J. Nonlinear Funct. Anal. 2018 (2018), Article ID 48 https://doi.org/10.23952/jnfa.2018.48

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

# NONEXISTENCE OF GLOBAL SOLUTIONS FOR COUPLED KIRCHHOFF-TYPE EQUATIONS WITH DEGENERATE DAMPING TERMS

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**Abstract.** In this paper, we investigate a system of coupled Kirchhoff-type equations with degenerate damping terms. We prove the nonexistence of global solutions with positive initial energy. We also give some estimates for lower bound of the blow up time.

**Keywords.** Blow up time; Degenerate damping term; Global solution; Kirchhoff-type equation; Nonexistence of solutions.

**2010 Mathematics Subject Classification.** 35B44, 35G61.

## 1. INTRODUCTION

In this paper, we study the following initial boundary value problem for the following coupled nonlinear Kirchhoff-type equations with degenerate damping and source terms

$$\begin{cases} |u_t|^j u_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + \left(|u|^k + |v|^l\right)|u_t|^{p-1} u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ |v_t|^j v_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + \left(|v|^\theta + |u|^\rho\right)|v_t|^{q-1} v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$ in $R^n$ $(n = 1,2,3)$, $p,q \geq 1$, $j,k,l,\theta,\rho \geq 0$, $f_i(.\,,.) : R^2 \longrightarrow R$ are given functions to be specified later, and $M(s)$ is a nonnegative locally Lipschitz function.

In the case of $j = 0$ and $M(s) \equiv 1$, Rammaha and Sakuntasathien [1] considered the following system

$$\begin{cases} u_{tt} - \triangle u + \left(|u|^k + |v|^l\right)|u_t|^{p-1} u_t = f_1(u,v), \\ v_{tt} - \triangle v + \left(|v|^\theta + |u|^\rho\right)|v_t|^{q-1} v_t = f_2(u,v). \end{cases} \tag{1.2}$$

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Received July 23, 2018; Accepted December 20, 2018.

They studied the global well posedness of solutions of problem (1.2). Recently, some authors [2, 3] considered the same problem treated in [1]. They studied the blow up and growth properties. Systems with degenerate damping terms have also been extensively investigated; see [4, 5, 6].

In 2016, Ye [7] considered the following system

$$\begin{cases} u_{tt}-M(\|\nabla u\|^2+\|\nabla v\|^2)\Delta u+|u_t|^{p-1}u_t=f_1(u,v), \\ v_{tt}-M(\|\nabla u\|^2+\|\nabla v\|^2)\Delta v+|v_t|^{q-1}v_t=f_2(u,v) \end{cases} \tag{1.3}$$

with initial boundary conditions. He proved the global existence and energy decay results. The model (1.3) is introduced in [8] for studying the nonlinear vibrations of an elastic string.

If both Kirchhoff-type terms ($M(s)$) and degenerate damping terms are present, then the analysis of their interaction is not easy. In this paper, motivated by the above studies, we prove the global nonexistence of solutions for (1.1). This paper is organized as follows: In Section 2, we present some lemmas and the local existence theorem. In Section 3, the blow up of solutions is given. In Section 4, some estimates for lower bound of the blow up time is given.

## 2. Preliminaries

In this section, we give some assumptions and lemmas which will be used throughout this paper. We use $\|.\|$ and $\|.\|_p$ to denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

To state and prove our main result, we need the following assumptions.

(**A1**) $M(s)$ is a nonnegative $C^1$ function for $s\geq 0$ satisfying

$$M(s)=1+s^{\gamma},\ \gamma>1.$$

(**A2**) For the nonlinearity, we suppose that

$$\begin{cases} p,q\geq 1 \ \text{ if } n=1,2, \\ 1\leq p,q\leq 5 \ \text{ if } n=3. \end{cases}$$

Concerning functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(r+1)}(u+v)+b|u|^r u|v|^{r+2}, \\ f_2(u,v) &= a|u+v|^{2(r+1)}(u+v)+b|v|^r v|u|^{r+2}, \end{aligned}$$

where $a,b>0$ are constants and $r$ satisfies

$$\begin{cases} -1<r \ \text{ if } n=1,2, \\ -1<r\leq 1 \text{ if } n=3. \end{cases} \tag{2.1}$$

One can easily verify that

$$uf_1(u,v)+vf_2(u,v)=2(r+2)F(u,v),\ \forall(u,v)\in R^2, \tag{2.2}$$

where

$$F(u,v)=\frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]. \tag{2.3}$$

We define the energy function as follows

$$\begin{aligned} E(t) &= \frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\frac{1}{2}(\|\nabla u\|^2+\|\nabla v\|^2) \\ &\quad +\frac{1}{2(\gamma+1)}(\|\nabla u\|^2+\|\nabla v\|^2)^{\gamma+1}-\int_\Omega F(u,v)\,dx. \end{aligned} \tag{2.4}$$

**Lemma 2.1.** [9]. *There exist two positive constants $c_0$ and $c_1$ such that*

$$c_0\left(|u|^{2(r+2)}+|v|^{2(r+2)}\right)\leq 2(r+2)F(u,v)\leq c_1\left(|u|^{2(r+2)}+|v|^{2(r+2)}\right). \tag{2.5}$$

**Lemma 2.2.** *(Sobolev-Poincare inequality)* [10]. *Let $q$ be a number with $2\leq q<\infty$ $(n=1,2)$ or $2\leq q\leq 2n/(n-2)$ $(n\geq 3)$. Then, there is a constant $C_*=C_*(\Omega,\ q)$ such that*

$$\|u\|_q\leq C_*\|\nabla u\| \text{ for } u\in H_0^1(\Omega).$$

**Lemma 2.3.** [11]. *Suppose that*

$$p\leq 2\frac{n-1}{n-2},\ n\geq 3$$

*holds. Then there exists a positive constant $C>1$ depending on $\Omega$ only such that*

$$\|u\|_p^s\leq C\left(\|\nabla u\|^2+\|u\|_p^p\right)$$

*for any $u\in H_0^1(\Omega)$, $2\leq s\leq p$.*

The next lemma shows that our energy functional (2.4) is a nonincreasing function along the solution of (1.1).

**Lemma 2.4.** *$E(t)$ is a nonincreasing function for $t\geq 0$ and*

$$\frac{d}{dt}E(t)=-\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{p+1}dx-\int_\Omega\left(|v|^\theta+|u|^\rho\right)|v_t|^{q+1}dx. \tag{2.6}$$

*Proof.* Multiplying the first equation of (1.1) by $u_t$, the second equation by $v_t$, and integrating them over $\Omega$, we obtain

$$\int_0^t E'(\tau)d\tau=-\int_0^t\int_\Omega(\left(|u|^k+|v|^l\right)|u_\tau|^{p+1}+\left(|v|^\theta+|u|^\rho\right)|v_\tau|^{q+1})dxd\tau$$

$$E(t)-E(0)=-\int_0^t\int_\Omega\left(\left(|u|^k+|v|^l\right)|u_\tau|^{p+1}+\left(|v|^\theta+|u|^\rho\right)|v_\tau|^{q+1}\right)dxd\tau \text{ for } t\geq 0. \tag{2.7}$$

This completes the proof. □

Next, we state the local existence theorem that can be established based on the arguments in [1, 12, 13, 14].

**Theorem 2.5.** *(Local existence). Suppose that (A1), (A2) and (2.1) hold. Let $u_0,v_0\in H_0^1(\Omega)\cap L^{r+1}(\Omega)$ and $u_1,v_1\in L^2(\Omega)$ are given. Then problem (1.1) has a unique solution satisfying*

$$u,v\in C\left([0,T);H_0^1(\Omega)\cap L^{r+1}(\Omega)\right),$$

$$u_t\in C\left([0,T);L^2(\Omega)\cap L^{p+1}(\Omega\times[0,T))\cap L^{j+2}(\Omega\times[0,T))\right),$$

$$v_t\in C\left([0,T);L^2(\Omega)\cap L^{q+1}(\Omega\times[0,T))\cap L^{j+2}(\Omega\times[0,T))\right)$$

*for some $T>0$.*

## 3. Blow up of solutions

In this section, we deal with the blow up results of the solution for the problem (1.1). Let us begin by stating the following two lemmas, which will be used later.

**Lemma 3.1.** [15]. *Suppose that (2.1) holds. Then there exists $\eta > 0$ such that, for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$,*

$$\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \leq \eta \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{r+2} \tag{3.1}$$

In order to prove our result, we take $a = b = 1$ for sake of simplicity. We introduce the following:

$$B = \eta^{\frac{1}{2(r+2)}},\ \alpha_1 = B^{-\frac{r+2}{r+1}},\ E_1 = \left(\frac{1}{2} - \frac{1}{2(r+2)}\right)\alpha_1^2 \tag{3.2}$$

where $\eta$ is the optimal constant in (3.1).

The following lemma will play an essential role in the proof of our main result, and it is similar to a lemma, which first used by Vitillaro [16].

**Lemma 3.2.** *Suppose that assumptions (A1), (A2) and (2.1) hold. Let $(u,v)$ be a solution of (1.1). Moreover, assume that $E(0) < E_1$ and*

$$\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\frac{1}{2}} > \alpha_1. \tag{3.3}$$

*Then there exists a constant $\alpha_2 > \alpha_1$ such that*

$$\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{\gamma+1}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1}\right)^{\frac{1}{2}} > \alpha_2,\ for\ t > 0, \tag{3.4}$$

$$\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right)^{\frac{1}{2(r+2)}} \geq B\alpha_2,\ for\ t > 0. \tag{3.5}$$

*for all $t \in [0,T)$.*

*Proof.* From (2.4), (3.1) and the definition of $B$, we have

$$\begin{aligned}
E(t) &\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} - \int_\Omega F(u,v)\,dx \\
&= \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\
&\quad - \frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}) \\
&\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\
&\quad - \frac{1}{2(r+2)}\eta(\|\nabla u\|^2 + \|\nabla v\|^2)^{r+2} \\
&\geq \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\
&\quad - \frac{B^{2(r+2)}}{2(r+2)}(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1})^{r+2} \\
&= \frac{1}{2}\alpha^2 - \frac{B^{2(r+2)}}{2(r+2)}\alpha^{2(r+2)} = G(\alpha),
\end{aligned} \tag{3.6}$$

where

$$\alpha = (\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1})^{1/2}.$$

It is not difficult to verify that $G$ is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$, $G(\alpha) \to -\infty$ as $\alpha \to \infty$ , and

$$G(\alpha_1) = \frac{1}{2}\alpha_1^2 - \frac{B^{2(r+2)}}{2(r+2)}\alpha_1^{2(r+2)} = E_1, \tag{3.7}$$

where $\alpha_1$ is given in (3.2). Since $E(0) < E_1$, we see that there exists $\alpha_2 > \alpha_1$ such that $G(\alpha_2) = E(0)$. Set $\alpha_0 = (\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{\gamma+1})^{1/2}$. Using (3.6), we get $G(\alpha_0) \leq E(0) = G(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$.

To show (3.4), we suppose by contradiction that

$$(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1})^{1/2} < \alpha_2,$$

for some $t_0 > 0$. By the continuity of

$$(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1})^{1/2},$$

we can obtain that,

$$(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1})^{1/2} > \alpha_1$$

It follows from (3.6) that

$$\begin{aligned} E(t_0) &\geq G(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2 + \frac{1}{(\gamma+1)}(\|\nabla u(t_0)\|^2 + \|\nabla v(t_0)\|^2)^{\gamma+1}) \\ &> G(\alpha_2) \\ &= E(0). \end{aligned}$$

This is impossible since $E(t) \leq E(0)$ for all $t \in [0,T)$. Hence (3.4) is established.

To prove (3.5), we make use of (2.4) to get

$$\begin{aligned} &\frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{1}{2(\gamma+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1} \\ \leq\ & E(0) + \frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}). \end{aligned}$$

Consequently, (3.4) yields

$$\begin{aligned} &\frac{1}{2(r+2)}(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}) \\ \geq\ & \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{\gamma+1}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1}) - E(0) \\ \geq\ & \frac{1}{2}\alpha_2^2 - E(0) \\ \geq\ & \frac{1}{2}\alpha_2^2 - G(\alpha_2) \\ =\ & \frac{B^{2(r+2)}}{2(r+2)}\alpha_2^{2(r+2)}. \end{aligned} \tag{3.8}$$

Therefore, (3.8) and (3.2) yield the desired result. This completes the proof of this lemma. □

Now, we are in a position to state and prove our main result.

**Theorem 3.3.** *Assume that (A1), (A2) and (2.1) hold. Assume further that*

$$2(r+2) > \max\{2\gamma+2,\ j+2,\ k+p+1,\ l+p+1,\ \theta+q+1,\ \rho+q+1\}.$$

*Then any solution of problem (1.1) with initial data satisfying*

$$\left(\|\nabla u_0\|^2+\|\nabla v_0\|^2\right)^{\frac{1}{2}} > \alpha_1,\ E(0) < E_1,$$

*cannot exist for all time, where $\alpha_1$ and $E_1$ are defined in (3.2).*

*Proof.* We suppose that the solution exists for all time and we reach to a contradiction.

For this purpose, we set

$$H(t) = E_1 - E(t). \tag{3.9}$$

By using (2.4) and (3.9), we have

$$\begin{aligned} 0 \quad < \quad & H(0) \le H(t) = E_1 - \frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right) - \frac{1}{2}(\|\nabla u\|^2+\|\nabla v\|^2) \\ & -\frac{1}{2(\gamma+1)}(\|\nabla u\|^2+\|\nabla v\|^2)^{\gamma+1} + \int_\Omega F(u,v)\,dx. \end{aligned} \tag{3.10}$$

From (3.8) and (2.5), we have

$$\begin{aligned} & E_1 - \frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right) - \frac{1}{2}(\|\nabla u\|^2+\|\nabla v\|^2) \\ & -\frac{1}{2(\gamma+1)}(\|\nabla u\|^2+\|\nabla v\|^2)^{\gamma+1} + \int_\Omega F(u,v)\,dx \\ \le \quad & E_1 - \frac{1}{2}\alpha_1^2 + \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right) \\ \le \quad & -\frac{1}{2(r+2)}\alpha_1^2 + \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right) \\ \le \quad & \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right). \end{aligned} \tag{3.11}$$

By combining (3.10) and (3.11), we have

$$0 < H(0) \le H(t) \le \frac{c_1}{2(r+2)}\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right). \tag{3.12}$$

We then define

$$\Psi(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{j+1}\left(\int_\Omega |u_t|^j u_t u\,dx + \int_\Omega |v_t|^j v_t v\,dx\right), \tag{3.13}$$

where $\varepsilon$ small to be chosen later and

$$\begin{aligned} 0 \quad < \quad & \sigma \le \min\left\{\frac{1}{j+2}, \frac{r+1}{2(r+2)}, \frac{2r+3-(k+p)}{2p(r+2)}, \frac{2r+3-(l+p)}{2p(r+2)},\right. \\ & \left.\frac{2r+3-(\rho+q)}{2q(r+2)}, \frac{2r+3-(\theta+q)}{2q(r+2)}\right\}. \end{aligned} \tag{3.14}$$

Our goal is to show that $\Psi(t)$ satisfies a differential inequality of the form

$$\Psi'(t) \ge \xi \Psi^{\zeta}(t),\ \zeta > 1.$$

This, of course, will lead to a blow up in finite time.

By taking a derivative of (3.13) and using Eq. (1.1), we obtain

$$\begin{aligned}
\Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\frac{\varepsilon}{j+1}\left(\int_\Omega |u_t|^{j+2}dx+\int_\Omega |v_t|^{j+2}dx\right)\\
&\quad +\varepsilon\left(\int_\Omega |u_t|^j u_{tt}u dx+\int_\Omega |v_t|^j v_{tt}v dx\right)\\
&= (1-\sigma)H^{-\sigma}(t)H'(t)+\frac{\varepsilon}{j+1}(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2})\\
&\quad -\varepsilon(\|\nabla u\|^2+\|\nabla v\|^2)-\varepsilon(\|\nabla u\|^2+\|\nabla v\|^2)^{\gamma+1}+2\varepsilon(r+2)\int_\Omega F(u,v)dx\\
&\quad -\varepsilon\left(\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{p-1}dx+\int_\Omega v\left(|v|^\theta+|u|^\rho\right)v_t|v_t|^{q-1}dx\right). \qquad (3.15)
\end{aligned}$$

From the definition of $H(t)$, it follows that

$$\begin{aligned}
&-(\|\nabla u\|^2+\|\nabla v\|^2)^{\gamma+1}\\
&= 2(\gamma+1)H(t)-2(\gamma+1)E_1+\frac{2(\gamma+1)}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\
&\quad +(\gamma+1)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-2(\gamma+1)\int_\Omega F(u,v)dx. \qquad (3.16)
\end{aligned}$$

Inserting (3.16) into (3.15), we get

$$\begin{aligned}
\Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\
&\quad +\varepsilon\gamma\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+2\varepsilon(\gamma+1)H(t)-2\varepsilon(\gamma+1)E_1\\
&\quad +\varepsilon\left(1-\frac{\gamma+1}{r+2}\right)(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2})\\
&\quad -\varepsilon\left(\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{p-1}dx+\int_\Omega v\left(|v|^\theta+|u|^\rho\right)v_t|v_t|^{q-1}dx\right).
\end{aligned}$$

In view of (3.5), we arrive at

$$\begin{aligned}
\Psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\
&\quad +\varepsilon\gamma\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\
&\quad +2(\gamma+1)\varepsilon H(t)+\varepsilon c'(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2})\\
&\quad -\varepsilon\left(\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{p-1}dx+\int_\Omega v\left(|v|^\theta+|u|^\rho\right)v_t|v_t|^{q-1}dx\right), \qquad (3.17)
\end{aligned}$$

where

$$c' = 1-\frac{\gamma+1}{r+2}-2(\gamma+1)E_1(B\alpha_2)^{-2(r+2)}>0,$$

since $\alpha_2 > B^{-\frac{r+2}{r+1}}$. In order to estimate the last two terms in (3.17), we make use of the following Young inequality

$$XY \leq \frac{\delta^k X^k}{k}+\frac{\delta^{-l}Y^l}{l},$$

where $X,Y\geq 0,\ \delta>0,\ k,l\in R^{+}$ such that $\frac{1}{k}+\frac{1}{l}=1$. Consequently, applying the above inequality, we find

$$\int_{\Omega} uu_t\left|u_t\right|^{p-1}dx\leq\frac{\delta_1^{p+1}}{p+1}\left\|u\right\|_{p+1}^{p+1}+\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\left\|u_t\right\|_{p+1}^{p+1}.$$

Therefore,

$$\begin{aligned}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)uu_t\left|u_t\right|^{p-1}dx &\leq \frac{\delta_1^{p+1}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u|^{p+1}dx\\ &\quad+\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u_t|^{p+1}dx.\end{aligned}$$

In the same way, we conclude that

$$\int_{\Omega} vv_t\left|v_t\right|^{q-1}dx\leq\frac{\delta_2^{q+1}}{q+1}\left\|v\right\|_{q+1}^{q+1}+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\left\|v_t\right\|_{q+1}^{q+1}.$$

It follows that

$$\begin{aligned}\int_{\Omega} v\left(|v|^{\theta}+|u|^{\rho}\right)v_t\left|v_t\right|^{q-1}dx &\leq \frac{\delta_2^{q+1}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\rho}\right)|v|^{q+1}dx\\ &\quad+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\rho}\right)|v_t|^{q+1}dx,\end{aligned}$$

where $\delta_1$, $\delta_2$ are constants depending on the time $t$ and specified later. Therefore, (3.17) becomes

$$\begin{aligned}\Psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right)\left(\left\|u_t\right\|_{j+2}^{j+2}+\left\|v_t\right\|_{j+2}^{j+2}\right)\\ &\quad+\varepsilon\gamma\left(\left\|\nabla u\right\|^{2}+\left\|\nabla v\right\|^{2}\right)+2\varepsilon(\gamma+1)H(t)+\varepsilon c'\left(\left\|u+v\right\|_{2(r+2)}^{2(r+2)}+2\left\|uv\right\|_{r+2}^{r+2}\right)\\ &\quad-\varepsilon\frac{\delta_1^{p+1}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u|^{p+1}dx-\varepsilon\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u_t|^{p+1}dx\\ &\quad-\varepsilon\frac{\delta_2^{q+1}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\rho}\right)|v|^{q+1}dx-\varepsilon\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\rho}\right)|v_t|^{q+1}dx. \qquad (3.18)\end{aligned}$$

Taking $\delta_1$ and $\delta_2$ such that

$$\begin{aligned}\delta_1^{-\frac{p+1}{p}} &= k_1H^{-\sigma}(t),\\ \delta_2^{-\frac{q+1}{q}} &= k_2H^{-\sigma}(t)\end{aligned}$$

where $k_1,\ k_2>0$ are specified later, we get

$$\begin{aligned}\Psi'(t) &\geq ((1-\sigma)-K\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right)\left(\left\|u_t\right\|_{j+2}^{j+2}+\left\|v_t\right\|_{j+2}^{j+2}\right)\\ &\quad+\varepsilon\gamma\left(\left\|\nabla u\right\|^{2}+\left\|\nabla v\right\|^{2}\right)+2\varepsilon(\gamma+1)H(t)+\varepsilon c'\left(\left\|u\right\|_{2(r+2)}^{2(r+2)}+\left\|v\right\|_{2(r+2)}^{2(r+2)}\right)\\ &\quad-\varepsilon\frac{k_1^{-p}H^{\sigma p}(t)}{p+1}\int_{\Omega}\left(|u|^{k}+|v|^{l}\right)|u|^{p+1}dx\\ &\quad-\varepsilon\frac{k_2^{-q}H^{\sigma q}(t)}{q+1}\int_{\Omega}\left(|v|^{\theta}+|u|^{\rho}\right)|v|^{q+1}dx, \qquad (3.19)\end{aligned}$$

where $K = \frac{k_1 p}{p+1} + \frac{k_2 q}{q+1}$. Thanks to Young's inequality, we obtain

$$\begin{aligned}\int_\Omega \left(|u|^k + |v|^l\right)|u|^{p+1}dx &\leq \int_\Omega |u|^{k+p+1}dx + \int_\Omega |v|^l |u|^{p+1}dx \\ &\leq \int_\Omega |u|^{k+p+1}dx + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\int_\Omega |v|^{l+p+1}dx \\ &\quad + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\int_\Omega |u|^{l+p+1}dx \\ &= \|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1} \\ &\quad + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1}. \end{aligned} \tag{3.20}$$

Similarly, one has

$$\begin{aligned}\int_\Omega \left(|v|^\theta + |u|^\rho\right)|v|^{q+1}dx &\leq \|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\rho}{\rho+q+1}\delta_2^{\frac{\rho+q+1}{\rho}}\|u\|_{\rho+q+1}^{\rho+q+1} \\ &\quad + \frac{q+1}{\rho+q+1}\delta_2^{-\frac{\rho+q+1}{q+1}}\|v\|_{\rho+q+1}^{\rho+q+1}. \end{aligned} \tag{3.21}$$

Inserting (3.21) and (3.20) into (3.19), we conclude that

$$\begin{aligned}\Psi'(t) \geq\ & ((1-\sigma)-K\varepsilon)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{1}{j+1} + \frac{2(\gamma+1)}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2}\right) \\ & +\varepsilon\gamma\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + 2\varepsilon(\gamma+1)H(t) + \varepsilon c'\left(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\right) \\ & -\varepsilon\frac{k_1^{-p}H^{\sigma p}(t)}{p+1}\left(\|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1} + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1}\right) \\ & -\varepsilon\frac{k_2^{-q}H^{\sigma q}(t)}{q+1}\left(\|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\rho}{\rho+q+1}\delta_2^{\frac{\rho+q+1}{\rho}}\|u\|_{\rho+q+1}^{\rho+q+1}\right. \\ & \left. + \frac{q+1}{\rho+q+1}\delta_2^{-\frac{\rho+q+1}{q+1}}\|v\|_{\rho+q+1}^{\rho+q+1}\right). \end{aligned} \tag{3.22}$$

Since

$$2(r+2) > \max\{2(\gamma+1), k+p+1, l+p+1, \theta+q+1, \rho+q+1\},$$

we have

$$H^{\sigma p}(t)\|u\|_{k+p+1}^{k+p+1} \leq C\left(\|u\|_{2(r+2)}^{2\sigma p(r+2)+k+p+1} + \|v\|_{2(r+2)}^{2\sigma p(r+2)}\|u\|_{k+p+1}^{k+p+1}\right), \tag{3.23}$$

$$H^{\sigma q}(t)\|v\|_{\theta+q+1}^{\theta+q+1} \leq C\left(\|v\|_{2(r+2)}^{2\sigma q(r+2)+\theta+q+1} + \|u\|_{2(r+2)}^{2\sigma q(r+2)}\|v\|_{\theta+q+1}^{\theta+q+1}\right), \tag{3.24}$$

$$\begin{aligned}&\frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}H^{\sigma p}(t)\|v\|_{l+p+1}^{l+p+1} \\ &\leq C\frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\left(\|v\|_{2(r+2)}^{2\sigma p(r+2)+l+p+1} + \|u\|_{2(r+2)}^{2\sigma p(r+2)}\|v\|_{l+p+1}^{l+p+1}\right),\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}&\frac{\rho}{\rho+q+1}\delta_2^{\frac{\rho+q+1}{\rho}}H^{\sigma q}(t)\|u\|_{\rho+q+1}^{\rho+q+1} \\ &\leq C\frac{\rho}{\rho+q+1}\delta_2^{\frac{\rho+q+1}{\rho}}\left(\|u\|_{2(r+2)}^{2\sigma q(r+2)+\rho+q+1} + \|v\|_{2(r+2)}^{2\sigma q(r+2)}\|u\|_{\rho+q+1}^{\rho+q+1}\right).\end{aligned} \tag{3.26}$$

By using (3.14) and the following algebraic inequality

$$z^{\upsilon} \leq z+1 \leq \left(1+\frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0<\upsilon \leq 1,\ a>0, \tag{3.27}$$

we have, for all $t \geq 0$,

$$\begin{aligned} \|u\|_{2(r+2)}^{2\sigma p(r+2)+k+p+1} &\leq d\left(\|u\|_{2(r+2)}^{2(r+2)}+H(0)\right) \\ &\leq d\left(\|u\|_{2(r+2)}^{2(r+2)}+H(t)\right), \end{aligned} \tag{3.28}$$

and

$$\|v\|_{2(r+2)}^{2\sigma q(r+2)+\theta+q+1} \leq d\left(\|v\|_{2(r+2)}^{2(r+2)}+H(t)\right), \tag{3.29}$$

where $d=1+\frac{1}{H(0)}$. Similarly

$$\|u\|_{2(r+2)}^{2\sigma q(r+2)+\rho+q+1} \leq d\left(\|u\|_{2(r+2)}^{2(r+2)}+H(t)\right), \tag{3.30}$$

$$\|v\|_{2(r+2)}^{2\sigma p(r+2)+l+p+1} \leq d\left(\|v\|_{2(r+2)}^{2(r+2)}+H(t)\right). \tag{3.31}$$

Also, since

$$(a+b)^{\lambda} \leq C\left(a^{\lambda}+b^{\lambda}\right), \quad a,b>0,$$

by Young inequality and using (3.14) and (3.27), we conclude that

$$\begin{aligned} \|v\|_{2(r+2)}^{2\sigma p(r+2)}\|u\|_{k+p+1}^{k+p+1} &\leq |\Omega|^{\frac{2(r+2)-(k+p+1)}{2(r+2)}}\left(\|v\|_{2(r+2)}^{2\sigma p(r+2)}\|u\|_{2(r+2)}^{k+p+1}\right) \\ &= |\Omega|^{\frac{2(r+2)-(k+p+1)}{2(r+2)}}\left(\|v\|_{2(r+2)}^{\sigma p}\|u\|_{2(r+2)}^{\frac{k+p+1}{2(r+2)}}\right)^{2(r+2)} \\ &\leq |\Omega|^{\frac{2(r+2)-(k+p+1)}{2(r+2)}}\left(c'\|v\|_{2(r+2)}^{\frac{2\sigma p(r+2)+k+p+1}{2(r+2)}}+c''\|u\|_{2(r+2)}^{\frac{2\sigma p(r+2)+k+p+1}{2(r+2)}}\right)^{2(r+2)} \\ &\leq C\left(\|v\|_{2(r+2)}^{2(r+2)}+\|u\|_{2(r+2)}^{2(r+2)}\right). \end{aligned} \tag{3.32}$$

Similarly, one has

$$\|u\|_{2(r+2)}^{2\sigma q(r+2)}\|v\|_{\theta+q+1}^{\theta+q+1} \leq C\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right), \tag{3.33}$$

$$\|u\|_{2(r+2)}^{2\sigma p(r+2)}\|v\|_{l+p+1}^{l+p+1} \leq C\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right), \tag{3.34}$$

and

$$\|v\|_{2(r+2)}^{2\sigma q(r+2)}\|u\|_{\rho+q+1}^{\rho+q+1} \leq C\left(\|v\|_{2(r+2)}^{2(r+2)}+\|u\|_{2(r+2)}^{2(r+2)}\right). \tag{3.35}$$

Inserting (3.23)-(3.26) and (3.28)-(3.35) into (3.22), we have

$$\begin{aligned}\Psi'(t) \geq\ & ((1-\sigma)-K\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\ &+\varepsilon\gamma\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &+\varepsilon\left[2(\gamma+1)-Ck_1^{-p}\left(1+\frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}+\frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\right)\right.\\ &\left.-Ck_2^{-q}\left(1+\frac{\rho}{\rho+q+1}\delta_2^{\frac{\rho+q+1}{\rho}}+\frac{q+1}{\rho+q+1}\delta_2^{-\frac{\rho+q+1}{q+1}}\right)\right]H(t)\\ &+\varepsilon\left[c'-Ck_1^{-p}\left(1+\frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}+\frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\right)\right.\\ &\left.-Ck_2^{-q}\left(1+\frac{\rho}{\rho+q+1}\delta_2^{\frac{\rho+q+1}{\rho}}+\frac{q+1}{\rho+q+1}\delta_2^{-\frac{\rho+q+1}{q+1}}\right)\right]\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right).\end{aligned} \tag{3.36}$$

At this point, and for large values of $k_1$ and $k_2$, we can find positive constats $K_1$ and $K_2$ such that (3.36) becomes

$$\begin{aligned}\Psi'(t) \geq\ & ((1-\sigma)-K\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\ &+\varepsilon\gamma\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon K_1 H(t)+\varepsilon K_2\left(\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right)\\ \geq\ & \beta\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}+H(t)+\|\nabla u\|^2+\|\nabla v\|^2+\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}\right),\end{aligned} \tag{3.37}$$

where $\beta=\min\left\{\varepsilon\left(\frac{1}{j+1}+\frac{2(\gamma+1)}{j+2}\right),\varepsilon\gamma,\varepsilon K_1,\varepsilon K_2\right\}$ and we pick $\varepsilon$ small enough so that $(1-\sigma)-K\varepsilon\geq 0$. Consequently, we have

$$\Psi(t)\geq\Psi(0)>0,\ \forall t\geq 0. \tag{3.38}$$

We now estimate $\Psi(t)^{\frac{1}{1-\sigma}}$. Applying Hölder inequality, we obtain

$$\left|\int_\Omega |u_t|^j u_t u dx\right|\leq \|u_t\|_{j+2}^{j+1}\|u\|_{j+2}\leq C_1\|u_t\|_{j+2}^{j+1}\|u\|_{2(r+2)} \tag{3.39}$$

and

$$\left|\int_\Omega |u_t|^j u_t u dx\right|^{\frac{1}{1-\sigma}}\leq C_2\|u_t\|_{j+2}^{\frac{j+1}{1-\sigma}}\|u\|_{2(r+2)}^{\frac{1}{1-\sigma}}\leq C_3(\|u_t\|_{j+2}^{\frac{(j+1)\mu}{1-\sigma}}+\|u\|_{2(r+2)}^{\frac{\theta}{1-\sigma}}), \tag{3.40}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. From (3.14), we choose $\mu=\frac{(1-\sigma)(j+2)}{j+1}\geq 1$. It follows that

$$2<\frac{\theta}{1-\sigma}=\frac{j+2}{(1-\sigma)(j+2)-(j+1)}<j+1. \tag{3.41}$$

Taking $s=\frac{\theta}{1-\sigma}$ in Lemma 2.3, we have

$$\|u\|_{2(r+2)}^{\frac{\theta}{1-\sigma}}\leq C_4(\|\nabla u\|^2+\|u\|_{2(r+2)}^{2(r+2)}). \tag{3.42}$$

Hence

$$\left|\int_\Omega |u_t|^j u_t u dx\right|^{\frac{1}{1-\sigma}}\leq C_5(\|u_t\|_{j+2}^{j+2}+\|\nabla u\|^2+\|u\|_{2(r+2)}^{2(r+2)}) \tag{3.43}$$

Similarly, one finds that

$$\left|\int_\Omega |v_t|^j v_t v dx\right|^{\frac{1}{1-\sigma}}\leq C_6(\|v_t\|_{j+2}^{j+2}+\|\nabla v\|^2+\|v\|_{2(r+2)}^{2(r+2)}). \tag{3.44}$$

By (3.12), it yields

$$\begin{aligned}\Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t)+\frac{\varepsilon}{j+1}\left(\int_\Omega |u_t|^j u_t u dx+\int_\Omega |v_t|^j v_t v dx\right)\right]^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}+H(t)+\|u\|_{2(r+2)}^{2(r+2)}+\|v\|_{2(r+2)}^{2(r+2)}+\|\nabla u\|^2+\|\nabla v\|^2\right). \quad (3.45)\end{aligned}$$

By combining of (3.37) and (3.45) we arrive at

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \quad (3.46)$$

where $\xi$ is a positive constant. This completes the proof. □

A simple integration of (3.46) over $(0,t)$ yields that $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time $T^*$, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

## 4. Lower bounds for blow up time

In this section, we discuss the lower bounds of the blow up time for the blow up solution of (1.1). First, we give following lemma, which can be found in [17, 18].

**Lemma 4.1.** *There exist two positive $c_1$ and $c_2$ such that*

$$\begin{aligned}\int_\Omega |f_1(u,v)|^2 dx &\leq c_1(\|\nabla u\|^2+\|\nabla v\|^2)^{2r+3}, \\ \int_\Omega |f_2(u,v)|^2 dx &\leq c_2(\|\nabla u\|^2+\|\nabla v\|^2)^{2r+3}. \quad (4.1)\end{aligned}$$

**Theorem 4.2.** *Suppose that (A1), (2.1) hold and $(u_0,u_1),(v_0,v_1) \in (H_0^1(\Omega)\cap L^{r+1}(\Omega))\times L^2(\Omega)$. Assume further that $j=0$ and $1<p,q<2r+1$. Then the finite blow-up time $T^*$ satisfies the following estimate*

$$\int_{\phi(0)}^{\infty} \frac{d\tau}{(E(0)+\tau)+2^{4(r+1)}(c_1+c_2)((E(0))^{2r+3}+\tau^{2r+3})} \leq T^*,$$

*where $\phi(0)=\int_\Omega F(u(0),v(0))dx$ and the positive constants $c_1$ and $c_2$ are specified in (4.1).*

*Proof.* Define

$$\phi(t)=\int_\Omega F(u,v)dx.$$

By differentiating $\phi(t)$ and using Young's inequality, we have

$$\begin{aligned}\phi'(t) &= \int_\Omega u_t F_u + v_t F_v dx \\ &\leq \frac{1}{2}\int_\Omega (u_t^2+v_t^2)dx+\frac{1}{2}\int_\Omega (F_u^2+F_v^2)dx. \quad (4.2)\end{aligned}$$

By using Lemma 4.1, we obtain

$$\phi'(t) \leq \frac{1}{2}\int_\Omega (u_t^2+v_t^2)dx+(\frac{c_1+c_2}{2})(\|\nabla u\|^2+\|\nabla v\|^2)^{2r+3}. \quad (4.3)$$

From (A1), (2.4) and Lemma 2.4, we have

$$\begin{aligned}\int_{\Omega}(u_t^2+v_t^2)dx+(\|\nabla u\|^2+\|\nabla v\|^2) &\leq 2E(t)+2\int_{\Omega}F(u,v)dx \\ &\leq 2E(0)+2\int_{\Omega}F(u,v)dx \end{aligned} \tag{4.4}$$

Combining (4.3)-(4.4), we get

$$\begin{aligned}\phi'(t) &\leq \phi(t)+E(0)+2^{2r+2}(c_1+c_2)\left[\phi(t)+E(0)\right]^{2r+3} \\ &\leq \phi(t)+E(0)+2^{4(r+1)}(c_1+c_2)\left[(\phi(t))^{2r+3}+(E(0))^{2r+3}\right]. \end{aligned} \tag{4.5}$$

Integrating (4.5) from 0 to $t$, we obtain

$$\int_{\phi(0)}^{\phi(t)}\frac{d\tau}{(E(0)+\tau)+2^{4(r+1)}(c_1+c_2)((E(0))^{2r+3}+\tau^{2r+3})}\leq t.$$

In view of $\lim\limits_{t\to T^{*-}}\phi(t)=\infty$, we can write

$$\int_{\phi(0)}^{\infty}\frac{d\tau}{(E(0)+\tau)+2^{4(r+1)}(c_1+c_2)((E(0))^{2r+3}+\tau^{2r+3})}\leq T^*.$$

Thus, we obtain the desired result. This completes the proof. □

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