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ON POSITIVE SOLUTIONS OF A SYSTEM OF $(2n,2m)$-ORDER p-LAPLACIAN EQUATIONS

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Abstract. The existence results of positive nontrivial solutions for a class of even order p-Laplacian systems are obtained. The proof of our main results is based on a well-known fixed point theorem in cones.
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1. Introduction

Equations of the p-Laplacian form occur in the study of non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium. Because of the important background, there are many papers devoted to the study of differential equations with p-Laplacian. For the scalar equation, we refer the readers to [1] (one-dimensional p-Laplacian), [2, 3, 4] (fourth order p-Laplacian), [5] (2nth-order p-Laplacian). For the system, Djebali, Moussaou and Precup [6] have shown some existence results for a fourth-order nonlinear p-system

$$\begin{cases} (\varphi_p(u_1''))'' = \lambda_1 a_1(t) f_1(u_1,u_2), & t\in(0,1),\\ (\varphi_p(u_2''))'' = \lambda_2 a_2(t) f_2(u_1,u_2), & t\in(0,1),\\ u_1(0)=u_1(1)=u_1''(0)=u_1''(1)=0, \\ u_2(0)=u_2(1)=u_2''(0)=u_2''(1)=0, \end{cases}$$

where $f_i\in C([0,+\infty)^2,[0,+\infty))$, $a_i\in C([0,1],[0,+\infty))$ and λ_i are positive parameters for $i=1,2$. Here $\varphi_p(s)=s|s|^{p-2}(p>1)$ refers to the p-Laplacian. A first existence result is obtained via the classical Krasnosel'skii's fixed point theorem of cone compression and expansion under the following notations and assumptions

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(H1) For $i=1,2$, there exist nonnegative constants f_i^0, f_i^∞ defined as

$$f_i^0=\lim_{u+v\to 0}\frac{f(u,v)}{(u+v)^{p-1}},\ and\ f_i^\infty=\lim_{u+v\to\infty}\frac{f(u,v)}{(u+v)^{p-1}}.$$

Another existence result is obtained via the vector versions of the Krasnoselskill fixed point theorem under the following notations and assumptions

(H2) For $\nu=0$ or $\nu=+\infty$, there exist nonnegative constants F_1^ν, F_2^ν defined as

$$F_1^\nu=\lim_{u\to\nu}\frac{f_1(u,v)}{u^{p-1}}\ uniformly\ with\ respect\ to\ v\ on\ compact\ subsets\ of\ R^+,$$

$$F_2^\nu=\lim_{v\to\nu}\frac{f_2(u,v)}{v^{p-1}}\ uniformly\ with\ respect\ to\ u\ on\ compact\ subsets\ of\ R^+.$$

A comparison of the obtained results to those from the literature is provided.

In addition, there are many authors studying a system of different order equations. An [7] and An and Fan [8] investigated a coupled system of second and fourth order equations. Kang, Xu and Wei [9], Prasad and Kameswararao [10], Ru and An [11] and Yang [12] studied the $2p$-$2q$ order nonlinear ordinary differential systems

$$\begin{cases}(-1)^p u^{(2p)} &= \lambda a(t)f(u(t),u(t)), \quad t\in[0,1],\\ (-1)^q v^{(2q)} &= \mu b(t)g(u(t),u(t)), \quad t\in[0,1],\end{cases}$$

via the classical Krasnosel'skii's fixed point theorem of cone compression and expansion under the following assumption

$$f_0=\lim_{(u,v)\to 0}\frac{f(u,v)}{u+v}, f_\infty=\lim_{(u,v)\to+\infty}\frac{f(u,v)}{u+v}.$$

Inspired by these results, this paper is mainly concerned with the existence of positive solutions of system of even-order p-Laplacian nonlinear equations

$$\begin{cases}(-1)^n[\varphi_{p_1}(u^{(2n_1)})]^{(2n_2)} &= f_1(u(t),v(t)), \quad t\in(0,1),\\ (-1)^m[\varphi_{p_2}(v^{(2m_1)})]^{(2m_2)} &= f_2(u(t),v(t)), \quad t\in(0,1),\\ u^{(2i)}(0)=u^{(2i)}(1)=0, \quad 0\le i\le n-1, \\ v^{(2j)}(0)=v^{(2j)}(1)=0, \quad 0\le j\le m-1,\end{cases} \tag{1.1}$$

where $n_1+n_2=n, m_1+m_2=m, n,m\in\mathbf{N}$, $f_i\in C([0,+\infty)^2,[0,+\infty)])$. Here $\varphi_p(s)=s|s|^{p-2}(p>1)$ refers to the p-Laplacian. We give some more weaker conditions of the nonlinearities than f_0 and f_∞, such as

(H1) There exist a constant $\alpha>0$ and a continuous function

$$\Phi_i:[0,\infty]\times[0,\infty]\to[0,\infty]$$

satisfying

(i) $\Phi_i(u,v)$ is a nondecreasing function with respective to u and v,

(ii) $\Phi_i(\alpha,\alpha)\le\kappa\alpha^{p_i-1}$, for some $\kappa>0$, κ will be given specific value in Section 3,

such that

$$f_i(u,v)\le\Phi_i(u,v),\ 0\le u+v\le\alpha;$$

(H2) There exist a constant $\beta>0$ and a continuous function

$$\psi_i:[0,\infty]\to[0,\infty]$$

satisfying
(i) $\psi_i(s)$ is a nondecreasing function with respective to s,
(ii) $\psi_i(\sigma\beta) \geq \mu\beta^{p_i-1}$, for some $\mu > 0$, μ will be given specific value in Section 3,
$\sigma = \min\{6^{n_1-1}\theta_{n_1}\left(\delta\right), 6^{m_1-1}\theta_{m_1}\left(\delta\right)\}$, $\theta_n(\delta) = \delta^n\left(\frac{4\delta^3-6\delta^2+1}{6}\right)^{n-1}$, and $\delta \in (0,\frac{1}{2})$, such that

$$f_i(u,v) \geq \psi_i(u+v),\ \sigma\beta \leq u+v \leq \beta.$$

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we give the main results in this paper.

2. Preliminaries

Let $G_n(t,s)$ be the Green's function of the linear boundary value problem:

$$\begin{cases} (-1)^{(n)}\omega^{(2n)}(t) = 0, & t \in [0,1], \\ \omega^{(2i)}(0) = \omega^{(2i)}(1) = 0, & 0 \leq i \leq n-1. \end{cases}$$

By the induction, the Green's function $G_n(t,s)$ can be expressed as (see [11])

$$G_i(t,s) = \int_0^1 G(t,\xi)G_{i-1}(\xi,s)d\xi, \quad 2 \leq i \leq n,$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

It is clear that

$$G_n(t,s) > 0,\ \ (t,s) \in (0,1)\times(0,1).$$

Lemma 2.1 ([11]). *$G_n(t,s)$ has the following properties*
(1) For any $(t,s) \in [0,1]\times[0,1]$, $G_n(t,s) \leq \frac{1}{6^{n-1}}s(1-s)$.
(2) Letting $\delta \in (0,\frac{1}{2})$, one finds, for any $(t,s) \in [\delta,1-\delta]\times[0,1]$, that

$$G_n(t,s) \geq \theta_n(\delta)s(1-s) \geq 6^{n-1}\theta_n(\delta)\max_{0\leq t\leq 1} G_n(t,s)$$

where $\theta_n(\delta) = \delta^n\left(\frac{4\delta^3-6\delta^2+1}{6}\right)^{n-1}$.

For $h(t) \in C[0,1]$, the solution of the boundary value problem

$$\begin{cases} (-1)^n[\varphi_p(u^{(2n_1)})]^{(2n_2)} = h(t), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & 0 \leq i \leq n-1, \end{cases} \tag{2.1}$$

can be expressed by

$$u(t) = \int_0^1 G_{n_1}(t,s)\varphi_q(\int_0^1 G_{n_2}(s,\tau)h(\tau)d\tau)ds,$$

where φ_q stands for the inverse function $\varphi_q = \varphi_p^{-1}$ with conjugates p,q, i.e., $\frac{1}{p}+\frac{1}{q} = 1$.

Lemma 2.2. *For $h(t) \in C[0,1]$ with $h(t) \geq 0$, the solution of (2.1) satisfies*

$$\min_{t\in[\delta,1-\delta]} u(t) \geq 6^{n_1-1}\theta_{n_1}\left(\delta\right)\|u\|,$$

where the norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$.

Proof. From Lemma 2.1, for $t \in [\delta, 1-\delta]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G_{n_1}(t,s)\varphi_q(\int_0^1 G_{n_2}(s,\tau)h(\tau)d\tau)ds \\ &\geq \max_{t\in[0,1]} \int_0^1 6^{n_1-1}\theta_{n_1}(\delta)G_{n_1}(t,s)\varphi_q(\int_0^1 G_{n_2}(s,\tau)h(\tau)d\tau)ds \\ &\geq 6^{n_1-1}\theta_{n_1}\Big(\delta\Big) \max_{t\in[0,1]} \int_0^1 G_{n_1}(t,s)\varphi_q(\int_0^1 G_{n_2}(s,\tau)h(\tau)d\tau)ds \\ &= 6^{n_1-1}\theta_{n_1}\Big(\delta\Big)\|u\|. \end{aligned}$$

Now we define a mapping $T : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)),$$

where

$$T_1(u,v)(t) = \int_0^1 G_{n_1}(t,s)\varphi_{q_1}(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau)ds,$$
$$T_2(u,v)(t) = \int_0^1 G_{m_1}(t,s)\varphi_{q_2}(\int_0^1 G_{m_2}(s,\tau)f_2(u(\tau),v(\tau))d\tau)ds.$$

Let X denote the Banach space $C[0,1]\times C[0,1]$ with the norm

$$\|(u,v)\| = \max_{0\leq t\leq 1}|u(t)| + \max_{0\leq t\leq 1}|v(t)|$$

and K be the sub-cone defined by

$$K = \{(u,v)\in X : u(t)\geq 0, v(t)\geq 0, \min_{\delta\leq t\leq 1-\delta}(u(t)+v(t)) \geq \sigma\|(u,v)\|\},$$

where $\sigma = \min\{6^{n_1-1}\theta_{n_1}\Big(\delta\Big), 6^{m_1-1}\theta_{m_1}\Big(\delta\Big)\}$. It is obvious that $0<\sigma<1$. This completes the proof. □

Lemma 2.3. *$T : K \to K$ is completely continuous.*

Proof. First, we show that $T(K)\subset K$. In fact, we find from Lemma 2.2, for any $(u,v)\in K$, that

$$\min_{t\in[\delta,1-\delta]} T_1(u,v)(t) \geq 6^{n_1-1}\theta_{n_1}\Big(\delta\Big)\|T_1(u,v)\| \geq \sigma\|T_1(u,v)\|,$$

$$\min_{t\in[\delta,1-\delta]} T_2(u,v)(t) \geq 6^{m_1-1}\theta_{m_1}\Big(\delta\Big)\|T_2(u,v)\| \geq \sigma\|T_2(u,v)\|.$$

Furthermore, we get

$$\min_{t\in[\delta,1-\delta]}(T_1(u,v)(t)+T_2(u,v)(t)) \geq \sigma\|T_1(u,v)\| + \sigma\|T_2(u,v)\| \geq \sigma\|(T_1(u,v),T_2(u,v))\|.$$

Now, we prove that T is completely continuous. For convenience, we only show that $T_1 : C[0,1]\times C[0,1] \to C[0,1]$ is completely continuous. Let $\{(u_k,v_k)\}$ be a sequence in $C[0,1]\times C[0,1]$, which converges to (u_0,v_0) uniformly on $[0,1]$. From the continuity of $f_1(u(t),v(t))$ and the Lebesgue dominated convergence theorem, we find that

$$\begin{aligned} \lim_{k\to\infty} T_1(u_k,v_k)(t) &= \lim_{k\to\infty}\int_0^1 G_{n_1}(t,s)\varphi_{q_1}(\int_0^1 G_{n_2}(s,\tau)f_1(u_k(\tau),v_k(\tau))d\tau)ds \\ &= \int_0^1 G_{n_1}(t,s)\varphi_{q_1}(\int_0^1 G_{n_2}(s,\tau)f_1(u_0(\tau),v_0(\tau))d\tau)ds \\ &= T_1(u_0,v_0)(t), \end{aligned}$$

which implies that T_1 is continuous.

Let Ω be a bounded subset in $C[0,1]\times C[0,1]$. Then there exists a $R>0$ such that $\|(u,v)\|\le R$, for all $(u,v)\in\Omega$. Since f_1 is continuous on (u,v), we set $M=\max_{\|(u,v)\|\le R} f_1(u(t),v(t))>0$. From Lemma 2.1, we have

$$\begin{aligned}
\|T_1(u,v)\| &= \max_{0\le t\le 1}|T_1(u,v)(t)| \\
&= \max_{0\le t\le 1}\left|\int_0^1 G_{n_1}(t,s)\varphi_{q_1}\Big(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau\Big)ds\right| \\
&\le \int_0^1 \frac{1}{6^{n_1-1}}s(1-s)\varphi_{q_1}\Big(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)Md\tau\Big)ds \\
&\le \frac{1}{6^{n_1}}\frac{1}{6^{n_2(q_1-1)}}M^{q_1-1}=M_1.
\end{aligned}$$

Hence $T_1(\Omega)$ is an equibounded set.

Finally, we prove the that equicontinuity of $T_1(\Omega)$ on $[0,1]$. Note that

$$\begin{aligned}
T_1(u,v)(t) &= \int_0^1 G_{n_1}(t,s)\varphi_{q_1}\Big(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau\Big)ds \\
&= \int_0^1\int_0^1 G(t,\xi)G_{n_1-1}(\xi,s)d\xi\,\varphi_{q_1}\Big(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau\Big)ds \\
&= \int_0^1\Big[\int_0^t (1-t)\xi G_{n_1-1}(\xi,s)d\xi \\
&\quad +\int_t^1 (1-\xi)tG_{n_1-1}(\xi,s)d\xi\Big]\varphi_{q_1}\Big(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau\Big)ds,
\end{aligned}$$

Using Lemma 2.1, we find, for any $t\in[0,1]$, that

$$\begin{aligned}
&|T_1'(u,v)(t)| \\
&= \Big|\int_0^1\Big[-\int_0^t \xi G_{n_1-1}(\xi,s)d\xi \\
&\quad +\int_t^1 (1-\xi)G_{n_1-1}(\xi,s)d\xi\Big]\varphi_{q_1}\Big(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau\Big)ds\Big| \\
&= \Big|\int_0^1\Big[\int_t^1 G_{n_1-1}(\xi,s)d\xi-\int_0^1 \xi G_{n_1-1}(\xi,s)d\xi\Big]\varphi_{q_1}\Big(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau\Big)ds\Big| \\
&\le \int_0^1\Big[\int_t^1 G_{n_1-1}(\xi,s)d\xi+\int_0^1 \xi G_{n_1-1}(\xi,s)d\xi\Big]\varphi_{q_1}\Big(\int_0^1 \frac{1}{6^{n_2-1}}\tau(1-\tau)Md\tau\Big)ds \\
&\le \int_0^1\int_0^1 (1+\xi)G_{n_1-1}(\xi,s)d\xi\,\varphi_{q_1}\Big(\frac{1}{6^{n_2}}M\Big)ds \\
&\le \int_0^1\int_0^1 (1+\xi)\frac{1}{6^{n_1-2}}s(1-s)d\xi\,\varphi_{q_1}\Big(\frac{1}{6^{n_2}}M\Big)ds \\
&= \frac{3}{2}\frac{1}{6^{n_1-1}}\varphi_{q_1}\Big(\frac{1}{6^{n_2}}M\Big):=M_2.
\end{aligned}$$

Then, for $\forall\, t_1,t_2\in[0,1], t_2>t_1$, we have

$$|T_1(u,v)(t_2)-T_1(u,v)(t_1)|\le M_2(t_2-t_1),$$

which implies that $T_1(\Omega)$ is equicontinuous on $[0,1]$. From the Ascoli-Arzelá theorem, we get $T_1 : C[0,1]\times C[0,1] \to C[0,1]$ is compact. The similar discussion is also hold for T_2. Therefore, T is completely continuous. □

For convenience, let D be a subset of X. We introduce the notions $D_K = D\bigcap K$ and $\partial_K D = (\partial D)\bigcap K$.

Lemma 2.4 ([13]). *Let X be a Banach space and K be a cone in X. Assume Ω^1 and Ω^2 are open bounded subsets of X with $\Omega^1{}_K \neq \emptyset, \overline{\Omega^1}_K \subset \Omega^2{}_K$. Let $T : \overline{\Omega^2}_K \to K$ be a completely continuous operator such that*

(A) $\| Tu \| \leq \| u \|, \forall u \in \partial_K \Omega^1$

(B) there exists $e \in K\setminus\{0\}$ such that

$$u \neq Tu + \lambda e, \ for\ u \in \partial_K \Omega^2 \ and\ \lambda > 0.$$

Then T has a fixed point in $\overline{\Omega^2{}_K}\setminus \Omega^1{}_K$. The same conclusion remains valid if (A) holds on $\partial_K\Omega^2$ and (B) holds on $\partial_K\Omega^1$.

For convenience, for any $\gamma > 0$, we define some adequate open sets by

$$\Omega^\gamma = \{(u,v)\in X : \min_{t\in[\delta,1-\delta]} (u(t)+v(t)) < \sigma\gamma\},$$

and

$$B^\gamma = \{(u,v)\in X : \| (u,v) \| < \gamma\}.$$

Referring to [14, 15], we can obtain the following.

Lemma 2.5. *Ω^γ and B^γ have the following properties:*
(a_1) Ω^γ_K and B^γ_K are open relative to K.
(a_2) $B^{\sigma\gamma}_K \subset \Omega^\gamma_K \subset B^\gamma_K$.
(a_3) $(u,v)\in \partial_K\Omega^\gamma$ if and only if $\min_{[\delta,1-\delta]} (u(t)+v(t)) = \sigma\gamma$.
(a_4) If $(u,v)\in\partial_K\Omega^\gamma$, then $\sigma\gamma \leq u(t)+v(t) \leq \gamma, \delta \leq t \leq 1-\delta$.

Proof. Obviously, (a_1) and (a_3) hold. Let $(u,v)\in B^{\sigma\gamma}_K$. Then

$$\sigma\|(u,v)\| \leq \min_{[\delta,1-\delta]} (u(t)+v(t)) \leq \|(u,v)\| < \sigma\gamma$$

and $(u,v)\in\Omega^\gamma_K$. If $(u,v)\in\Omega^\gamma_K$, then

$$\sigma\|(u,v)\| \leq \min_{[\delta,1-\delta]} (u(t)+v(t)) < \sigma\gamma.$$

This implies that $\| (u,v) \| < \gamma$ and $(u,v)\in B^\gamma_K$. Hence, (a_2) holds. If $(u,v)\in\partial_K\Omega^\gamma$, we find from (a_3) that

$$\sigma\|(u,v)\| \leq \min_{[\delta,1-\delta]} (u(t)+v(t)) = \sigma\gamma \leq u(t)+v(t),$$

and $u(t)+v(t) \leq \|(u,v)\| \leq \gamma$, for all $t\in[\delta,1-\delta]$. So (a_4) holds. This completes the proof. □

Remark 2.6. It is easy to see that Ω^γ is unbounded for each $\gamma > 0$. So we can not apply Lemma 2.4 to Ω^γ, however, we can apply Lemma 2.4 by taking into account that, for each $\rho > \gamma$, the following relations

$$\Omega^\gamma_K = (\Omega^\gamma\bigcap B^\rho)_K,\ \ \overline{\Omega^\gamma_K} = \overline{(\Omega^\gamma\bigcap B^\rho)}_K.$$

3. Main results

Set

$$\kappa=\min\{2^{\frac{1}{1-q_1}}6^{\frac{n_1}{q_1-1}+n_2},2^{\frac{1}{1-q_2}}6^{\frac{m_1}{q_2-1}+m_2}\},$$

$$\mu=\max\{\nu_1,\nu_2\},$$

$$\nu_1=\frac{\sigma^{\frac{1}{q_1-1}}}{(2\theta_{n_1}(\delta))^{\frac{1}{q_1-1}}\theta_{n_2}(\delta)(\frac{2}{3}\delta^3-\delta^2+\frac{1}{6})^{\frac{q_1}{q_1-1}}},$$

and

$$\nu_2=\frac{\sigma^{\frac{1}{q_2-1}}}{(2\theta_{m_1}(\delta))^{\frac{1}{q_2-1}}\theta_{m_2}(\delta)(\frac{2}{3}\delta^3-\delta^2+\frac{1}{6})^{\frac{q_2}{q_2-1}}},$$

where q_i are the conjugates of p_i, $i=1,2$, i.e., $\frac{1}{p_i}+\frac{1}{q_i}=1$.

Theorem 3.1. *Assume that (H1) and (H2) hold.*
(I) If $\alpha<\sigma\beta$, then (1.1) has at least one positive solution $(u(t),v(t))$ satisfying

$$\alpha\le\|(u,v)\|\le\beta \text{ and } \sigma\alpha\le\min_{t\in[\delta,1-\delta]}(u(t)+v(t))\le\sigma\beta.$$

(II) If $\alpha>\beta$, then (1.1) has at least one positive solution $(u(t),v(t))$ satisfying

$$\sigma\beta\le\|(u,v)\|\le\alpha \text{ and } \sigma\beta\le\min_{t\in[\delta,1-\delta]}(u(t)+v(t)).$$

Proof. From Lemma 2.4 and Lemma 2.5, we just need to prove
(A) $\| A(u,v)\|\le\|(u,v)\|$, $\forall(u,v)\in\partial_K B^\alpha$,
(B) there exists $e\in K\setminus\{0\}$ such that

$$(u,v)\neq T(u,v)+\lambda e,\ for\ (u,v)\in\partial_K\Omega^\beta \text{ and } \lambda>0.$$

Part (I). For any $(u,v)\in\partial_K B^\alpha$, we find from (H1) and Lemma 2.1 that

$$\begin{aligned}
T_1(u,v)(t) &= \int_0^1 G_{n_1}(t,s)\varphi_{q_1}(\int_0^1 G_{n_2}(s,\tau)f_1(u(\tau),v(\tau))d\tau)ds\\
&\le \int_0^1\frac{1}{6^{n_1-1}}s(1-s)\varphi_{q_1}(\int_0^1\frac{1}{6^{n_2-1}}\tau(1-\tau)f_1(u(\tau),v(\tau))d\tau)ds\\
&\le \frac{1}{6^{n_1-1}}\int_0^1 s(1-s)\varphi_{q_1}(\int_0^1\frac{1}{6^{n_2-1}}\tau(1-\tau)\Phi_1(u(\tau),v(\tau))d\tau)ds\\
&\le \frac{1}{6^{n_1-1}}\int_0^1 s(1-s)\varphi_{q_1}(\int_0^1\frac{1}{6^{n_2-1}}\tau(1-\tau)\Phi_1(\alpha,\alpha)d\tau)ds\\
&\le \frac{1}{6^{n_1-1}}\int_0^1 s(1-s)\varphi_{q_1}(\int_0^1\frac{1}{6^{n_2-1}}\tau(1-\tau)2^{\frac{1}{1-q_1}}6^{\frac{n_1}{q_1-1}+n_2}\alpha^{p_1-1}d\tau)ds\\
&\le \frac{1}{6^{n_1}}\varphi_{q_1}(2^{\frac{1}{1-q_1}}6^{\frac{n_1}{q_1-1}}\alpha^{p_1-1})\\
&= \frac{1}{2}\alpha.
\end{aligned}$$

In a similar way, we also get

$$T_2(u,v)(t)\le\frac{1}{2}\alpha,\quad\forall(u,v)\in\partial_K B^\alpha.$$

It follows that

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| \le \alpha = \|(u,v)\|, \quad \forall (u,v) \in \partial_K B^\alpha,$$

which implies that (A) holds. Let $e=(1,1)$. Now we prove that $(u,v) \neq T(u,v) + (\lambda,\lambda)$, for $(u,v) \in \partial_K \Omega^\beta$ and $\lambda > 0$.

On the contrary, if there exist a pair of $(u_0,v_0) \in \partial_K \Omega^\beta$ and $\lambda_0 > 0$ such that

$$(u_0,v_0) = (T_1(u_0,v_0), T_2(u_0,v_0)) + (\lambda_0,\lambda_0),$$

then we from (a_4) of Lemma 2.5 that

$$\sigma\beta = \sigma\|(u_0,v_0)\| \le (u_0(t)+v_0(t)) \le \beta, \delta \le t \le 1-\delta.$$

Furthermore, for $\delta \le t \le 1-\delta$, we find from (H2) and Lemma 2.1 that

$$\begin{aligned}
u_0(t) &= T_1(u_0,v_0)(t) + \lambda_0 \\
&= \int_0^1 G_{n_1}(t,s)\varphi_{q_1}\left(\int_0^1 G_{n_2}(s,\tau) f_1(u_0(\tau),v_0(\tau))d\tau\right)ds + \lambda_0 \\
&\ge \theta_{n_1}(\delta)\varphi_{q_1}(\theta_{n_2}(\delta)) \int_\delta^{1-\delta} s(1-s)\varphi_{q_1}\left(\int_0^1 \tau(1-\tau) f_1(u_0(\tau),v_0(\tau))d\tau\right)ds + \lambda_0 \\
&\ge \theta_{n_1}(\delta)\varphi_{q_1}(\theta_{n_2}(\delta))\left(\tfrac{2}{3}\delta^3 - \delta^2 + \tfrac{1}{6}\right)\varphi_{q_1}\left(\int_\delta^{1-\delta} \tau(1-\tau)\psi_1(u_0(\tau)+v_0(\tau))d\tau\right) + \lambda_0 \\
&\ge \theta_{n_1}(\delta)\varphi_{q_1}(\theta_{n_2}(\delta))\left(\tfrac{2}{3}\delta^3 - \delta^2 + \tfrac{1}{6}\right)^{q_1}\varphi_{q_1}(\nu_1\beta^{p_1-1}) + \lambda_0 \\
&= \tfrac{1}{2}\sigma\beta + \lambda_0.
\end{aligned}$$

In a similar way, we also have

$$v_0(t) \ge T_2(u_0,v_0)(t) + \lambda_0 \ge \frac{1}{2}\sigma\beta + \lambda_0.$$

Furthermore, we get

$$\min_{t\in[\delta,1-\delta]} (u_0(t)+v_0(t)) \ge \sigma\beta + 2\lambda_0 > \sigma\beta,$$

which contradicts with the statement (a_3) of Lemma 2.5. So (B) holds.

Now, we suppose that $\alpha < \sigma\beta$. Using Lemma 2.5, one has $\overline{B_K^\alpha} \subset B_K^{\sigma\beta} \subset \Omega_K^\beta$. Using Lemma 2.4, we get that T has at least one positive fixed point $(u(t),v(t)) \in \overline{\Omega_K^\beta} \setminus B_K^\alpha$. Hence, the inequalities hold

$$\|u,v)\| \ge \alpha,$$

$$\sigma\alpha \le \min_{t\in[\delta,1-\delta]} (u(t)+v(t)) \le \sigma\beta.$$

On the other hand, since $\sigma\|(u,v)\| \le \min_{t\in[\delta,1-\delta]} (u(t)+v(t)) \le \sigma\beta$, we have $\|(u,v)\| \le \beta$. This implies that the conclusion I holds.

Finally, if $\beta < \alpha$, one has $\overline{\Omega_K^\beta} \subset B_K^\alpha$. Furthermore, Lemma 2.4 guarantees that T has at least one fixed point $(u,v) \in \overline{B_K^\alpha} \setminus \Omega_K^\beta$. Analogously, we can obtain the inequalities

$$\sigma\beta \le \| (u,v) \| \le \alpha \; and \; \sigma\beta \le \min_{t\in[\delta,1-\delta]} (u(t)+v(t)).$$

Therefore, the conclusion II holds. □

Example 3.2. Consider the following problem

$$\begin{cases} [\varphi_4(u'')]'' = \frac{u^4}{v+1} + v\arctan u^3 + (u+v)^3, \quad t\in(0,1), \\ [\varphi_4(v'')]'' = \frac{v^5}{e^u} + u\arctan v^4 + (u+v)^4, \quad t\in(0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0\le i\le n-1, n=2, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, \quad 0\le j\le m-1, m=2. \end{cases} \tag{3.1}$$

It is clear that

$$\psi_1(u+v) = (u+v)^3 < f_1(u,v) < u^4 + v\arctan u^3 + (u+v)^3 = \Phi_1(u,v),$$

and

$$\psi_2(u+v) = (u+v)^4 < f_2(u,v) < v^5 + u\arctan v^4 + (u+v)^4 = \Phi_2(u,v).$$

Since $\lim_{\alpha\to 0} = \frac{\Phi_i(\alpha,\alpha)}{\alpha^3}$ exists, there exists a sufficiently small $\alpha > 0$ such that (H1) holds. Since $\psi_1(\sigma\beta) = \sigma^3\beta^3$, $\psi_2(\sigma\beta) = \sigma^4\beta^4$, there exists a sufficiently large $\beta > 0$ such that (H2) holds. Therefore, by (I) of Theorem 3.1, problem (3.1) has a positive solution.

Example 3.3. Consider the following problem

$$\begin{cases} [\varphi_4(u'')]'' = \frac{u^2}{v+1} + v\arctan u^{\frac{1}{3}} + (u+v)^{\frac{1}{3}}, \quad t\in(0,1), \\ [\varphi_4(v'')]'' = \frac{v}{e^u} + u\arctan v^{\frac{1}{4}} + (u+v)^{\frac{1}{4}}, \quad t\in(0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 0\le i\le n-1, n=2, \\ v^{(2j)}(0) = v^{(2j)}(1) = 0, \quad 0\le j\le m-1, m=2. \end{cases} \tag{3.2}$$

It is clear that

$$\psi_1(u+v) = (u+v)^{\frac{1}{3}} < f_1(u,v) < u^2 + v\arctan u^{\frac{1}{3}} + (u+v)^{\frac{1}{3}} = \Phi_1(u,v),$$

and

$$\psi_2(u+v) = (u+v)^{\frac{1}{4}} < f_2(u,v) < v + u\arctan v^{\frac{1}{4}} + (u+v)^{\frac{1}{4}} = \Phi_2(u,v).$$

Since

$$\Phi_1(\alpha,\alpha) = \alpha^2 + \alpha\arctan\alpha^{\frac{1}{3}} + (2\alpha)^{\frac{1}{3}} \le \alpha^2 + \alpha^{\frac{4}{3}} + (2\alpha)^{\frac{1}{3}},$$

and

$$\Phi_2(\alpha,\alpha) = \alpha + \alpha\arctan\alpha^{\frac{1}{4}} + (2\alpha)^{\frac{1}{4}} \le \alpha + \alpha^{\frac{5}{4}} + (2\alpha)^{\frac{1}{4}},$$

we find that there exists a sufficiently large $\alpha > 0$ such that $\Phi_i(\alpha,\alpha) \le \alpha^3$. Since

$$\psi_1(\sigma\beta) = (\sigma\beta)^{\frac{1}{3}}, \ \psi_2(\sigma\beta) = (\sigma\beta)^{\frac{1}{4}},$$

there exists a sufficiently small $\beta > 0$ such that $\psi_i(\sigma\beta) > \beta^3$. Therefore, by (II) of Theorem 3.1, problem (3.2) has a positive solution.

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