2.\end{aligned}\tag{3.6}$$

Similarly, we have

$$\begin{aligned}\|t_k-y^*\|^2=&\|y_k-y^*\|^2-\gamma_k\|Ax_k-By_k\|^2-\gamma_k\|By_k-By^*\|^2\\ &+\gamma_k\|Ax_k-By^*\|^2+\gamma_k^2\|B^*(Ax_k-By_k)\|^2.\end{aligned}\tag{3.7}$$

Noting $Ax^*=By^*$, we see that

$$\begin{aligned}&\|s_k-x^*\|^2+\|t_k-y^*\|^2\\ \le&\|x_k-x^*\|^2+\|y_k-y^*\|^2-\gamma_k[2\|Ax_k-By_k\|^2-\gamma_k(\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2)].\end{aligned}\tag{3.8}$$

By using (3.2), we find that

$$\|s_k-x^*\|^2+\|t_k-y^*\|^2\le\|x_k-x^*\|^2+\|y_k-y^*\|^2.\tag{3.9}$$

Since $U_i$ is $\beta_i$-demicontractive $(1\le i\le p)$, from Lemma 2.5, we have

$$\begin{aligned}&\|x_{k+1}-x^*\|^2\\ \le&\alpha_k^0\|s_k-x^*\|^2+\sum_{i=1}^p\alpha_k^i\|U_i(s_k)-x^*\|^2-\sum_{i=1}^p\alpha_k^0\alpha_k^i\|U_i(s_k)-s_k\|^2\\ \le&\alpha_k^0\|s_k-x^*\|^2+\sum_{i=1}^p\alpha_k^i(\|s_k-x^*\|^2+\beta\|U_i(s_k)-s_k\|^2)-\sum_{i=1}^p\alpha_k^0\alpha_k^i\|U_i(s_k)-s_k\|^2\\ =&\|s_k-x^*\|^2-(\alpha_k^0-\beta)(\sum_{i=1}^p\alpha_k^i\|U_i(s_k)-s_k\|^2).\end{aligned}\tag{3.10}$$

Similarly, since $T_j$ is $\mu_j$-demicontractive $(1\le j\le r)$, we can get

$$\|y_{k+1}-y^*\|^2\le\|t_k-y^*\|^2-(\sigma_k^0-\mu)(\sum_{j=1}^r\sigma_k^j\|T_j(t_k)-t_k\|^2).\tag{3.11}$$

Adding up (3.10) and (3.11), and setting $l_k(x^*,y^*) = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$\begin{aligned} &l_{k+1}(x^*,y^*) \\ \le &l_k(x^*,y^*) - (\alpha_k^0 - \beta)(\sum_{i=1}^{p} \alpha_k^i \|U_i(s_k) - s_k\|^2) - (\sigma_k^0 - \mu)(\sum_{j=1}^{r} \sigma_k^j \|T_j(t_k) - t_k\|^2) \\ &- \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]. \end{aligned} \tag{3.12}$$

For the case $Ax_k - By_k = 0$, we have

$$l_{k+1}(x^*,y^*) \le l_k(x^*,y^*) - (\alpha_k^0 - \beta)(\sum_{i=1}^{p} \alpha_k^i \|U_i(s_k) - s_k\|^2) - (\sigma_k^0 - \mu)(\sum_{j=1}^{r} \sigma_k^j \|T_j(t_k) - t_k\|^2). \tag{3.13}$$

Otherwise, we deduce from (3.2) and (3.12) that

$$\begin{aligned} l_{k+1}(x^*,y^*) \le l_k(x^*,y^*) &- (\alpha_k^0 - \beta)(\sum_{i=1}^{p} \alpha_k^i \|U_i(s_k) - s_k\|^2) \\ &- (\sigma_k^0 - \mu)(\sum_{j=1}^{r} \sigma_k^j \|T_j(t_k) - t_k\|^2) \\ &- \rho_k(2-\rho_k)\frac{\|Ax_k - By_k\|^4}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}. \end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we see that sequence $\{l_k(x^*,y^*)\}$ is non-increasing and lower bounded by 0. So, $\{l_k(x^*,y^*)\}$ converges to some finite limit, say, $l(x^*,y^*)$. Hence, $\{x_k\}$ and $\{y_k\}$ are bounded.

Next, we show that

$$\lim_{k\to\infty} \|Ax_k - By_k\| = \lim_{k\to\infty} \|s_k - U_i(s_k)\| = \lim_{k\to\infty} \|t_k - T_j(t_k)\| = 0$$

for each $1 \le i \le p$ and $1 \le j \le r$. From (3.14), for $Ax_k - By_k \ne 0$, we have

$$\rho_k(2-\rho_k)\frac{\|Ax_k - By_k\|^4}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} \le l_k(x^*,y^*) - l_{k+1}(x^*,y^*),$$

which implies that

$$\lim_{k\to\infty} \frac{\|Ax_k - By_k\|^4}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} = 0 \tag{3.15}$$

with the assumption on $\{\rho_k\}$. Similarly, by the conditions on $\{\alpha_k^i\}$ $(0 \le i \le p)$ and $\{\sigma_k^j\}$ $(0 \le j \le r)$, we obtain that, for each $1 \le i \le p$, $1 \le j \le r$,

$$\lim_{k\to\infty} \|s_k - U_i(s_k)\| = 0 \tag{3.16}$$

and

$$\lim_{k\to\infty} \|t_k - T_j(t_k)\| = 0. \tag{3.17}$$

If $Ax_k - By_k = 0$, it is clear that

$$s_k - x_k = \gamma_k A^*(Ax_k - By_k) = 0. \tag{3.18}$$

Otherwise, it follows from (3.15) that

$$\begin{aligned}\frac{1}{\|A\|^2+\|B\|^2}\|Ax_k-By_k\|^2 &= \|Ax_k-By_k\|^2\frac{\|Ax_k-By_k\|^2}{(\|A\|^2+\|B\|^2)\|Ax_k-By_k\|^2}\\ &\leq \|Ax_k-By_k\|^2\frac{\|Ax_k-By_k\|^2}{\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2}\\ &= \frac{\|Ax_k-By_k\|^4}{\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2}\to 0,\end{aligned} \tag{3.19}$$

which implies that

$$\lim_{k\to\infty}\|Ax_k-By_k\|=0.$$

Moreover, from (3.15), we also have

$$\begin{aligned}&\frac{\|Ax_k-By_k\|^2}{\max\{\|A^*(Ax_k-By_k)\|,\|B^*(Ax_k-By_k)\|\}}\\ &=\sqrt{\frac{\|Ax_k-By_k\|^4}{\max\{\|A^*(Ax_k-By_k)\|^2,\|B^*(Ax_k-By_k)\|^2\}}}\\ &\leq\sqrt{\frac{2\|Ax_k-By_k\|^4}{\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2}}\to 0.\end{aligned} \tag{3.20}$$

So, we have

$$\begin{aligned}\|s_k-x_k\| &= \|\gamma_k A^*(Ax_k-By_k)\|\\ &=\frac{\rho_k\|Ax_k-By_k\|^2}{\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2}\|A^*(Ax_k-By_k)\|\\ &\leq\frac{\rho_k\|Ax_k-By_k\|^2\max\{\|A^*(Ax_k-By_k)\|,\|B^*(Ax_k-By_k)\|\}}{\max\{\|A^*(Ax_k-By_k)\|^2,\|B^*(Ax_k-By_k)\|^2\}}\\ &=\frac{\rho_k\|Ax_k-By_k\|^2}{\max\{\|A^*(Ax_k-By_k)\|,\|B^*(Ax_k-By_k)\|\}}\to 0.\end{aligned} \tag{3.21}$$

It follows from (3.16) that $\lim_{k\to\infty}\|x_k-U_i(s_k)\|=0$ for all $1\leq i\leq p$. So, from

$$\|x_{k+1}-x_k\|\leq\alpha_k^0\|s_k-x_k\|+\alpha_k^1\|U_1(s_k)-x_k\|+\cdots+\alpha_k^p\|U_p(s_k)-x_k\|,$$

we have

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=0,$$

which infers that $\{x_k\}$ is asymptotically regular. Similarly, we have $\lim_{k\to\infty}\|t_k-y_k\|=0$ as $k\to\infty$ and $\{y_k\}$ is asymptotically regular.

Now, we prove that $\omega_w(x_k,y_k)\subseteq\Gamma$. Taking $(\tilde{x},\tilde{y})\in\omega_\omega(x_k,y_k)$, from $\lim_{k\to\infty}\|s_k-x_k\|=0$ and $\lim_{k\to\infty}\|t_k-y_k\|=0$, we have $(\tilde{x},\tilde{y})\in\omega_\omega(s_k,t_k)$. For $1\leq i\leq p$ and $1\leq j\leq r$, combined with the demiclosednesses of $U_i-I$ and $T_j-I$ at 0, we find from (3.16) and (3.17) that $U_i(\tilde{x})=\tilde{x}$ and $T_j(\tilde{y})=\tilde{y}$. So $\tilde{x}\in\bigcap_{i=1}^p F(U_i)$ and $\tilde{y}\in\bigcap_{j=1}^r F(T_j)$. On the other hand, $A\tilde{x}-B\tilde{y}\in\omega_w(Ax_k-By_k)$ and weakly lower semicontinuity of norms imply that

$$\|A\tilde{x}-B\tilde{y}\|\leq\liminf_{k\to\infty}\|Ax_k-By_k\|=0.$$

Hence $(\tilde{x},\tilde{y})\in\Gamma$. So $\omega_w(x_k,y_k)\subseteq\Gamma$.

Finally, we show the uniquess of the weak cluster point $\{(x_k, y_k)\}$. Indeed, let $(\tilde{x}, \tilde{y})$ be other weak cluster point of $\{(x_k, y_k)\}$. Then $(\tilde{x}, \tilde{y}) \in \Gamma$. From the definition of $l_k(x, y)$, we have

$$\begin{aligned} & l_k(\tilde{x},\tilde{y}) \\ =& \|x_k-\bar{x}\|^2+\|\bar{x}-\tilde{x}\|^2+2\langle x_k-\bar{x},\bar{x}-\tilde{x}\rangle+\|y_k-\bar{y}\|^2+\|\bar{y}-\tilde{y}\|^2+2\langle y_k-\bar{y},\bar{y}-\tilde{y}\rangle \\ =& l_k(\bar{x},\bar{y})+\|\bar{x}-\tilde{x}\|^2+2\langle x_k-\bar{x},\bar{x}-\tilde{x}\rangle+\|\bar{y}-\tilde{y}\|^2+2\langle y_k-\bar{y},\bar{y}-\tilde{y}\rangle. \end{aligned} \tag{3.22}$$

Without loss of generality, we may assume that $x_k \rightharpoonup \bar{x}$ and $y_k \rightharpoonup \bar{y}$. By passing to the limit in relation (3.22), we obtain that

$$l(\tilde{x},\tilde{y}) = l(\bar{x},\bar{y}) + \|\bar{x}-\tilde{x}\|^2 + \|\bar{y}-\tilde{y}\|^2.$$

Reversing the role of $(\tilde{x}, \tilde{y})$ and $(\bar{x}, \bar{y})$, we also have

$$l(\bar{x},\bar{y}) = l(\tilde{x},\tilde{y}) + \|\tilde{x}-\bar{x}\|^2 + \|\tilde{y}-\bar{y}\|^2.$$

By adding the two last equalities, we obtain $\tilde{x} = \bar{x}$ and $\tilde{y} = \bar{y}$, which implies that the whole $\{(x_k, y_k)\}$ weakly converges to the solution of (1.8). This completes the proof. □

Next, we propose a self-adaptive cyclic iterative algorithm for solving MSECFP (1.8) of demicontractive operators.

**Algorithm 3.4.** Assume the sequences $\{\alpha_k\} \subset [0,1]$ and $\{\sigma_k\} \subset [0,1]$. Take $x_0 \in H_1$, $y_0 \in H_2$, and calculate

$$\begin{cases} s_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = (1-\alpha_k)s_k + \alpha_k U_{i(k)}(s_k), \\ t_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = (1-\sigma_k)t_k + \sigma_k T_{j(k)}(t_k), \end{cases} \tag{3.23}$$

where $i(k) = k(\mod p) + 1$, $j(k) = k(\mod r) + 1$ and the stepsize $\{\gamma_k\}$ is chosen as (3.2).

**Theorem 3.5.** *Let $\{U_i,\ 1 \le i \le p\} : H_1 \to H_1$ and $\{T_j,\ 1 \le j \le r\} : H_2 \to H_2$ be $\beta_i$-demicontractive and $\mu_j$-demicontractive operators, respectively. Assume that $U_i - I (1 \le i \le p)$, $T_j - I (1 \le j \le r)$ are demiclosed at origin and $\Gamma$ is nonempty. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.4 weakly converges to a solution $(x^*, y^*)$ of (1.8), provided that $0 < \liminf_{k\to\infty} \alpha_k \le \limsup_{k\to\infty} \alpha_k < 1-\beta$ and $0 < \liminf_{k\to\infty} \sigma_k \le \limsup_{k\to\infty} \sigma_k < 1-\mu$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

*Proof.* Take $(x^*, y^*) \in \Gamma$. Similar to the proof of Theorem 3.3, we obtain that (3.8) and (3.9) hold. Since, for each $1 \le i \le p$ and $1 \le j \le r$, $U_i$ and $T_j$ are demicontractive, from Lemma 2.6, we get

$$\|x_{k+1} - x^*\|^2 \le \|s_k - x^*\|^2 - \alpha_k(1-\beta-\alpha_k)\|s_k - U_{i(k)}(s_k)\|^2 \tag{3.24}$$

and

$$\|y_{k+1} - y^*\|^2 \le \|t_k - y^*\|^2 - \sigma_k(1-\mu-\sigma_k)\|t_k - T_{j(k)}(t_k)\|^2. \tag{3.25}$$

Adding up (3.24) and (3.25), and setting $l_k(x^*, y^*) = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$\begin{aligned} & l_{k+1}(x^*,y^*) \\ \le& l_k(x^*,y^*) - \alpha_k(1-\beta-\alpha_k)\|s_k - U_{i(k)}(s_k)\|^2 - \sigma_k(1-\mu-\sigma_k)\|t_k - T_{j(k)}(t_k)\|^2 \\ & - \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)] \end{aligned} \tag{3.26}$$

by (3.8). We see that $\{l_k(x^*,y^*)\}$ is non-increasing and lower bounded by 0. So, $\{l_k(x^*,y^*)\}$ converges to some finite limit, say, $l(x^*,y^*)$. Similar to the proof of Theorem 3.3, we get

$$\lim_{k\to\infty}\|Ax_k - By_k\| = \lim_{k\to\infty}\|s_k - U_{i(k)}(s_k)\| = \lim_{k\to\infty}\|t_k - T_{j(k)}(t_k)\| = 0. \tag{3.27}$$

and

$$\lim_{k\to\infty}\|s_k - x_k\| = \lim_{k\to\infty}\|t_k - y_k\| = 0. \tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\lim_{k\to\infty}\|x_k - U_{i(k)}(s_k)\| = \lim_{k\to\infty}\|y_k - T_{j(k)}(t_k)\| = 0. \tag{3.29}$$

From

$$\|x_{k+1} - x_k\| \le (1-\alpha_k)\|s_k - x_k\| + \alpha_k\|U_{i(k)}(s_k) - x_k\|$$

and

$$\|y_{k+1} - y_k\| \le (1-\sigma_k)\|t_k - y_k\| + \sigma_k\|T_{j(k)}(t_k) - y_k\|,$$

we have

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = \lim_{k\to\infty}\|y_{k+1} - y_k\| = 0,$$

which infers that $\{x_k\}$ and $\{y_k\}$ are asymptotically regular. It follows from

$$\|s_{k+1} - s_k\| \le \|s_{k+1} - x_{k+1}\| + \|x_{k+1} - x_k\| + \|x_k - s_k\|$$

and

$$\|t_{k+1} - t_k\| \le \|t_{k+1} - y_{k+1}\| + \|y_{k+1} - y_k\| + \|y_k - t_k\|$$

that

$$\lim_{k\to\infty}\|s_{k+1} - s_k\| = \lim_{k\to\infty}\|t_{k+1} - t_k\| = 0.$$

So

$$\lim_{k\to\infty}\|s_{k+i} - s_k\| = \lim_{k\to\infty}\|t_{k+j} - t_k\| = 0, \;\; 1 \le i \le p, \; 1 \le j \le r. \tag{3.30}$$

Taking $(\tilde{x},\tilde{y}) \in \omega_\omega(x_k,y_k)$, we have $(\tilde{x},\tilde{y}) \in \omega_\omega(s_k,t_k)$. Let an index $i \in \{1,2,\cdots,p\}$ be fixed. Noting that the pool of indexes is finite, from (3.30), we can find a subsequence $\{s_{k_m}\} \subset \{s_k\}$ such that $s_{k_m} \rightharpoonup \tilde{x}$ as $m \to \infty$ and $i(k_m) = i$ for all $m \in \mathbb{N}$. It turns out from (3.27) that

$$\lim_{m\to\infty}\|s_{k_m} - U_i(s_{k_m})\| = \lim_{m\to\infty}\|s_{k_m} - U_{i(k_m)}(s_{k_m})\| = 0. \tag{3.31}$$

Combined with the demiclosednesses of $U_i - I$ at 0, it follows from (3.31) that $U_i(\tilde{x}) = \tilde{x}$. So $\tilde{x} \in F(U_i)$. Hence $\tilde{x} \in \bigcap_{i=1}^{p} F(U_i)$. By the same reason, we get $\tilde{y} \in \bigcap_{j=1}^{r} F(T_j)$. Similar to the proof of Theorem 3.3, we have $\omega_w(x_k,y_k) \subseteq \Gamma$ and the sequence $\{(x_k,y_k)\}$ weakly converges to the solution of (1.8). This completes the proof. □

**Remark 3.6.** Algorithm 3.1 and Algorithm 3.4 not only extend the iteration methods of Moudafi [28] from the stepsizes relying on operator norms to self-adaptive stepsizes, but also generalize the results of Zhao and He [35] from quasi-nonexpansive operators to demicontractive operators for solving the MSECFP.

**Remark 3.7.** For the particular case “p=r=1”, Algorithm 3.1 and Algorithm 3.4 solve SECFP (1.5) governed by demicontractive operators without prior knowledge of operator norms.

## 4. SELF-ADAPTIVE MIXED ITERATIVE ALGORITHMS FOR THE MSECFP

In this section, we introduce two mixed parallel and cyclic iterative algorithms for solving MSECFP (1.8) of demicontractive operators where the stepsizes do not depend on the operator norms $\|A\|$ and $\|B\|$ and prove the weak convergence of the algorithms.

**Algorithm 4.1.** Let $\{\alpha_k^i\}_{k=0}^{\infty} \subset [0,1]$ $(0 \le i \le p)$ be sequences such that $\Sigma_{i=0}^p \alpha_k^i = 1$ for every $k \ge 0$ and the sequence $\{\sigma_k\} \subset [0,1]$. Take $x_0 \in H_1, y_0 \in H_2$, and calculate

$$\begin{cases} s_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k^0 s_k + \alpha_k^1 U_1(s_k) + \cdots + \alpha_k^p U_p(s_k), \\ t_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = (1-\sigma_k)t_k + \sigma_k T_{j(k)}(t_k), \end{cases} \tag{4.1}$$

where $j(k) = k(\mod r) + 1$ and the stepsize $\{\gamma_k\}$ is chosen as (3.2).

**Theorem 4.2.** *Let $\{U_i,\ \ 1 \le i \le p\} : H_1 \to H_1$ and $\{T_j,\ \ 1 \le j \le r\} : H_2 \to H_2$ be $\beta_i$-demicontractive and $\mu_j$-demicontractive operators, respectively. Assume that $U_i - I(1 \le i \le p)$, $T_j - I(1 \le j \le r)$ are demiclosed at origin and $\Gamma$ is nonempty. Let $\beta = \max_{1\le i\le p}\{\beta_i\}$ and $\mu = \max_{1\le j\le r}\{\mu_j\}$. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 4.1 weakly converges to a solution $(x^*, y^*)$ of (1.8), provided that $\liminf_{k\to\infty} \alpha_k^0 > \beta$, $\liminf_{k\to\infty} \alpha_k^i > 0$ for all $1 \le i \le p$ and $0 < \liminf_{k\to\infty} \sigma_k \le \limsup_{k\to\infty} \sigma_k < 1-\mu$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

*Proof.* Taking $(x^*, y^*) \in \Gamma$, similar to the proof of Theorem 3.3, we have that (3.8) and (3.9) hold. From the proof of Theorem 3.3, we have (3.10) holds, i.e.,

$$\|x_{k+1} - x^*\|^2 \le \|s_k - x^*\|^2 - (\alpha_k^0 - \beta)(\sum_{i=1}^p \alpha_k^i \|U_i(s_k) - s_k\|^2).$$

From the proof of Theorem 3.5, we have (3.25) holds, i.e.,

$$\|y_{k+1} - y^*\|^2 \le \|t_k - y^*\|^2 - \sigma_k(1-\mu-\sigma_k)\|t_k - T_{j(k)}(t_k)\|^2.$$

Adding up the last two inequalities, and setting $l_k(x^*, y^*) = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$\begin{aligned} & l_{k+1}(x^*, y^*) \\ \le & l_k(x^*, y^*) - (\alpha_k^0 - \beta)(\sum_{i=1}^p \alpha_k^i \|U_i(s_k) - s_k\|^2) - \sigma_k(1-\mu-\sigma_k)\|t_k - T_{j(k)}(t_k)\|^2 \\ & - \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)] \end{aligned} \tag{4.2}$$

by (3.8). We see that $\{l_k(x^*, y^*)\}$ is non-increasing and lower bounded by 0. $\{l_k(x^*, y^*)\}$ converges to some finite limit, say $l(x^*, y^*)$. Similar to the proof of Theorem 3.3, we have

$$\lim_{k\to\infty} \|Ax_k - By_k\| = \lim_{k\to\infty} \|t_k - T_{j(k)}(t_k)\| = 0 \tag{4.3}$$

and

$$\lim_{k\to\infty} \|s_k - U_i(s_k)\| = 0 \tag{4.4}$$

for each $1 \le i \le p$. Moreover, we have

$$\lim_{k\to\infty} \|s_k - x_k\| = \lim_{k\to\infty} \|t_k - y_k\| = 0. \tag{4.5}$$

It follows from (4.3), (4.4) and (4.5) that

$$\lim_{k\to\infty}\|x_k-U_i(s_k)\|=\lim_{k\to\infty}\|y_k-T_{j(k)}(t_k)\|=0. \tag{4.6}$$

From

$$\|x_{k+1}-x_k\|\le \alpha_k^0\|s_k-x_k\|+\alpha_k^1\|U_1(s_k)-x_k\|+\cdots+\alpha_k^p\|U_p(s_k)-x_k\|$$

and

$$\|y_{k+1}-y_k\|\le(1-\sigma_k)\|t_k-y_k\|+\sigma_k\|T_{j(k)}(t_k)-y_k\|,$$

we have

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}\|y_{k+1}-y_k\|=0,$$

which infers that $\{x_k\}$ and $\{y_k\}$ are asymptotically regular and

$$\lim_{k\to\infty}\|s_{k+1}-s_k\|=\lim_{k\to\infty}\|t_{k+1}-t_k\|=0.$$

Taking $(\tilde{x},\tilde{y})\in\omega_\omega(x_k,y_k)$, we have $(\tilde{x},\tilde{y})\in\omega_\omega(s_k,t_k)$. Combing the proof of Theorem 3.3 and Theorem 3.5, we have $\omega_w(x_k,y_k)\subseteq\Gamma$ and the sequence $\{(x_k,y_k)\}$ weakly converges to the solution of (1.8). This completes the proof. □

Next, we propose another mixed self-adaptive parallel and cyclic iterative algorithm for solving MSECFP (1.8) of demicontractive operators.

**Algorithm 4.3.** Let $\{\alpha_k\}\subset[0,1]$ $\{\sigma_k^j\}_{k=0}^\infty\subset[0,1]$ $(0\le j\le r)$ be sequences such that $\Sigma_{j=0}^r\sigma_k^j=1$ for every $k\ge 0$. Take $x_0\in H_1, y_0\in H_2$, and calculate

$$\begin{cases} s_k=x_k-\gamma_kA^*(Ax_k-By_k),\\ x_{k+1}=(1-\alpha_k)s_k+\alpha_kU_{i(k)}(s_k),\\ t_k=y_k+\gamma_kB^*(Ax_k-By_k),\\ y_{k+1}=\sigma_k^0t_k+\sigma_k^1T_1(t_k)+\cdots+\sigma_k^rT_r(t_k),\end{cases} \tag{4.7}$$

where $i(k)=k(\mod p)+1$ and the stepsize $\gamma_k$ is chosen as (3.2).

Using a similar argument in the proof of Theorem 4.2, we conclude that the following result.

**Theorem 4.4.** *Let $\{U_i,\ \ 1\le i\le p\}:H_1\to H_1$ and $\{T_j,\ \ 1\le j\le r\}:H_2\to H_2$ be $\beta_i$-demicontractive and $\mu_j$-demicontractive operators, respectively. Assume that $U_i-I(1\le i\le p)$, $T_j-I(1\le j\le r)$ are demiclosed at origin and $\Gamma$ is nonempty. Let $\beta=\max_{1\le i\le p}\{\beta_i\}$ and $\mu=\max_{1\le j\le r}\{\mu_j\}$. Then the sequence $\{(x_k,y_k)\}$ generated by Algorithm 4.3 weakly converges to a solution $(x^*,y^*)$ of (1.8), provided that $0<\liminf_{k\to\infty}\alpha_k\le\limsup_{k\to\infty}\alpha_k<1-\beta$, $\liminf_{k\to\infty}\sigma_k^0>\mu$ and $\liminf_{k\to\infty}\sigma_k^i>0$ for all $1\le j\le r$. Moreover, $\|Ax_k-By_k\|\to 0$, $\|x_k-x_{k+1}\|\to 0$ and $\|y_k-y_{k+1}\|\to 0$ as $k\to\infty$.*

**Remark 4.5.** For the particular case “p=r=1”, Algorithm 4.1 and Algorithm 4.3 become mixed iterative algorithms to solve SECFP (1.5) governed by demicontractive operators without prior knowledge of operator norms.

From Lemma 2.8, we have the following results on MSECFP (1.8) of strictly pseudocontractive operators.

**Corollary 4.6.** *Let $\{U_i, \ \ 1 \le i \le p\} : H_1 \to H_1$ and $\{T_j, \ \ 1 \le j \le r\} : H_2 \to H_2$ be $\beta_i$-strictly pseucontractive and $\mu_j$-strictly pseucontractive operators, respectively. Assume that $\Gamma$ is nonempty. Let $\beta = \max_{1\le i\le p}\{\beta_i\}$ and $\mu = \max_{1\le j\le r}\{\mu_j\}$. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 4.1 weakly converges to a solution $(x^*, y^*)$ of (1.8), provided that $\liminf_{k\to\infty} \alpha_k^0 > \beta$, $\liminf_{k\to\infty} \alpha_k^i > 0$ for all $1 \le i \le p$ and $0 < \liminf_{k\to\infty} \sigma_k \le \limsup_{k\to\infty} \sigma_k < 1-\mu$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

**Corollary 4.7.** *Let $\{U_i, \ \ 1 \le i \le p\} : H_1 \to H_1$ and $\{T_j, \ \ 1 \le j \le r\} : H_2 \to H_2$ be $\beta_i$-strictly pseucontractive and $\mu_j$-strictly pseucontractive operators, respectively. Assume that $\Gamma$ is nonempty. Let $\beta = \max_{1\le i\le p}\{\beta_i\}$ and $\mu = \max_{1\le j\le r}\{\mu_j\}$. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 4.3 weakly converges to a solution $(x^*, y^*)$ of (1.8), provided that $0 < \liminf_{k\to\infty} \alpha_k \le \limsup_{k\to\infty} \alpha_k < 1-\beta$, $\liminf_{k\to\infty} \sigma_k^0 > \mu$ and $\liminf_{k\to\infty} \sigma_k^i > 0$ for all $1 \le j \le r$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

## 5. Numerical experiments

In this section, we provide some numerical experiments and show the performance of the proposed self-adaptive iterative algorithms with stepsize (3.2) for solving the MSECFP (1.8). All the codes are written in MATLAB and are performed on a personal Lenovo computer with Pentium(R) Dual-Core CPU @ 2.4GHz and RAM 2.00GB.

In this part, we take the following experiment parameters $\rho_k = 1.95$ in all iterative algorithms, and the stopping criteria is $f(x,y) < 10^{-5}$. Here function $f(x,y)$ measures the distance given as follows

$$f(x,y) = \Sigma_{i=1}^{p}\|x - U_i x\| + \Sigma_{j=1}^{r}\|y - T_j y\|.$$

Let $R^n$ be $n$ dimensional Euclidean space with inner product

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$$

and norm $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for all $x = (x_1, \cdots, x_n)^T$, $y = (y_1, \cdots, y_n)^T \in R^n$.

**Example 5.1.** Define operators $U_i : R^2 \to R^2$ and $T_i : R^2 \to R^2$ $(i = 1,2)$ as follows:

$$U_1(x) = (z_1, z_2)^T, \ \ U_2(x) = (x_1, x_2)^T, \ \ T_1(x) = (\frac{x_1}{3}, \frac{x_2}{3})^T, \ \ T_2(x) = (\frac{x_1}{2}, \frac{x_2}{2})^T,$$

where $x = (x_1, x_2)^T$ and

$$z_i = \begin{cases} x_i, & x_i < 0, \\ -2x_i, & x_i \ge 0, \end{cases} \quad (i = 1,2).$$

Note that $U_1$ is $\frac{1}{3}$-demicontractive and $U_2$, $T_1$, $T_2$ are nonexpansive operators. Obviously,

$$F(U_1) \cap F(U_2) = \{(x_1, x_2)^T : x_1 \le 0, x_2 \le 0\}, \ \ F(T_1) \cap F(T_2) = \{(0,0)^T\}.$$

Let

$$A = \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix}, \ \ B = \begin{pmatrix} 3 & 8 \\ 1 & 1 \end{pmatrix}.$$

We consider the following MSECFP:

$$\text{finding } \ x^* \in F(U_1) \cap F(U_2), \ y^* \in F(T_1) \cap F(T_2), \ \text{ such that } \ Ax^* = By^*.$$

TABLE 1. Numerical results for solving Example 5.1 with different iterative algorithms.

$x_0 = (-1,5)^T$, $y_0 = (3,1)^T$

| | k | $x_k$ | $y_k$ |
|---|---|---|---|
| Algorithm 3.1 | 108 | $(-0.7707,-0.1919)^T \times 10^{-5}$ | $(-0.1973,0.8577)^T \times 10^{-6}$ |
| Algorithm 3.4 | 81 | $(0.1390,-0.3408)^T \times 10^{-5}$ | $(-0.1571,-0.3707)^T \times 10^{-5}$ |
| Algorithm 4.1 | 125 | $(-0.5490,0)^T \times 10^{-5}$ | $(-0.1556,-0.3228)^T \times 10^{-5}$ |
| Algorithm 4.3 | 69 | $(0.0262,-0.4602)^T \times 10^{-5}$ | $(-0.1983,-0.4614)^T \times 10^{-5}$ |

We apply the proposed self-adaptive simultaneous iterative Algorithm 3.1, cyclic iterative Algorithm 3.4, and two mixed iterative Algorithm 4.1 and 4.3 to solve Example 5.1. We take initial points $x_0 = (-1,5)^T$, $y_0 = (3,1)^T$. In self-adaptive simultaneous iterative Algorithm 3.1, we take $\alpha_k^0 = \alpha_k^1 = \alpha_k^2 = \sigma_k^0 = \sigma_k^1 = \sigma_k^2 = \frac{1}{3}$. In self-adaptive cyclic iterative Algorithm 3.4, we take $\alpha_k = \sigma_k = \frac{1}{3}$. In self-adaptive mixed iterative Algorithm 4.1 and Algorithm 4.3, we take $\alpha_k^0 = \alpha_k^1 = \alpha_k^2 = \sigma_k = \frac{1}{3}$ and $\alpha_k = \sigma_k^0 = \sigma_k^1 = \sigma_k^2 = \frac{1}{3}$, respectively. The numerical results are given in Table 1. Denote $x_k$ and $y_k$ as the kth iterative sequences.

**Example 5.2.** Define the operators $U_i : R^3 \to R^3$ and $T_i : R^2 \to R^2$ $(i = 1,2)$ as follows:

$$U_1(x) = (z_1,z_2,z_3)^T, \ \ U_2(x) = (x_1,x_2,x_3)^T, \ \ T_1(y) = (w_1,w_2)^T, \ \ T_2(y) = (\frac{y_1}{2}, \frac{y_2}{2})^T,$$

where $x = (x_1,x_2,x_3)^T$, $y = (y_1,y_2)^T$ and

$$z_i = \begin{cases} x_i, & x_i < 0, \\ -2x_i, & x_i \geq 0, \end{cases} \quad (i = 1,2,3), \quad w_j = \begin{cases} y_j, & y_j < 0, \\ -3y_j, & y_j \geq 0, \end{cases} \quad (j = 1,2).$$

Note that $U_1$ is $\frac{1}{3}$-demicontractive, $T_1$ is $\frac{1}{2}$-demicontractive and $U_2$, $T_2$ are nonexpansive operators. Obviously,

$$F(U_1) \cap F(U_2) = \{(x_1,x_2)^T : x_1 \leq 0, x_2 \leq 0\}, \ \ F(T_1) \cap F(T_2) = \{(0,0)^T\}.$$

Let

$$A = \begin{pmatrix} 1 & -6 & -1 \\ 2 & 2 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 3 & -8 \\ -1 & 1 \end{pmatrix}.$$

We consider the following MSECFP:

$$\text{finding } \ x^* \in F(U_1) \cap F(U_2), \ y^* \in F(T_1) \cap F(T_2), \ \text{ such that } \ Ax^* = By^*.$$

We apply the proposed self-adaptive simultaneous iterative Algorithm 3.1, cyclic iterative Algorithm 3.4, and two mixed iterative Algorithm 4.1 and 4.3 to solve Example 5.2. We take different initial points $x_0 = (5,-6,-1)^T$, $y_0 = (3,0)^T$ and $x_0 = (3,4,-1)^T$, $y_0 = (-3,0)^T$. In self-adaptive simultaneous iterative Algorithm 3.1, we take $\alpha_k^0 = \alpha_k^1 = \alpha_k^2 = \frac{1}{3}$, $\sigma_k^0 = \frac{1}{2}$ and $\sigma_k^1 = \sigma_k^2 = \frac{1}{4}$. In self-adaptive cyclic iterative Algorithm 3.4, we take $\alpha_k = \frac{1}{3}$ and $\sigma_k = \frac{1}{2}$. In self-adaptive mixed iterative Algorithm 4.1 and Algorithm 4.3, we take $\alpha_k^0 = \alpha_k^1 = \alpha_k^2 = \frac{1}{3}$, $\sigma_k = \frac{1}{2}$ and $\alpha_k = \frac{1}{3}$, $\sigma_k^0 = \frac{1}{2}$, $\sigma_k^1 = \sigma_k^2 = \frac{1}{4}$, respectively. The numerical results are given in Table 2-3. Denote $x_k$ and $y_k$ as the kth iterative sequences.

TABLE 2. Numerical results for solving Example 5.2 with different iterative algorithms.

| $x_0 = (5,-6,-1)^T$, $y_0 = (3,0)^T$ | | | |
|---|---|---|---|
| | k | $x_k$ | $y_k$ |
| Algorithm 3.1 | 173 | $(0,-0.2425,-0.0097)^T \times 10^{-4}$ | $(-0.4193,-0.8367)^T \times 10^{-5}$ |
| Algorithm 3.4 | 327 | $(0.0017,-0.2485,-0.0493)^T \times 10^{-4}$ | $(-0.8625,-0.1308)^T \times 10^{-5}$ |
| Algorithm 4.1 | 307 | $(0,-0.2960,-0.0381)^T \times 10^{-4}$ | $(-0.8566,-0.4243)^T \times 10^{-5}$ |
| Algorithm 4.3 | 147 | $(-0.0014,-0.2162,-0.0375)^T \times 10^{-4}$ | $(-0.6753,-0.2237)^T \times 10^{-5}$ |

TABLE 3. Numerical results for solving Example 5.2 with different iterative algorithms.

| $x_0 = (3,4,-1)^T$, $y_0 = (-3,0)^T$ | | | |
|---|---|---|---|
| | k | $x_k$ | $y_k$ |
| Algorithm 3.1 | 144 | $(0,-0.2402,-0.0096)^T \times 10^{-4}$ | $(-0.4153,-0.8287)^T \times 10^{-5}$ |
| Algorithm 3.4 | 111 | $(-0.2759,-0.2050,0.1505)^T \times 10^{-5}$ | $(0.1863,-0.4865)^T \times 10^{-5}$ |
| Algorithm 4.1 | 87 | $(0,0,0)^T$ | $(-0.5918,-0.5465)^T \times 10^{-5}$ |
| Algorithm 4.3 | 191 | $(-0.0012,-0.3678,0.1122)^T \times 10^{-5}$ | $(-0.0061,-0.6484)^T \times 10^{-5}$ |

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