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EXISTENCE RESULTS AND THE MONOTONE ITERATIVE TECHNIQUES FOR SYSTEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, based on the upper and lower solutions method and monotone iterative techniques, we study the existence of solutions for fractional differential systems with continuous nonlinearities and integral boundary conditions. The construction of the monotone sequences and the definition of upper and lower solutions depend on the quasimonotone property of the reaction functions. We prove the existence of maximal and minimal solutions for quasimonotone increasing systems. Also, we prove the existence of maximal-minimal and minimal-maximal solutions for quasimonotone decreasing systems and for mixed quasimonotone systems and the existence of at least one solution. Finally, we give some examples to illustrate our results.

Keywords. Riemann-Liouville fractional derivative; Fractional differential systems; Integral boundary condition; Upper and lower solutions; Monotone iterative technique.

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1. INTRODUCTION

The purpose of this paper is to study the existence of solutions for a class of first order fractional differential systems subject to integral boundary conditions. More precisely, we consider the following nonlinear boundary value problem

$$\begin{cases} D^{\alpha_1}u(t) = f(t,u,v),\ t \in (0,1], \\ D^{\alpha_2}v(t) = g(t,u,v),\ t \in (0,1], \\ t^{1-\alpha_1}u(t)\,|_{t=0} = \int_0^1 g_1(s)u(s)ds, \\ t^{1-\alpha_2}v(t)\,|_{t=0} = \int\limits_0^1 g_2(s)\,v(s)\,ds, \end{cases} \tag{1.1}$$

where D^{α_i} is the Riemann-Liouville fractional derivative of order α_i with $0 < \alpha_i < 1$ for $i = 1,\ 2$, $f : (0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $g : (0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $g_i : [0,1] \to \mathbb{R}_+$ $(i = 1,2)$ are continuous functions.

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Fractional differential equations or fractional differential systems arise in many scientific fields such as plasticity, viscoelasticity, electrical circuits, electroanalytical chemistry, biology, control theory, pharmacodynamics, pharmacokinetics, electromagnetic theory, biomedical problems, psychological and life sciences; see [1], [2], [3], [4], [5], [6] and the references therein. Fractional differential systems with nonlocal boundary conditions have been studied by several authors using the upper and lower solutions method, the monotone iterative methods and fixed point theorems in cones; see [7], [8], [9], [10] and the references therein.

It is well know that the method of upper and lower solutions coupled with monotone iterative techniques has been used to prove existence of solutions of nonlinear boundary value problems by various authors; see [10], [11], [12], [13], [14], [15], [16] and [17]. Observe that the definition itself of upper and lower solutions of (1.1) and the construction of the monotone sequences depend on the quasimonotone property of the reaction functions f and g. Thus, following Pao's notation in [16], we can classify (1.1) according to their relative monotony as follows:

Type 1. Quasimonotone increasing systems: f is increasing in v and g in u.

Type 2. Quasimonotone decreasing systems: f is decreasing in v and g in u.

Type 3. Mixed quasimonotone systems: f is increasing in v and g is decreasing in u or vice versa.

In this paper, we prove the existence of maximal and minimal solutions for systems of Type 1. These solutions are the limits of two monotone sequences. Also, we prove the existence of maximal-minimal and minimal-maximal solutions for systems of Type 2. When the system is of type 3 and if α_1=α_2, we prove the existence of at least one solution. Our results improve and generalize the results obtained in [10], [13] and [17]. We note also that to the best of our knowledge this is the first paper which uses the method of upper and lower solutions coupled with monotone iterative techniques to prove the existence of solutions for mixed quasimonotone systems with integral boundary conditions.

The plan of this paper is as follows: In Section 2, we give some preliminary results that will be used throughout the paper. In Section 3, we study the existence of maximal and minimal solutions for quasimonotone increasing systems. Section 4 is concerned with the existence of maximal-minimal and minimal-maximal solutions for quasimonotone decreasing systems. In Section 5, we study the existence of solutions for mixed quasimonotone systems. Finally in Section 6, we give some examples to illustrate our results.

2. Preliminaries

In this section, we give some definitions and preliminary results that will be used in the remainder of this paper.

Definition 2.1. Let $0 < q < 1$. We denote by $C_{1-q}([0,1])$, the function space

$$C_{1-q}([0,1]) = \left\{u \in C((0,1]);\ t^{1-q}u(t) \in C([0,1])\right\}.$$

Definition 2.2. For $u \in C_{1-q}([0,1])$, we define the weighted norm by

$$\|u\| = \max_{t\in[0,1]} \left|t^{1-q}u(t)\right|.$$

Remark 2.3. $(C_{1-q}([0,1]), \|.\|)$ is a Banach space.

Definition 2.4. Let $0<q<1$ and let $h\in C_{1-q}([0,1])$. The Riemann-Liouville integral of order q of h is defined by

$$I^{q}h(t):=\frac{1}{\Gamma(q)}\int_{0}^{t}(t-s)^{q-1}h(s)\,ds,$$

where Γ is the Gamma Euler function defined by

$$\Gamma(z)=\int_{0}^{+\infty}\exp(-t)\,t^{z-1}dz,$$

where $z\in\mathbb{C}$ with $\Re(z)>0$.

Definition 2.5. Let $0<q<1$ and let $h\in C_{1-q}([0,1])$. The Riemann-Liouville fractional derivative of order q of h is defined by $D^{q}h(t)=\frac{d}{dt}I^{1-q}h(t)$.

Now, we consider this following initial problem

$$\begin{cases} D^{q}u(t)+Mu(t)=\widetilde{h}(t),\ t\in(0,1],\\ t^{1-q}u(t)|_{t=0}=\widetilde{u_0}, \end{cases} \tag{2.1}$$

where $0<q<1$, M is a positive constant, $\widetilde{h}\in C_{1-q}([0,1])$ and $\widetilde{u_0}\in\mathbb{R}$.

We have the following results.

Lemma 2.6. [5] *Problem (2.1) admits a unique solution u which is given by*

$$u(t)=\Gamma(q)\widetilde{u_0}t^{q-1}E_{q,q}(-Mt^{q})+\int_0^t(t-s)^{q-1}E_{q,q}(-M(t-s)^{q})\widetilde{h}(s)ds,$$

where $E_{q,q}$ is the Mittag-Leffler function defined by

$$E_{q,q}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(q(n+1))},\ z\in\mathbb{C}.$$

Remark 2.7. If u is a solution of (2.1), then $u\in C_{1-q}([0,1])$.

Now, for $0<q<1$, we define the Mittag-Leffler function E_q by

$$E_{q}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(qn+1)},\ z\in\mathbb{C}.$$

We have the following result.

Theorem 2.8. [18, Theorem 4.2.] *For $0<q<1$, E_q has no zeros on the real axis; $0<E_q(x)<1$ for $x<0$ and $\frac{d}{dx}E_q(x)>0$ for the whole real axis.*

Lemma 2.9. *For $0<q<1$, one has*

(i) $E_{q,q}(-x)=-q\frac{d}{dx}E_q(-x)$, *for* $x\geq 0$,
(ii) $E_{q,q}(-x)>0$, *for* $x\geq 0$.

Proof. Letting $0<q<1$ and $x\geq 0$, we have

(i)

$$\begin{aligned}\frac{d}{dx}E_q(-x) &= \frac{d}{dx}\sum_{n=0}^{+\infty}\frac{(-x)^n}{\Gamma(qn+1)}\\ &= \sum_{n=1}^{+\infty}\frac{-n(-x)^{n-1}}{\Gamma(qn+1)}\\ &= \sum_{n=1}^{+\infty}\frac{-n(-x)^{n-1}}{nq\Gamma(qn)}\\ &= \frac{-1}{q}\sum_{n=0}^{+\infty}\frac{(-x)^n}{\Gamma(q(n+1))}\\ &= \frac{-E_{q,q}(-x)}{q}.\end{aligned}$$

Then

$$E_{q,q}(-x) = -q\frac{d}{dx}E_q(-x).$$

(ii) By Theorem 2.8, we have

$$\frac{d}{dx}E_q(-x) = -\frac{d}{d(-x)}E_q(-x) < 0.$$

Using (i), one finds that $E_{q,q}(-x) > 0$.

□

Lemma 2.10. *Let $u \in C_{1-q}([0,1])$ with $0 < q < 1$ and assume that u satisfies*

$$\begin{cases} D^q u(t) + Mu(t) \geq 0, \ t \in (0,1], \\ t^{1-q}u(t)|_{t=0} \geq 0. \end{cases} \tag{2.2}$$

Then $t^{1-q}u(t) \geq 0$ for all $t \in [0,1]$.

Proof. The proof of this Lemma is a consequence of Lemma 2.6 and (ii) of Lemma 2.9. □

Lemma 2.11. *Let $u \in C_{1-q}([0,1])$ with $0 < q < 1$ and assume that u satisfies*

$$\begin{cases} D^q u(t) + \widetilde{M}u(t) \leq 0, \ t \in (0,1], \\ t^{1-q}u(t)|_{t=0} \leq \int\limits_0^1 \widetilde{g}(s)u(s)\,ds, \end{cases} \tag{2.3}$$

where $\widetilde{M} \leq 0$ and $\widetilde{g} : [0,1] \to \mathbb{R}_+$ is a continuous function such that

$$\Gamma(q)\int_0^1 s^{q-1}E_{q,q}(-\widetilde{M}s^q)\widetilde{g}(s)\,ds < 1.$$

Then $t^{1-q}u(t) \leq 0$ for all $t \in [0,1]$.

Proof. Let $u \in C_{1-q}([0,1])$ with $0 < q < 1$ and assume the hypothesis of the lemma are satisfied. By Lemma 2.6, we have

$$u(t) \leq \Gamma(q)\int_0^1 \widetilde{g}(s)u(s)\,ds t^{q-1}E_{q,q}(-\widetilde{M}t^q), \tag{2.4}$$

which implies that

$$\int_0^1 \widetilde{g}(s)u(s)\,ds \leq \Gamma(q)\int_0^1 \widetilde{g}(s)u(s)\,ds\int_0^1 s^{q-1}E_{q,q}(-\widetilde{M}s^q)\widetilde{g}(s)\,ds.$$

Then, we have

$$\int_0^1 \widetilde{g}(s)\,u(s)\,ds\left(1-\Gamma(q)\int_0^1 s^{q-1}E_{q,q}(-\widetilde{M}s^q)\widetilde{g}(s)\,ds\right)\leq 0.$$

Since $\Gamma(q)\int_0^1 s^{q-1}E_{q,q}(-\widetilde{M}s^q)\widetilde{g}(s)\,ds<1$, we obtain $\int_0^1 \widetilde{g}(s)\,u(s)\,ds\leq 0$. Using (2.4), one finds $t^{1-q}u(t)\leq 0$ for all $t\in[0,1]$. □

3. Existence of minimal and maximal solutions for systems with quasimonotone increasing functions

In this section, we study the existence of minimal and maximal solutions of system (1.1) when the nonlinearities f and g are quasimonotone increasing.

On the nonlinearities f and g, we impose the following conditions

- (H1) $f:(0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function and there exists $M_1>0$ such that the function $u\longmapsto f(t,u,v)+M_1u$ is increasing for all $t\in(0,1]$ and $v\in\mathbb{R}$.
- (H2) $g:(0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function and there exists $M_2>0$ such that the function $v\longmapsto g(t,u,v)+M_2v$ is increasing for all $t\in(0,1]$ and $u\in\mathbb{R}$.
- (H3) $f(t,u,v)$ is increasing in v for all fixed $t\in(0,1]$, and $u\in\mathbb{R}$.
- (H4) $g(t,u,v)$ is increasing in u for all fixed $t\in(0,1]$, and $v\in\mathbb{R}$.
- (H5) The function $t\mapsto t^{1-\alpha_1}f(t,u(t),v(t))$ is a continuous function in $[0,1]$ for all $u\in C_{1-\alpha_1}([0,1])$ and $v\in C_{1-\alpha_2}([0,1])$.
- (H6) The function $t\mapsto t^{1-\alpha_2}g(t,u(t),v(t))$ is a continuous function in $[0,1]$ for all $u\in C_{1-\alpha_1}([0,1])$ and $v\in C_{1-\alpha_2}([0,1])$.

Definition 3.1. We say that (u,v) is a solution of (1.1) if

- (i) $(u,v)\in C_{1-\alpha_1}([0,1])\times C_{1-\alpha_2}([0,1])$,
- (ii) (u,v) satisfies (1.1).

Definition 3.2. We say that $(\underline{u},\underline{v})$,$(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 1 for problem (1.1) if

(i) $(\underline{u},\overline{u})\in(C_{1-\alpha_1}([0,1]))^2$ and $(\underline{v},\overline{v})\in(C_{1-\alpha_2}([0,1]))^2$,

(ii) $$\begin{cases} D^{\alpha_1}\underline{u}(t)\leq f(t,\underline{u},\underline{v}),\ t\in(0,1],\\ D^{\alpha_1}\overline{u}(t)\geq f(t,\overline{u},\overline{v}),\ t\in(0,1],\\ D^{\alpha_2}\underline{v}(t)\leq g(t,\underline{u},\underline{v}),\ t\in(0,1],\\ D^{\alpha_2}\overline{v}(t)\geq g(t,\overline{u},\overline{v}),\ t\in(0,1],\end{cases}$$

(iii) $$\begin{cases} t^{1-\alpha_1}\underline{u}(t)|_{t=0}\leq\int_0^1 g_1(s)\underline{u}(s)ds,\\ t^{1-\alpha_1}\overline{u}(t)|_{t=0}\geq\int_0^1 g_1(s)\overline{u}(s)ds,\\ t^{1-\alpha_2}\underline{v}(t)|_{t=0}\leq\int_0^1 g_2(s)\underline{v}(s)ds,\\ t^{1-\alpha_2}\overline{v}(t)|_{t=0}\geq\int_0^1 g_2(s)\overline{v}(s)ds.\end{cases}$$

Theorem 3.3. *Assume that the hypothesis (Hi) for i=1,...,6 are satisfied and let $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ be a pair of lower-upper solutions of type 1 for problem (1.1) such that $t^{1-\alpha_1}\underline{u}(t)\leq t^{1-\alpha_1}\overline{u}(t)$ and $t^{1-\alpha_2}\underline{v}(t)\leq t^{1-\alpha_2}\overline{v}(t)$, $\forall t\in[0,1]$. Then problem (1.1) has a minimal solution (u_*,v_*) and a maximal solution (u^*,v^*) such that for every solution (u,v) of (1.1) with $t^{1-\alpha_1}\underline{u}(t)\leq t^{1-\alpha_1}u(t)\leq t^{1-\alpha_1}\overline{u}(t)$ and $t^{1-\alpha_2}\underline{v}(t)\leq t^{1-\alpha_2}v(t)\leq t^{1-\alpha_2}\overline{v}(t)$, $\forall t\in[0,1]$, we have*

$$t^{1-\alpha_1}\underline{u}(t)\leq t^{1-\alpha_1}u_*(t)\leq t^{1-\alpha_1}u(t)\leq t^{1-\alpha_1}u^*(t)\leq t^{1-\alpha_1}\overline{u}(t)\ \text{in}\ [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}(t) \le t^{1-\alpha_2}v_*(t) \le t^{1-\alpha_2}v(t) \le t^{1-\alpha_2}v^*(t) \le t^{1-\alpha_2}\overline{v}(t) \text{ in } [0,1].$$

Proof. The proof will be given in several steps.

We take $\underline{u}_0 = \underline{u}$, $\underline{v}_0 = \underline{v}$ and we define the sequences of functions $\{\underline{u}_n\}_{n\ge1}$, $\{\underline{v}_n\}_{n\ge1}$ by

$$\begin{cases} D^{\alpha_1}\underline{u}_{n+1}(t) + M_1\underline{u}_{n+1}(t) = \widetilde{f}_n(t),\ t \in (0,1], \\ t^{1-\alpha_1}\underline{u}_{n+1}(t)|_{t=0} = \int_0^1 g_1(s)\underline{u}_n(s)ds, \end{cases} \tag{3.1}$$

$$\begin{cases} D^{\alpha_2}\underline{v}_{n+1}(t) + M_2\underline{v}_{n+1}(t) = \widetilde{g}_n(t),\ t \in (0,1], \\ t^{1-\alpha_2}\underline{v}_{n+1}(t)|_{t=0} = \int_0^1 g_2(s)\underline{v}_n(s)ds, \end{cases} \tag{3.2}$$

where

$$\widetilde{f}_n(t) := f(t,\underline{u}_n(t),\underline{v}_n(t)) + M_1\underline{u}_n(t),$$

and

$$\widetilde{g}_n(t) := g(t,\underline{u}_n(t),\underline{v}_n(t)) + M_2\underline{v}_n(t).$$

Analogously, we take $\overline{u}_0 = \overline{u}$, $\overline{v}_0 = \overline{v}$ and we define the sequences of functions $\{\overline{u}_n\}_{n\ge1}$, $\{\overline{v}_n\}_{n\ge1}$ by

$$\begin{cases} D^{\alpha_1}\overline{u}_{n+1}(t) + M_1\overline{u}_{n+1}(t) = \widehat{f}_n(t),\ t \in (0,1], \\ t^{1-\alpha_1}\overline{u}_{n+1}(t)|_{t=0} = \int_0^1 g_1(s)\overline{u}_n(s)ds, \end{cases} \tag{3.3}$$

$$\begin{cases} D^{\alpha_2}\overline{v}_{n+1}(t) + M_2\overline{v}_{n+1}(t) = \widehat{g}_n(t),\ t \in (0,1], \\ t^{1-\alpha_2}\overline{v}_{n+1}(t)|_{t=0} = \int_0^1 g_2(s)\overline{v}_n(s)ds, \end{cases} \tag{3.4}$$

where

$$\widehat{f}_n(t) := f(t,\overline{u}_n(t),\overline{v}_n(t)) + M_1\overline{u}_n(t),$$

and

$$\widehat{g}_n(t) := g(t,\overline{u}_n(t),\overline{v}_n(t)) + M_2\overline{v}_n(t).$$

Step 1. The sequences of functions $\{(\underline{u}_n,\underline{v}_n)\}_{n\in\mathbb{N}}$ is well defined. For all $n \in \mathbb{N}$, we have

$$(\underline{u}_n,\underline{v}_n) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1]).$$

For $n = 0$, we have $(\underline{u}_0,\underline{v}_0) = (\underline{u},\underline{v})$ and $(\underline{u},,\underline{v}) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1])$. Assume that for fixed $n \ge 1$, $(\underline{u}_n,\underline{v}_n)$ is well defined. It follows that

$$(\underline{u}_n,\underline{v}_n) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1]).$$

We show that $(\underline{u}_{n+1},\underline{v}_{n+1})$ is well defined and we have

$$(\underline{u}_{n+1},\underline{v}_{n+1}) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1]).$$

By (3.1) and (3.2), we have

$$\begin{cases} D^{\alpha_1}\underline{u}_{n+1}(t) + M_1\underline{u}_{n+1}(t) = \widetilde{f}_n(t),\ t \in (0,1], \\ t^{1-\alpha_1}\underline{u}_{n+1}(t)|_{t=0} = \int_0^1 g_1(s)\underline{u}_n(s)ds, \end{cases}$$

$$\begin{cases} D^{\alpha_2}\underline{v}_{n+1}(t) + M_2\underline{v}_{n+1}(t) = \widetilde{f}_n(t),\ t \in (0,1], \\ t^{1-\alpha_2}\underline{v}_{n+1}(t)|_{t=0} = \int_0^1 g_2(s)\underline{v}_n(s)ds. \end{cases}$$

Since $(\underline{u}_n, \underline{v}_n) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1])$ and using hypothesis (H1), (H2), (H5) and (H6), we obtain that the functions $F_n \in C([0,1])$ and $G_n \in C([0,1])$, where

$$F_n(t) := t^{1-\alpha_1} \widetilde{f}_n(t),$$

and

$$G_n(t) := t^{1-\alpha_2} \widetilde{g}_n(t).$$

Then by Lemma 2.6, we find that $\left(\underline{u}_{n+1}, \underline{v}_{n+1}\right)$ is well defined. Hence,

$$\left(\underline{u}_{n+1}, \underline{v}_{n+1}\right) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1]).$$

For all $n \in \mathbb{N}$, the sequence of functions $\{(\underline{u}_n, \underline{v}_n)\}_{n\in\mathbb{N}}$ is well defined. So

$$(\underline{u}_n, \underline{v}_n) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1]).$$

Step 2. The sequence of functions $\{(\overline{u}_n, \overline{v}_n)\}_{n\in\mathbb{N}}$ is well defined. For all $n \in \mathbb{N}$, we have

$$(\overline{u}_n, \overline{v}_n) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1]).$$

The proof is similar to that of Step 1, so it is omitted here.

Step 3. For all $n \in \mathbb{N}$, we have

$$t^{1-\alpha_1} \underline{u}_n(t) \le t^{1-\alpha_1} \underline{u}_{n+1}(t) \le t^{1-\alpha_1} \overline{u}_{n+1}(t) \le t^{1-\alpha_1} \overline{u}_n(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2} \underline{v}_n(t) \le t^{1-\alpha_2} \underline{v}_{n+1}(t) \le t^{1-\alpha_2} \overline{v}_{n+1}(t) \le t^{1-\alpha_2} \overline{v}_n(t) \text{ in } [0,1].$$

Let

$$w_0(t) := \underline{u}_1(t) - \underline{u}_0(t) \text{ and } z_0(t) := \underline{v}_1(t) - \underline{v}_0(t),\, t \in (0,1],$$
$$t^{1-\alpha_1} w_0(t)|_{t=0} = t^{1-\alpha_1} \underline{u}_1(t)|_{t=0} - t^{1-\alpha_1} \underline{u}_0(t)|_{t=0},$$

and

$$t^{1-\alpha_2} z_0(t)|_{t=0} = t^{1-\alpha_2} \underline{v}_1(t)|_{t=0} - t^{1-\alpha_2} \underline{v}_0(t)|_{t=0}.$$

By (3.1), (3.2) and using the Definition 3.2, we have

$$\begin{cases} D^{\alpha_1} w_0(t) + M_1 w_0(t) \ge 0,\, t \in (0,1], \\ t^{1-\alpha_1} w_0(t)|_{t=0} \ge 0, \end{cases}$$

and

$$\begin{cases} D_0^{\alpha_2} z_0(t) + M_1 z_0(t) \ge 0,\, t \in (0,1], \\ t^{1-\alpha_2} z_0(t)|_{t=0} \ge 0. \end{cases}$$

Using Lemma 2.10, we obtain

$$t^{1-\alpha_1} w_0(t) \ge 0 \text{ and } t^{1-\alpha_2} z_0(t) \ge 0 \text{ in } [0,1],$$

that is,

$$t^{1-\alpha_1} \underline{u}_0(t) \le t^{1-\alpha_1} \underline{u}_1(t) \text{ in } [0,1], \tag{3.5}$$

and

$$t^{1-\alpha_2} \underline{v}_0(t) \le t^{1-\alpha_2} \underline{v}_1(t) \text{ in } [0,1]. \tag{3.6}$$

Similarly, we can prove that

$$t^{1-\alpha_1} \overline{u}_1(t) \le t^{1-\alpha_1} \overline{u}_0(t) \text{ in } [0,1], \tag{3.7}$$

and

$$t^{1-\alpha_2}\overline{v}_1(t) \leq t^{1-\alpha_2}\overline{v}_0(t) \text{ in } [0,1]. \tag{3.8}$$

Now, we put by definition

$$p_1(t) = \underline{u}_1(t) - \overline{u}_1(t) \text{ and } y_1(t) = \underline{v}_1(t) - \overline{v}_1(t), t \in (0,1],$$
$$t^{1-\alpha_1} p_1(t)|_{t=0} = t^{1-\alpha_1}\underline{u}_1(t)|_{t=0} - t^{1-\alpha_1}\overline{u}_1(t)|_{t=0},$$

and

$$t^{1-\alpha_2} y_1(t)|_{t=0} = t^{1-\alpha_2}\underline{v}_1(t)|_{t=0} - t^{1-\alpha_2}\overline{v}_1(t)|_{t=0}.$$

By (3.1), (3.2), (3.3) and (3.4), we have

$$\begin{cases} D^{\alpha_1} p_1(t) + M_1 p_1(t) = \widetilde{f}_0(t) - \widehat{f}_0(t), t \in (0,1], \\ t^{1-\alpha_1} p_1(t)|_{t=0} = \int_0^1 g_1(s)\left(\underline{u}_0(s) - \overline{u}_0(s)\right) ds, \end{cases}$$

and

$$\begin{cases} D^{\alpha_2} y_1(t) + M_2 y_1(t) = \widetilde{g}_0(t) - \widehat{g}_0(t), t \in (0,1], \\ t^{1-\alpha_2} y_1(t)|_{t=0} = \int_0^1 g_2(s)\left(\underline{v}_0(s) - \overline{v}_0(s)\right) ds. \end{cases}$$

Since

$$t^{1-\alpha_1}\underline{u}_0(t) = t^{1-\alpha_1}\underline{u}(t) \leq t^{1-\alpha_1}\overline{u}(t) = t^{1-\alpha_1}\overline{u}_0(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}_0(t) = t^{1-\alpha_2}\underline{v}(t) \leq t^{1-\alpha_2}\overline{v}(t) = t^{1-\alpha_2}\overline{v}_0(t) \text{ in } [0,1],$$

and using the hypothesis (Hi) for i=1,...,4, we obtain

$$\begin{cases} D^{\alpha_1} p_1(t) + M_1 p_1(t) \leq 0,\ t \in (0,1], \\ t^{1-\alpha_1} p_1(t)|_{t=0} = \int_0^1 g_1(s) s^{\alpha_1-1} s^{1-\alpha_1}\left(\underline{u}_0(s) - \overline{u}_0(s)\right) ds \leq 0, \end{cases}$$

and

$$\begin{cases} D^{\alpha_2} y_1(t) + M_2 y_1(t) \leq 0,\ t \in (0,1], \\ t^{1-\alpha_1} y_1(t)|_{t=0} = \int_0^1 g_2(s) s^{\alpha_2-1} s^{1-\alpha_2}\left(\underline{v}_0(s) - \overline{v}_0(s)\right) ds \leq 0. \end{cases}$$

Using Lemma 2.10, we obtain

$$t^{1-\alpha_1} p_1(t) \leq 0 \text{ and } t^{1-\alpha_2} y_1(t) \leq 0 \text{ in } [0,1],$$

that is,

$$t^{1-\alpha_1}\underline{u}_1(t) \leq t^{1-\alpha_1}\overline{u}_1(t) \text{ in } [0,1], \tag{3.9}$$

and

$$t^{1-\alpha_2}\underline{v}_1(t) \leq t^{1-\alpha_2}\overline{v}_1(t) \text{ in } [0,1]. \tag{3.10}$$

In view of (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10), we have

$$t^{1-\alpha_1}\underline{u}_0(t) \leq t^{1-\alpha_1}\underline{u}_1(t) \leq t^{1-\alpha_1}\overline{u}_1(t) \leq t^{1-\alpha_1}\overline{u}_0(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}_0(t) \leq t^{1-\alpha_2}\underline{v}_1(t) \leq t^{1-\alpha_2}\overline{v}_1(t) \leq t^{1-\alpha_2}\overline{v}_0(t) \text{ in } [0,1].$$

Assume, for fixed $n \geq 1$, that

$$t^{1-\alpha_1}\underline{u}_n(t) \leq t^{1-\alpha_1}\underline{u}_{n+1}(t) \leq t^{1-\alpha_1}\overline{u}_{n+1}(t) \leq t^{1-\alpha_1}\overline{u}_n(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}_n(t) \leq t^{1-\alpha_2}\underline{v}_{n+1}(t) \leq t^{1-\alpha_2}\overline{v}_{n+1}(t) \leq t^{1-\alpha_2}\overline{v}_n(t) \text{ in } [0,1].$$

We show that

$$t^{1-\alpha_1}\underline{u}_{n+1}(t) \le t^{1-\alpha_1}\underline{u}_{n+2}(t) \le t^{1-\alpha_1}\overline{u}_{n+2}(t) \le t^{1-\alpha_1}\overline{u}_{n+1}(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}_{n+1}(t) \le t^{1-\alpha_2}\underline{v}_{n+2}(t) \le t^{1-\alpha_2}\overline{v}_{n+2}(t) \le t^{1-\alpha_2}\overline{v}_{n+1}(t) \text{ in } [0,1].$$

We put by definition

$$w_{n+1}(t) := \underline{u}_{n+2}(t) - \underline{u}_{n+1}(t) \text{ and } z_{n+1}(t) := \underline{v}_{n+2}(t) - \underline{v}_{n+1}(t),\, t \in (0,1],$$
$$t^{1-\alpha_1} w_{n+1}(t)|_{t=0} = t^{1-\alpha_1}\underline{u}_{n+2}(t)|_{t=0} - t^{1-\alpha_1}\underline{u}_{n+1}(t)|_{t=0},$$

and

$$t^{1-\alpha_2} z_{n+1}(t)|_{t=0} = t^{1-\alpha_2}\underline{v}_{n+2}(t)|_{t=0} - t^{1-\alpha_2}\underline{v}_{n+1}(t)|_{t=0}.$$

By (3.1) and (3.2), we have

$$\begin{cases} D^{\alpha_1} w_{n+1}(t) + M_1 w_{n+1}(t) = \widetilde{f}_{n+1}(t) - \widetilde{f}_n(t),\, t \in (0,1], \\ t^{1-\alpha_1} w_{n+1}(t)|_{t=0} = \int_0^1 g_1(s)\left(\underline{u}_{n+1}(s) - \underline{u}_n(s)\right) ds, \end{cases}$$

and

$$\begin{cases} D^{\alpha_2} z_{n+1}(t) + M_2 z_{n+1}(t) = \widetilde{g}_{n+1}(t) - \widetilde{g}_n(t),\, t \in (0,1], \\ t^{1-\alpha_2} z_{n+1}(t)|_{t=0} = \int_0^1 g_2(s)\left(\underline{v}_{n+1}(s) - \underline{v}_{n+1}(s)\right) ds. \end{cases}$$

Since by the hypothesis of recurrence, we have $t^{1-\alpha_1}\underline{u}_n(t) \le t^{1-\alpha_1}\underline{u}_{n+1}(t)$ and $t^{1-\alpha_2}\underline{v}_n(t) \le t^{1-\alpha_2}\underline{v}_{n+1}(t)$ in $[0,1]$. Using the hypothesis (Hi) for $i = 1,...,4$, we obtain

$$\begin{cases} D^{\alpha_1} w_{n+1}(t) + M_1 w_{n+1}(t) \ge 0,\ t \in (0,1], \\ t^{1-\alpha_1} w_{n+1}(t)|_{t=0} \ge 0, \end{cases}$$

and

$$\begin{cases} D^{\alpha_2} z_{n+1}(t) + M_2 z_{n+1}(t) \ge 0,\ t \in (0,1], \\ t^{1-\alpha_2} z_{n+1}(t)|_{t=0} \ge 0. \end{cases}$$

Using Lemma 2.10, we obtain

$$t^{1-\alpha_1} w_{n+1}(t) \ge 0 \text{ and } t^{1-\alpha_2} z_{n+1}(t) \ge 0 \text{ in } [0,1],$$

that is,

$$t^{1-\alpha_1}\underline{u}_{n+1}(t) \le t^{1-\alpha_1}\underline{u}_{n+2}(t) \text{ and } t^{1-\alpha_2}\underline{v}_{n+1}(t) \le t^{1-\alpha_2}\underline{v}_{n+2}(t) \text{ in } [0,1]. \tag{3.11}$$

Similarly, we can prove that

$$t^{1-\alpha_1}\overline{u}_{n+2}(t) \le t^{1-\alpha_1}\overline{u}_{n+1}(t) \text{ and } t^{1-\alpha_2}\overline{v}_{n+2}(t) \le t^{1-\alpha_2}\overline{v}_{n+1}(t) \text{ in } [0,1], \tag{3.12}$$

and

$$t^{1-\alpha_1}\underline{u}_{n+2}(t) \le t^{1-\alpha_1}\overline{u}_{n+2}(t) \text{ and } t^{1-\alpha_2}\underline{v}_{n+2}(t) \le t^{1-\alpha_2}\overline{v}_{n+2}(t) \text{ in } [0,1]. \tag{3.13}$$

In view of (3.11), (3.12) and (3.13), we have

$$t^{1-\alpha_1}\underline{u}_{n+1}(t) \le t^{1-\alpha_1}\underline{u}_{n+2}(t) \le t^{1-\alpha_1}\overline{u}_{n+2}(t) \le t^{1-\alpha_1}\overline{u}_{n+1}(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}_{n+1}(t) \le t^{1-\alpha_2}\underline{v}_{n+2}(t) \le t^{1-\alpha_2}\overline{v}_{n+2}(t) \le t^{1-\alpha_2}\overline{v}_{n+1}(t) \text{ in } [0,1].$$

Hence for all $n \in \mathbb{N}$, we have

$$t^{1-\alpha_1}\underline{u}_n(t) \le t^{1-\alpha_1}\underline{u}_{n+1}(t) \le t^{1-\alpha_1}\overline{u}_{n+1}(t) \le t^{1-\alpha_1}\overline{u}_n(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}_n(t) \le t^{1-\alpha_2}\underline{v}_{n+1}(t) \le t^{1-\alpha_2}\overline{v}_{n+1}(t) \le t^{1-\alpha_2}\overline{v}_n(t) \text{ in } [0,1].$$

The proof of Step 3 is complete.

Step 4. The sequences of functions $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$, $\{(t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ are uniformly bounded on $[0,1]$.

Letting $n\in\mathbb{N}$ and using Step 3, we have

$$t^{1-\alpha_1}\underline{u}(t) \le t^{1-\alpha_1}\underline{u}_n(t) \le t^{1-\alpha_1}\overline{u}(t) \text{ in } [0,1]. \tag{3.14}$$

Since the functions $t \longmapsto t^{1-\alpha_1}\underline{u}(t)$ and $t \longmapsto t^{1-\alpha_1}\overline{u}(t)$ are continuous on $[0,1]$, we find that they are bounded on $[0,1]$. Hence, the sequence of functions $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$ is uniformly bounded on $[0,1]$. Similarly, we can prove that the sequences of functions $\{(t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ is uniformly bounded on $[0,1]$.

Step 5. The sequences of functions $\{t^{1-\alpha_1}\overline{u}_n\}_{n\in\mathbb{N}}$ and $\{t^{1-\alpha_2}\overline{v}_n\}_{n\in\mathbb{N}}$ are uniformly bounded on $[0,1]$.

The proof is similar to that of Step 4, so it is omitted.

Now we are in a position to prove that the sequences of functions $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$ and $\{t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ are equicontinuous on $[0,1]$. The idea of the proof is similar to that used in [19, Page 5].

Step 6. The sequences of functions $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$ and $\{t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ are equicontinuous on $[0,1]$.

Let $\varepsilon>0$, $t_1, t_2\in[0,1]$ such that $t_1<t_2$ and $n\in\mathbb{N}$. First, we have

$$\lim_{t\to 0} t^{1-\alpha_1}\int_0^t (t-s)^{\alpha_1-1}s^{\alpha_1-1}ds = \lim_{t\to 0} t^{\alpha_1}\beta(\alpha_1,\alpha_1) = 0,$$

where β is the Beta Euler function defined by

$$\beta(z_1,z_2) = \int_0^1 (1-s)^{z_1-1}s^{z_2-1}ds,$$

where $z_1\in\mathbb{C}$, $z_2\in\mathbb{C}$ with $\Re(z_1)>0$ and $\Re(z_2)>0$. Then there exists $\delta_1>0$ such that for $0<t<\delta_1$, we have

$$t^{1-\alpha_1}\int_0^t (t-s)^{\alpha_1-1}s^{\alpha_1-1}ds < \varepsilon.$$

We distinguish two cases

Case 1. $0\le t_1<t_2<\delta_1$. By (3.1) and using Theorem 3.10 in [4], we have

$$\begin{aligned} &t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2) - t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1) \\ &= \frac{t_2^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_1-1}\widetilde{F}_n(s)\,ds - \frac{t_1^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_1-1}\widetilde{F}_n(s)\,ds, \end{aligned}$$

where

$$\widetilde{F}_n(s) := f(s,\underline{u}_n(s),\underline{v}_n(s)) + M_1\left(\underline{u}_n(s)-\underline{u}_{n+1}(s)\right).$$

It follows that

$$\begin{aligned} &\left|t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2) - t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1)\right| \\ &\le \frac{t_2^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_1-1}\left|\widetilde{F}_n(s)\right|ds + \frac{t_1^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_1-1}\left|\widetilde{F}_n(s)\right|ds, \end{aligned}$$

In view of Step 4 and hypothesis (H5), we have

$$\exists \widehat{M}>0, \forall n\in\mathbb{N}, \forall s\in[0,1], \left|s^{1-\alpha_1}\widetilde{F}_n(s)\right| \le \widehat{M},$$

which implies that

$$
\begin{aligned}
&\left|t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2)-t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1)\right|\\
&\leq \frac{\widetilde{M}}{\Gamma(\alpha_1)}\left(t_2^{1-\alpha_1}\int_0^{t_2}(t_2-s)^{\alpha_1-1}s^{\alpha_1-1}ds+t_1^{1-\alpha_1}\int_0^{t_1}(t_1-s)^{\alpha_1-1}s^{\alpha_1-1}ds\right)\\
&\leq \frac{2\widetilde{M}}{\Gamma(\alpha_1)}\varepsilon.
\end{aligned}
$$

Case 2. $\delta_1 \leq t_1 < t_2 \leq 1$. In this case, we have

$$
\begin{aligned}
&t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2)-t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1)\\
&=\tfrac{1}{\Gamma(\alpha_1)}\int_0^{t_1}\left(t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}-t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}\right)\widetilde{F}_n(s)\,ds+\tfrac{t_2^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\widetilde{F}_n(s)\,ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left|t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2)-t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1)\right| &\leq \frac{\widehat{M}}{\Gamma(\alpha_1)}\int_0^{t_1}\left|t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}-t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}\right|s^{\alpha_1-1}ds\\
&+\frac{\widehat{M}t_2^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}s^{\alpha_1-1}ds\\
&\leq \frac{\widehat{M}}{\Gamma(\alpha_1)}\int_0^{t_1}\left|t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}-t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}\right|s^{\alpha_1-1}ds\\
&+\frac{\widehat{M}t_2^{1-\alpha_1}}{\Gamma(\alpha_1+1)}\delta_1^{\alpha_1-1}(t_2-t_1)^{\alpha_1}\\
&\leq \frac{\widehat{M}}{\Gamma(\alpha_1)}\int_0^{\delta_1}\left|t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}-t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}\right|s^{\alpha_1-1}ds\\
&+\frac{\widehat{M}\delta_1^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_{\delta_1}^{t_1}\left(t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}-t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}\right)ds\\
&+\frac{\widehat{M}t_2^{1-\alpha_1}}{\Gamma(\alpha_1+1)}\delta_1^{\alpha_1-1}(t_2-t_1)^{\alpha_1}\\
&\leq \frac{\widehat{M}}{\Gamma(\alpha_1)}\int_0^{\delta_1}\left|t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}-t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}\right|s^{\alpha_1-1}ds\\
&+\frac{\widehat{M}\delta_1^{\alpha_1-1}}{\Gamma(\alpha_1+1)}\left(t_1^{1-\alpha_1}(t_1-\delta_1)^{\alpha_1}-t_2^{1-\alpha_1}(t_2-\delta_1)^{\alpha_1}\right)\\
&+2\frac{\widehat{M}t_2^{1-\alpha_1}}{\Gamma(\alpha_1+1)}\delta_1^{\alpha_1-1}(t_2-t_1)^{\alpha_1},
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\left|t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2)-t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1)\right|\\
&\leq \frac{\widehat{M}}{\Gamma(\alpha_1)}\int_0^{\delta_1}\left|t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1}-t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1}\right|s^{\alpha_1-1}ds\\
&+\frac{\widehat{M}\delta_1^{\alpha_1-1}}{\Gamma(\alpha_1+1)}\left(t_1^{1-\alpha_1}(t_1-\delta_1)^{\alpha_1}-t_2^{1-\alpha_1}(t_2-\delta_1)^{\alpha_1}\right)+2\frac{\widehat{M}t_2^{1-\alpha_1}}{\Gamma(\alpha_1+1)}\delta_1^{\alpha_1-1}(t_2-t_1)^{\alpha_1}.
\end{aligned}
$$

Note that the functions $t \mapsto t^{1-\alpha_1}(t-s)^{\alpha_1-1}$ and $t \mapsto t^{\alpha}$ are continuous. If $t_2 - t_1 < \delta_2$, there exists $\delta_2 > 0$ such that

$$\left| t_2^{1-\alpha_1}(t_2-s)^{\alpha_1-1} - t_1^{1-\alpha_1}(t_1-s)^{\alpha_1-1} \right| < \frac{\Gamma(\alpha_1+1)}{3\widehat{M}\delta_1^{\alpha_1-1}}\varepsilon, \tag{3.15}$$

and

$$(t_2-t_1)^{\alpha_1} < \frac{\Gamma(\alpha_1+1)}{6\widehat{M}t_2^{1-\alpha_1}\delta_1^{\alpha_1-1}}\varepsilon. \tag{3.16}$$

If $t_2 - t_1 < \delta_2$, we obtain from (3.15) and (3.16) that

$$\left| t_2^{1-\alpha_1}\underline{u}_{n+1}(t_2) - t_1^{1-\alpha_1}\underline{u}_{n+1}(t_1) \right| < \frac{\delta_1}{3}\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

Therefore the sequence of functions $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$. Similarly, we can prove that the sequence of functions $\{t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

Step 7. The sequences of functions $\{t^{1-\alpha_1}\overline{u}_n\}_{n\in\mathbb{N}}$ and $\{t^{1-\alpha_2}\overline{v}_n\}_{n\in\mathbb{N}}$ are equicontinuous on $[0,1]$.

The proof is similar to that of Step 6, so it is omitted.

Step 8. The consequence $\{(\underline{u}_n, \underline{v}_n)\}_{n\in\mathbb{N}}$ converges to a minimal solution of (1.1).

By Step 4 and Step 6, the sequence of functions $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$ and $\{t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ are uniformly bounded on $C([0,1])$ and equicontinuous on $[0,1]$. Using the Arzéla-Ascoli theorem, we see that there exists a subsequence $\{(t^{1-\alpha_1}\underline{u}_{n_j}, t^{1-\alpha_2}\underline{v}_{n_j})\}_{n_j\in\mathbb{N}}$ of $\{(t^{1-\alpha_1}\underline{u}_n, t^{1-\alpha_2}\underline{v}_n)\}_{n\in\mathbb{N}}$, which converges in $C([0,1]) \times C([0,1])$. Let

$$(t^{1-\alpha_1}u, t^{1-\alpha_2}v) := \lim_{n_j\to+\infty}(t^{1-\alpha_1}\underline{u}_{n_j}, t^{1-\alpha_2}\underline{v}_{n_j}).$$

From Step 3, the sequences $\{t^{1-\alpha_1}\underline{u}_n\}_{n\in\mathbb{N}}$ and $\{t^{1-\alpha_2}\underline{v}_n\}_{n\in\mathbb{N}}$ are increasing and bounded from above. Hence the pointwise limit of $\{(t^{1-\alpha_1}\underline{u}_n, t^{1-\alpha_2}\underline{v}_n)\}_{n\in\mathbb{N}}$ exists, denoted by $(t^{1-\alpha_1}u_*, t^{1-\alpha_2}v_*)$. Hence we have

$$(t^{1-\alpha_1}u, t^{1-\alpha_2}v) = (t^{1-\alpha_1}u_*, t^{1-\alpha_2}v_*).$$

Moreover, the whole sequence converges in $C([0,1]) \times C([0,1])$ to $(t^{1-\alpha_1}u_*, t^{1-\alpha_2}v_*)$. Using Lemma 2.6, we have

$$t^{1-\alpha_1}\underline{u}_{n+1}(t) = a_{n,\alpha_1}E_{\alpha_1,\alpha_1}(-M_1t^{\alpha_1}) + t^{1-\alpha_1}\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-M_1(t-s)^{\alpha_1})\widetilde{f}_n(s)\,ds,$$

and

$$t^{1-\alpha_2}\underline{v}_{n+1}(t) = b_{n,\alpha_2}E_{\alpha_2,\alpha_2}(-M_2t^{\alpha_2}) + t^{1-\alpha_2}\int_0^t (t-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-M_2(t-s)^{\alpha_2})\widetilde{g}_n(s)\,ds,$$

where

$$\begin{aligned} a_{n,\alpha_1} &= \Gamma(\alpha_1)\int_0^1 g_1(s)\underline{u}_n(s)ds \\ &= \Gamma(\alpha_1)\int_0^1 g_1(s)s^{\alpha_1-1}s^{1-\alpha_1}\underline{u}_n(s)ds, \end{aligned}$$

and

$$\begin{aligned} b_{n,\alpha_2} &= \Gamma(\alpha_2)\int_0^1 g_1(s)\underline{v}_n(s)ds \\ &= \Gamma(\alpha_2)\int_0^1 g_1(s)s^{\alpha_2-1}s^{1-\alpha_2}\underline{v}_n(s)ds. \end{aligned}$$

Letting $n \to \infty$ and $s \neq 0$, we obtain

$$\widetilde{f_n}(s) \to f(s, u_*(s), v_*(s)),$$

and

$$\widetilde{g_n}(s) \to g(s, u_*(s), v_*(s)).$$

Also, we have

$$\exists c_1 > 0, \forall n \in \mathbb{N}, \forall s \in [0,1], \left| s^{1-\alpha_1} \widehat{f_n}(s) \right| \leq c_1,$$

and

$$\exists c_2 > 0, \forall n \in \mathbb{N}, \forall s \in [0,1], \left| s^{1-\alpha_2} \widehat{g}_n(s) \right| \leq c_2.$$

Hence, the dominated convergence theorem of Lebesgue implies that

$$t^{1-\alpha_1} u_*(t) = a_{\alpha_1} E_{\alpha_1,\alpha_1}(-M_1 t^{\alpha_1}) + t^{1-\alpha_1} \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-M_1 (t-s)^{\alpha_1}) \widetilde{f}(s)\, ds,$$

and

$$t^{1-\alpha_2} v_*(t) = b_{\alpha_2} E_{\alpha_2,\alpha_2}(-M_2 t^{\alpha_2}) + t^{1-\alpha_2} \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2,\alpha_2}(-M_2 (t-s)^{\alpha_2}) \widetilde{g}(s)\, ds,$$

where

$$a_{\alpha_1} = \Gamma(\alpha_1) \int_0^1 g_0(s) u^*(s) ds,$$

$$b_{\alpha_2} = \Gamma(\alpha_2) \int_0^1 g_2(s) v^*(s) ds,$$

$$\widetilde{f}(s) = f(s, u_*(s), v_*(s)) + M_1 u_*(s),$$

and

$$\widetilde{g}(s) = g(s, u_*(s), v_*(s)) + M_2 v_*(s).$$

Using Lemma 2.6, we have

$$\begin{cases} D^{\alpha_1} u_*(t) = f(t, u_*, v_*), t \in (0,1], \\ D^{\alpha_2} v_*(t) = g(t, u_*, v_*), t \in (0,1], \\ \quad t^{1-\alpha_1} u_*(t)\,|_{t=0} = \int_0^1 g_1(s) u_*(s) ds \\ \quad t^{1-\alpha_2} v_*(t)\,|_{t=0} = \int\limits_0^1 g_2(s)\, v_*(s)\, ds, \end{cases} \tag{3.17}$$

which means that (u_*, v_*) is a solution of (1.1). Now, we prove that if (u, v) is another solution of (1.1) such that $t^{1-\alpha_1} \underline{u}(t) \leq t^{1-\alpha_1} u(t) \leq t^{1-\alpha_1} \overline{u}(t)$ and $t^{1-\alpha_2} \underline{v}(t) \leq t^{1-\alpha_2} v(t) \leq t^{1-\alpha_2} \overline{v}(t)$, $\forall t \in [0,1]$, then $t^{1-\alpha_1} u_*(t) \leq t^{1-\alpha_1} u(t)$ and $t^{1-\alpha_2} v_*(t) \leq t^{1-\alpha_2} v(t)$, $\forall t \in [0,1]$. Since $(\underline{u}, \overline{u})$, $(\underline{v}, \overline{v})$ is a pair of lower-upper solutions of (1.1), we find from Step 3 that

$$\forall n \in \mathbb{N},\ \forall t \in [0,1], t^{1-\alpha_1} \underline{u}_n(t) \leq t^{1-\alpha_1} u(t) \text{ and } t^{1-\alpha_2} \underline{v}_n(t) \leq t^{1-\alpha_2} v(t).$$

Letting $n \to +\infty$, we obtain

$$\forall t \in [0,1], t^{1-\alpha_1} u_*(t) \leq t^{1-\alpha_1} u(t) \text{ and } t^{1-\alpha_2} v_*(t) \leq t^{1-\alpha_2} v(t),$$

which means that (u_*, v_*) is a minimal solution of problem (1.1). The proof of Step 8 is complete.

Step 9. The sequence $\{(\overline{u}_n, \overline{v}_n)\}_{n\in\mathbb{N}}$ converges to a maximal solution (u^*, v^*) of (1.1).

The proof is similar to that of Step 8, so it is omitted. Hence, the whole proof is complete. □

4. Existence of minimal- maximal and maximal- minimal solutions for systems with quasimonotone decreasing functions

In this section, we replace the hypothesis (H3) and (H4) with the following hypothesis

(H7) $f(t,u,v)$ is decreasing in v for all fixed $t \in (0,1]$, and $u \in \mathbb{R}$.

(H8) $g(t,u,v)$ is decreasing in u for all fixed $t \in (0,1]$, and $v \in \mathbb{R}$.

Definition 4.1. We say that $(\underline{u},\underline{v}),(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 2 for problem (1.1) if

(i) $(\underline{u},\overline{u}) \in (C_{1-\alpha_1}([0,1]))^2$ and $(\underline{v},\overline{v}) \in (C_{1-\alpha_2}([0,1]))^2$,

(ii) $\begin{cases} D^{\alpha_1}\underline{u}(t) \leq f(t,\underline{u},\overline{v}), t \in (0,1], \\ D^{\alpha_1}\overline{u}(t) \geq f(t,\overline{u},\underline{v}), t \in (0,1], \\ D^{\alpha_2}\underline{v}(t) \leq g(t,\overline{u},\underline{v}), t \in (0,1], \\ D^{\alpha_2}\overline{v}(t) \geq g(t,\underline{u},\overline{v}), t \in (0,1], \end{cases}$

(iii) $\begin{cases} t^{1-\alpha_1}\underline{u}(t)|_{t=0} \leq \int_0^1 g_1(s)\underline{u}(s)ds, t \in (0,1], \\ t^{1-\alpha_1}\overline{u}(t)|_{t=0} \geq \int_0^1 g_1(s)\overline{u}(s)ds, t \in (0,1], \\ t^{1-\alpha_2}\underline{v}(t)|_{t=0} \leq \int_0^1 g_1(s)\underline{v}(s)ds, t \in (0,1], \\ t^{1-\alpha_2}\overline{v}(t)|_{t=0} \geq \int_0^1 g_1(s)\overline{v}(s)ds, t \in (0,1]. \end{cases}$

Theorem 4.2. *Assume that hypothesis(H1), (H2), (Hi) for $i = 5,...,8$ are satisfied and $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 2 for problem (1.1) such that $t^{1-\alpha_1}\underline{u}(t) \leq t^{1-\alpha_1}\overline{u}(t)$ and $t^{1-\alpha_2}\underline{v}(t) \leq t^{1-\alpha_2}\overline{v}(t)$ in $[0,1]$. Then problem (1.1) admits a maximal-minimal solution (u^*,v_*) and a minimal-maximal solution (u_*,v^*) such that for every solution (u,v) of (1.1) with $t^{1-\alpha_1}\underline{u}(t) \leq t^{1-\alpha_1}u(t) \leq t^{1-\alpha_1}\overline{u}(t)$ and $t^{1-\alpha_2}\underline{v}(t) \leq t^{1-\alpha_2}v(t) \leq t^{1-\alpha_2}\overline{v}(t)$ in $[0,1]$,*

$$t^{1-\alpha_1}\underline{u}(t) \leq t^{1-\alpha_1}u_*(t) \leq t^{1-\alpha_1}u(t) \leq t^{1-\alpha_1}u^*(t) \leq t^{1-\alpha_1}\overline{u}(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}(t) \leq t^{1-\alpha_2}v_*(t) \leq t^{1-\alpha_2}v(t) \leq t^{1-\alpha_2}v^*(t) \leq t^{1-\alpha_2}\overline{v}(t) \text{ in } [0,1].$$

Proof. We take $\overline{u}_0 = \overline{u}$, $\underline{v}_0 = \underline{v}$ and define the sequences of functions $\{\overline{u}_n\}_{n\geq 1}$, $\{\underline{v}_n\}_{n\geq 1}$in the following way

$$\begin{cases} D^{\alpha_1}\overline{u}_{n+1}(t) + M_1\overline{u}_{n+1}(t) = f(t,\overline{u}_n(t),\underline{v}_n(t)) + M_1\overline{u}_n(t), t \in (0,1], \\ t^{1-\alpha_1}\overline{u}_{n+1}(t)|_{t=0} = \int_0^1 g_1(s)\overline{u}_n(s)ds, \end{cases}$$

$$\begin{cases} D^{\alpha_2}\underline{v}_{n+1}(t) + M_2\underline{v}_n(t) = g(t,\overline{u}_n(t),\underline{v}_n(t)) + M_2\underline{v}_n(t), t \in (0,1], \\ t^{1-\alpha_2}\underline{v}_n(t)|_{t=0} = \int_0^1 g_2(s)\underline{v}_n(s)ds. \end{cases}$$

Analogously, we take $\underline{u}_0 = \underline{u}$, $\overline{v}_0 = \overline{v}$ and define the sequences of functions $\{\underline{u}_n\}_{n\geq 1}$, $\{\overline{v}_n\}_{n\geq 1}$ in the following way

$$\begin{cases} D^{\alpha_1}\underline{u}_{n+1}(t) + M_1\underline{u}_{n+1}(t) = f(t,\underline{u}_n(t),\overline{v}_n(t)) + M_1\underline{u}_n(t), t \in (0,1], \\ t^{1-\alpha_1}\underline{u}_{n+1}(t)|_{t=0} = \int_0^1 g_1(s)\underline{u}_n(s)ds, \end{cases}$$

$$\begin{cases} D^{\alpha_2}\overline{v}_{n+1}(t) + M_2\overline{v}_{n+1}(t) = g(t,\underline{u}_n(t),\overline{v}_n(t)) + M_2\overline{v}_n(t), t \in (0,1], \\ t^{1-\alpha_2}\overline{v}_{n+1}(t)|_{t=0} = \int_0^1 g_2(s)\overline{v}_n(s)ds. \end{cases}$$

The rest of the proof is similar to that of Theorem 3.3. □

5. Existence of solutions for systems with mixed quasimonotone functions

In this section, we assume that $f(t,u,v)$ is increasing in v for all fixed $t \in (0,1]$, $u \in \mathbb{R}$ and $g(t,u,v)$ is decreasing in u for all fixed $t \in (0,1]$, $v \in \mathbb{R}$.

Definition 5.1. We say that $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 3 for problem (1.1) if

(i) $(\underline{u},\overline{u}) \in (C_{1-\alpha_1}([0,1]))^2$ and $(\underline{v},\overline{v}) \in (C_{1-\alpha_2}([0,1]))^2$,

(ii) $$\begin{cases} D^{\alpha_1}\underline{u}(t) \leq f(t,\underline{u},\underline{v}), t \in (0,1], \\ D^{\alpha_1}\overline{u}(t) \geq f(t,\overline{u},\overline{v}), t \in (0,1], \\ D^{\alpha_2}\underline{v}(t) \leq g(t,\overline{u},\underline{v}), t \in (0,1], \\ D^{\alpha_2}\overline{v}(t) \geq g(t,\underline{u},\overline{v}), t \in (0,1], \end{cases}$$

(iii) $$\begin{cases} t^{1-\alpha_1}\underline{u}(t)|_{t=0} \leq \int_0^1 g_1(s)\underline{u}(s)ds, \\ t^{1-\alpha_1}\overline{u}(t)|_{t=0} \geq \int_0^1 g_1(s)\overline{u}(s)ds, \\ t^{1-\alpha_2}\underline{v}(t)|_{t=0} \leq \int_0^1 g_2(s)\underline{v}(s)ds, \\ t^{1-\alpha_1}\overline{v}(t)|_{t=0} \geq \int_0^1 g_2(s)\overline{v}(s)ds. \end{cases}$$

Definition 5.2. The pair of functions (u^*,v^*), (u_*,v_*) are called a quasisolutions of (1.1) if

(i) $(u^*,v^*) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1])$ and $(u_*,v_*) \in C_{1-\alpha_1}([0,1]) \times C_{1-\alpha_2}([0,1])$,

(ii) $$\begin{cases} D^{\alpha_1}u^*(t) = f(t,u^*,v^*), t \in (0,1], \\ D^{\alpha_1}u_*(t) = f(t,u_*,v_*), t \in (0,1], \\ D^{\alpha_2}v^*(t) = g(t,u_*,v^*), t \in (0,1], \\ D^{\alpha_2}v_*(t) = g(t,u^*,v_*), t \in (0,1], \\ t^{1-\alpha_1}u^*(t)|_{t=0} = \int_0^1 g_1(s)u^*(s)ds, \\ t^{1-\alpha_1}u_*(t)|_{t=0} = \int_0^1 g_1(s)u_*(s)ds, \\ t^{1-\alpha_2}v^*(t)|_{t=0} = \int_0^1 g_2(s)v^*(s)ds, \\ t^{1-\alpha_1}v_*(t)|_{t=0} = \int_0^1 g_2(s)v_*(s)ds. \end{cases}$$

We have the following result.

Theorem 5.3. *Assume that hypothesis (Hi) for $i = 1,2,3$, (H5), (H6) and (H8) are satisfied and $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 3 for problem (1.1) such that $t^{1-\alpha_1}\underline{u}(t) \leq t^{1-\alpha_1}\overline{u}(t)$ and $t^{1-\alpha_2}\underline{v}(t) \leq t^{1-\alpha_2}\overline{v}(t)$ in $[0,1]$. Then problem (1.1) admits a pair of quasisolutions (u^*,v^*), (u_*,v_*) such that*

$$t^{1-\alpha_1}\underline{u}(t) \leq t^{1-\alpha_1}u_*(t) \leq t^{1-\alpha_1}u^*(t) \leq t^{1-\alpha_1}\overline{u}(t) \text{ in } [0,1],$$

and

$$t^{1-\alpha_2}\underline{v}(t) \leq t^{1-\alpha_2}v_*(t) \leq t^{1-\alpha_2}v^*(t) \leq t^{1-\alpha_2}\overline{v}(t) \text{ in } [0,1].$$

Proof. We take $u_0 = \overline{u}$, $u_1 = \underline{u}$, $v_0 = \overline{v}$, $v_1 = \underline{v}$ and define the sequences of functions $\{u_n\}_{n\geq 2}$ and $\{v_n\}_{n\geq 2}$ in the following way

$$\begin{cases} D^{\alpha_1}u_{n+2}(t) + M_1u_{n+2}(t) = f(t,u_n(t),v_n(t)) + M_1u_n(t), t \in (0,1], \\ t^{1-\alpha_1}u_{n+2}(t)|_{t=0} = \int_0^1 g_1(s)u_n(s)ds, \end{cases}$$

$$\begin{cases} D^{\alpha_2}v_{n+2}(t) + M_2v_{n+2}(t) = f(t,u_{n+1}(t),v_n(t)) + M_2v_n(t), t \in (0,1], \\ t^{1-\alpha_2}v_{n+2}(t)|_{t=0} = \int_0^1 g_2(s)v_n(s)ds. \end{cases}$$

By using a proof, which is similar to that of Step 2 of theorem 3.3, we obtain

$$\begin{aligned} t^{1-\alpha_1}\underline{u}(t) &= t^{1-\alpha_1}u_1(t) \leq t^{1-\alpha_1}u_3(t) \leq \dots \leq t^{1-\alpha_1}u_{2n+1}(t) \leq \dots. \\ \dots &\leq t^{1-\alpha_1}u_{2n}(t) \leq \dots \leq t^{1-\alpha_1}u_2(t) \leq t^{1-\alpha_1}u_0(t) = t^{1-\alpha_1}\overline{u}(t), \text{ in } [0,1], \end{aligned}$$

and

$$\begin{aligned} t^{1-\alpha_2}\underline{v}(t) &= t^{1-\alpha_2}v_1(t) \leq t^{1-\alpha_2}v_3(t) \leq \dots \leq t^{1-\alpha_2}v_{2n+1}(t) \leq \dots. \\ \dots &\leq t^{1-\alpha_2}v_{2n}(t) \leq \dots \leq t^{1-\alpha_2}v_2(t) = t^{1-\alpha_2}v_0(t) = t^{1-\alpha_2}\overline{v}(t), \text{ in } [0,1]. \end{aligned}$$

The previous inequalities show that the sequences of functions $\{(u_{2n}, v_{2n})\}_{n\in\mathbb{N}}$ and $\{(u_{2n+1}, v_{2n+1})\}_{n\in\mathbb{N}}$ converge to (u^*, v^*) and (u_*, v_*). Using a proof, which is similar to that of Step 5 of Theorem 3.3, we prove that these functions are quasisolutions of (1.1). □

Since the quasisolutions are not a true solutions, it is necessary to impose additional conditions on f, g and g_i for $i = 1,2$ which ensures that $u^* = u_*$ and $v^* = v_*$ and consequently the problem (1.1) admits at least one solution.

We assume that $\alpha_1 = \alpha_2 = \alpha$ and on the nonlinearities f and g and the functions g_i for $i = 1,2$, we shall impose the following additional conditions

- (H9) There exists $M_3 < 0$ such that the function $u \longmapsto f(t,u,v) + M_3 u$ is decreasing for all $t \in (0,1]$ and $v \in \mathbb{R}$.
- (H10) There exists $M_4 < 0$ such that the function $v \longmapsto f(t,u,v) + M_4 v$ is decreasing for all $t \in (0,1]$ and $u \in \mathbb{R}$.
- (H11) There exists $M_5 > 0$ such that the function $u \longmapsto g(t,u,v) + M_5 u$ is increasing for all $t \in (0,1]$ and $v \in \mathbb{R}$.
- (H12) There exists $M_6 < 0$ such that the function $v \longmapsto g(t,u,v) + M_6 v$ is decreasing for all $t \in (0,1]$ and $u \in \mathbb{R}$.
- (H13) $\Gamma(\alpha)\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-M_7 s^{\alpha_1})g_3(s)\,ds < 1$, where

$$M_7 = \max(M_3 - M_5, M_4 + M_6),$$

and

$$g_3(s) = \max(g_1(s), g_2(s)), \text{ for all } s \in [0,1].$$

We have the following result.

Theorem 5.4. *Assume that hypothesis (Hi) for $i = 1,2,3$, (H5), (H6) and (Hi) for $i = 8,\dots,13$ are satisfied and $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower -upper solutions of type 3 for the problem (1.1) such that $t^{1-\alpha}\underline{u}(t) \leq t^{1-\alpha}\overline{u}(t)$ and $t^{1-\alpha}\underline{v}(t) \leq t^{1-\alpha}\overline{v}(t)$ in $[0,1]$. Then problem (1.1) admits at least one solution (u,v) such that $t^{1-\alpha}\underline{u}(t) \leq t^{1-\alpha}u(t) \leq t^{1-\alpha}\overline{u}(t)$ and $t^{1-\alpha}\underline{v}(t) \leq t^{1-\alpha}v(t) \leq t^{1-\alpha}\overline{v}(t)$ in $[0,1]$.*

Proof. By Theorem 5.3, problem (1.1) admits a pair of quasisolutions (u^*, v^*), (u_*, v_*) such that

$$t^{1-\alpha}\underline{u}(t) \leq t^{1-\alpha}u_*(t) \leq t^{1-\alpha}u^*(t) \leq t^{1-\alpha}\overline{u}(t), \text{ in } [0,1], \tag{5.1}$$

and

$$t^{1-\alpha}\underline{v}(t) \leq t^{1-\alpha}v_*(t) \leq t^{1-\alpha}v^*(t) \leq t^{1-\alpha}\overline{v}(t), \text{ in } [0,1]. \tag{5.2}$$

Now, we put by definition

$$z^*(t) = u^*(t) - u_*(t) \text{ and } w^*(t) = v^*(t) - v_*(t), t \in (0,1],$$
$$t^{1-\alpha} z^*(t)|_{t=0} = t^{1-\alpha} u^*(t)|_{t=0} - t^{1-\alpha} u_*(t)|_{t=0},$$

and

$$t^{1-\alpha} w^*(t)|_{t=0} = t^{1-\alpha} v^*(t)|_{t=0} - t^{1-\alpha} v_*(t)|_{t=0}.$$

Using (5.1) and (5.2), we have

$$t^{1-\alpha} z^*(t) \geq 0 \text{ and } t^{1-\alpha} w^*(t) \geq 0, \text{ for all } t \in [0,1]. \tag{5.3}$$

Now, we are in a position to prove that

$$t^{1-\alpha} z^*(t) \leq 0 \text{ and } t^{1-\alpha} w^*(t) \leq 0, \text{ for all } t \in [0,1].$$

We have

$$\begin{cases} D^\alpha z^*(t) = f(t, u^*(t), v^*(t)) - f(t, u_*(t), v_*(t)), \ t \in (0,1], \\ D^\alpha w^*(t) = g(t, u_*(t), v^*(t)) - g(t, u^*(t), v_*(t)), \ t \in (0,1], \\ t^{1-\alpha} z^*(t)|_{t=0} = \int_0^1 g_1(s) z^*(s) ds, \\ t^{1-\alpha} w^*(t)|_{t=0} = \int_0^1 g_2(s) w^*(s) ds. \end{cases} \tag{5.4}$$

From hypothesis (H9) and (H10), one has

$$\begin{aligned} & D^\alpha z^*(t) + M_3 z^*(t) + M_4 w^*(t) \\ & = f(t, u^*(t), v^*(t)) - f(t, u_*(t), v_*(t)) + M_3 z^*(t) + M_4 w^*(t) \\ & \leq f(t, u_*(t), v^*(t)) - f(t, u_*(t), v_*(t)) + M_4 w^*(t) \\ & \leq 0, \end{aligned}$$

that is,

$$D^\alpha z^*(t) + M_3 z^*(t) + M_4 w^*(t) \leq 0, \ t \in (0,1]. \tag{5.5}$$

Similarly, we find from hypothesis (H11) and (H12) that

$$D^\alpha w^*(t) - M_5 z^*(t) + M_6 w^*(t) \leq 0, \ t \in (0,1], \tag{5.6}$$

which implies that

$$D^\alpha (z^* + w^*)(t) + M_7.(z^* + w^*)(t) \leq 0, \ t \in (0,1].$$

Using the initial conditions in (5.4), we have

$$\begin{cases} D^\alpha (z^* + w^*)(t) + M_7.(z^* + w^*)(t) \leq 0, \ t \in (0,1], \\ t^{1-\alpha} (z^* + w^*)(t)|_{t=0} \leq \int_0^1 g_3(s)(z^* + w^*)(s) ds. \end{cases} \tag{5.7}$$

From hypothesis (H13) and Lemma 2.11, we obtain

$$t^{1-\alpha} (z^* + w^*)(t) \leq 0, \text{ for all } t \in [0,1].$$

Consequently, by (5.3), it follows that

$$t^{1-\alpha} z^*(t) = 0 \text{ and } t^{1-\alpha} w^*(t) = 0 \text{ for all } t \in [0,1],$$

that is,

$$u^*(t) = u_*(t) \text{ and } v^*(t) = v_*(t) \text{ for all } t \in (0,1], \tag{5.8}$$

and

$$t^{1-\alpha} u^*(t)|_{t=0} = t^{1-\alpha} u_*(t)|_{t=0} \text{ and } t^{1-\alpha} v^*(t)|_{t=0} = t^{1-\alpha} v_*(t)|_{t=0}. \tag{5.9}$$

It follows that problem (1.1) admits at least one solution (u,v) such that $t^{1-\alpha}\underline{u}(t) \leq t^{1-\alpha}u(t) \leq t^{1-\alpha}\overline{u}(t)$ and $t^{1-\alpha}\underline{v}(t) \leq t^{1-\alpha}v(t) \leq t^{1-\alpha}\overline{v}(t)$ in $[0,1]$. □

6. Examples

In this section, we give some examples illustrating the application of our results.

Example 6.1. We consider the following problem

$$\begin{cases} D^{\alpha_1}u(t) = f_1(t,u,v),\ t \in (0,1], \\ D^{\alpha_2}v(t) = g_4(t,u,v),\ t \in (0,1], \\ u(t) > 0 \text{ and } v(t) > 0 \text{ in } (0,1], \\ t^{1-\alpha_1}u(t)|_{t=0} = \int_0^1 \frac{s^{1-\alpha_1}}{10}u(s)ds, \\ t^{1-\alpha_2}v(t)|_{t=0} = \int_0^1 \frac{s^{1-\alpha_2}}{10}v(s)ds, \end{cases} \tag{6.1}$$

where

$$f_1(t,u,v) = a_1(t)t^{(1-\alpha_1)k_1}u^{k_1} + a_2(t)t^{(1-\alpha_2)k_2}v^{k_2} + a_3(t)t^{(1-\alpha_1)k_3}t^{1-\alpha_2)k_4}u^{k_3}v^{k_4} + h_1(t),$$

and

$$g_4(t,u,v) = b_1(t)t^{(1-\alpha_1)k_5}u^{k_5} + b_2(t)t^{(1-\alpha_2)k_6}v^{k_6} + b_3(t)t^{(1-\alpha_1)k_7}t^{(1-\alpha_2)}k_8u^{k_7}v^{k_8} + h_2(t),$$

with $a_i : (0,1] \to \mathbb{R}_+$ $(i = 1,...,3)$, $b_i : (0,1] \to \mathbb{R}_+$ $(i = 1,...,3)$ and $h_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$ are continuous and bounded functions such that $t \mapsto t^{1-\alpha_1}a_i(t)$ $(i = 1,...,3)$, $t \mapsto t^{1-\alpha_1}h_1(t)$, $t \mapsto t^{1-\alpha_2}b_i(t)$ $(i = 1,...,3)$, $t \mapsto t^{1-\alpha_2}h_2(t)$ are continuous on $[0,1]$, $0 < k_i < 1$ for $i = 1,2,5,6$ and $0 < k_i + k_{i+1} < 1$ for $i = 3,7$.

Theorem 6.2. *Problem (6.1) admits a maximal solution (u^*,v^*) and a minimal solution (u_*,v_*).*

Proof. We put $(\underline{u},\underline{v}) = (0,0)$ and $(\overline{u},\overline{v}) = \left(Lt^{\alpha_1-1}(1+t), Lt^{\alpha_2-1}(1+t)\right)$, where L is a positive constant. $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 1 for problem (6.1) if

$$\begin{cases} D^{\alpha_1}\underline{u}(t) \leq f_1(t,\underline{u},\underline{v}),\ t \in (0,1], \\ D^{\alpha_1}\overline{u}(t) \geq f_1(t,\overline{u},\overline{v}),\ t \in (0,1], \\ D^{\alpha_2}\underline{v}(t) \leq g_4(t,\underline{u},\underline{v}),\ t \in (0,1], \\ D^{\alpha_2}\overline{v}(t) \geq g_4(t,\overline{u},\overline{v}),\ t \in (0,1], \end{cases}$$

and

$$\begin{cases} t^{1-\alpha_1}\underline{u}(t)|_{t=0} \leq \int_0^1 \frac{s^{1-\alpha_1}}{10}\underline{u}(s)ds, \\ t^{1-\alpha_1}\overline{u}(t)|_{t=0} \geq \int_0^1 \frac{s^{1-\alpha_1}}{10}\overline{u}(s)ds, \\ t^{1-\alpha_2}\underline{v}(t)|_{t=0} \leq \int_0^1 \frac{s^{1-\alpha_2}}{10}\underline{v}(s)ds, \\ t^{1-\alpha_2}\overline{v}(t)|_{t=0} \geq \int_0^1 \frac{s^{1-\alpha_2}}{10}\overline{v}(s)ds, \end{cases}$$

that is,

$$\begin{cases} 0 \leq h_1(t),\ t \in (0,1], \\ L\Gamma(\alpha_1+1) \geq a_1(t)L^{k_1}(1+t)^{k_1} + a_2(t)L^{k_2}(1+t)^{k_2} \\ +a_3(t)L^{k_3+k_4}(1+t)^{k_3+k_4} + h_1(t),\ t \in (0,1], \\ 0 \leq h_2(t),\ t \in (0,1], \\ L\Gamma(\alpha_2+1) \geq b_1(t)L^{k_5}(1+t)^{k_5} + b_2(t)L^{k_6}(1+t)^{k_6} \\ +b_3(t)L^{k_7+k_8}(1+t)^{k_7+k_8} + h_2(t),\ t \in (0,1],\ t \in (0,1]. \end{cases}$$

Since $0 < k_i < 1$ for $i = 1,2,\ 5,6$ and $0 < k_i + k_{i+1} < 1$ for $i = 3,7$, it is not difficult to prove that if we choose L such that it is sufficiently large, then $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower -upper solutions of type 1 for problem (6.1). Consequently, by Theorem 3.3, it follows that this problem admits a maximal solution (u^*,v^*) and a minimal solution (u_*,v_*). □

Example 6.3. We consider the following problem

$$\begin{cases} D^{\alpha_1}u(t) = f_2(t,u,v),\ t \in (0,1], \\ D^{\alpha_2}v(t) = g_5(t,u,v),\ t \in (0,1], \\ u(t) > 0 \text{ and } v(t) > 0,\ t \in (0,1], \\ t^{1-\alpha_1}u(t)|_{t=0} = \int_0^1 \frac{s^{1-\alpha_1}}{10}u(s)ds, \\ t^{1-\alpha_2}v(t)|_{t=0} = \int_0^1 \frac{s^{1-\alpha_2}}{10}v(s)ds, \end{cases} \tag{6.2}$$

where

$$f_2(t,u,v) = a_1(t)t^{(1-\alpha_1)k_1}u^{k_1} - a_2(t)t^{(1-\alpha_2)k_2}v^{k_2} + h_1(t),$$

and

$$g_5(t,u,v) = -b_1(t)t^{(1-\alpha_1)k_5}u^{k_5} + b_2(t)t^{(1-\alpha_2)k_6}v^{k_6} + h_2(t),$$

with $a_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$, $b_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$ and $h_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$ are continuous and bounded functions such that $t \mapsto t^{1-\alpha_1}a_i(t)$ $(i = 1,\ 2)$, $t \mapsto t^{1-\alpha_1}h_1(t)$, $t \mapsto t^{1-\alpha_2}b_i(t)$ $(i = 1,\ 2)$, $t \mapsto t^{1-\alpha_2}h_2(t)$ are continuous on $[0,1]$, $0 < k_i < 1$ for $i = 1,2,5,6$.

Theorem 6.4. *Problem (6.2) admits a maximal-minimal solution (u^*,v_*) and a minimal-maximal solution (u_*,v^*).*

Proof. We put $(\underline{u},\underline{v}) = (0,0)$ and $(\overline{u},\overline{v}) = \left(L_1 t^{\alpha_1-1}(1+t), L_2 t^{\alpha_2-1}(1+t)\right)$, where L_1 and L_2 are two positive real numbers. $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 2 for problem (6.2) if

$$\begin{cases} D^{\alpha_1}\underline{u}(t) \leq f_2(t,\underline{u},\overline{v}),\ t \in (0,1], \\ D^{\alpha_1}\overline{u}(t) \geq f_2(t,\overline{u},\underline{v}),\ t \in (0,1], \\ D^{\alpha_2}\underline{v}(t) \leq g_5(t,\overline{u},\underline{v}),\ t \in (0,1], \\ D^{\alpha_2}\overline{v}(t) \geq g_5(t,\underline{u},\overline{v}),\ t \in (0,1], \end{cases}$$

and

$$\begin{cases} t^{1-\alpha_1}\underline{u}(t)|_{t=0} \le \int_0^1 \dfrac{s^{1-\alpha_1}}{10}\underline{u}(s)ds, \\ t^{1-\alpha_1}\overline{u}(t)|_{t=0} \ge \int_0^1 \dfrac{s^{1-\alpha_1}}{10}\overline{u}(s)ds, \\ t^{1-\alpha_2}\underline{v}(t)|_{t=0} \le \int_0^1 \dfrac{s^{1-\alpha_2}}{10}\underline{v}(s)ds, \\ t^{1-\alpha_2}\overline{v}(t)|_{t=0} \ge \int_0^1 \dfrac{s^{1-\alpha_2}}{10}\overline{v}(s)ds, \end{cases}$$

that is,

$$\begin{cases} 0 \le -a_2(t)L_2^{k_2}(1+t)^{k_2} + h_1(t),\ t \in (0,1], \\ L_1\Gamma(\alpha_1+1) \ge a_1(t)L_1^{k_1}(1+t)^{k_1} + h_1(t),\ t \in (0,1], \\ 0 \le -b_1(t)L_1^{k_5}(1+t)^{k_5} + h_2(t),\ t \in (0,1], \\ L_2\Gamma(\alpha_2+1) \ge b_2(t)L_2^{k_6}(1+t)^{k_6} + h_2(t),\ t \in (0,1],\ t \in (0,1]. \end{cases}$$

Put

$$C_1 = \max\left(1, \sup_{t\in(0,1]} \frac{h_1(t)}{\Gamma(\alpha_1+1) - a_1(t)(1+t)^{k_1}}\right),$$

$$C_2 = \inf_{t\in(0,1]} \left(\frac{h_2(t)}{b_1(t)(1+t)^{k_5}}\right)^{\frac{1}{k_5}},$$

$$C_3 = \max\left(1, \sup_{t\in(0,1]} \frac{h_2(t)}{\Gamma(\alpha_2+1) - b_2(t)(1+t)^{k_6}}\right),$$

and

$$C_4 = \inf_{t\in(0,1]} \left(\frac{h_1(t)}{a_2(t)(1+t)^{k_2}}\right)^{\frac{1}{k_2}}.$$

If we suppose that the constants C_i $(i = 1,\dots,4)$ are well defined, $C_1 \le C_2$ and $C_3 \le C_4$ and $C_1 \le L_1 \le C_2$ and $C_3 \le L_2 \le C_4$, then $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower -upper solutions of type 2 for problem (6.2). Using Theorem 4.2, we find that that this problem admits a maximal-minimal solution (u^*,v_*) and a minimal-maximal solution (u_*,v^*). □

Example 6.5. We consider the following problem

$$\begin{cases} D^{\alpha}u(t) = f_3(t,u,v),\ t \in (0,1], \\ D^{\alpha}v(t) = g_6(t,u,v),\ t \in (0,1], \\ u(t) > 0 \text{ and } v(t) > 0,\ t \in (0,1], \\ t^{1-\alpha}u(t)|_{t=0} = \dfrac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(A)} \int_0^1 \dfrac{s^{1-\alpha}}{10}u(s)ds, \\ t^{1-\alpha}v(t)|_{t=0} = \dfrac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(B)} \int_0^1 \dfrac{s^{1-\alpha}}{10}v(s)ds, \end{cases} \tag{6.3}$$

where

$$f_3(t,u,v) = a_1(t)t^{1-\alpha_1}u + a_2(t)t^{(1-\alpha_2)k_2}v^{k_2} + h_1(t),$$

and

$$g_6(t,u,v) = -b_1(t)t^{1-\alpha_1}u + b_2(t)t^{1-\alpha_2}v + h_2(t),$$

with $a_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$, $b_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$ and $h_i : (0,1] \to \mathbb{R}_+$ $(i = 1,2)$ are continuous and bounded functions such that $t \mapsto t^{1-\alpha}a_i(t)$ $(i = 1,\ 2)$, $t \mapsto t^{1-\alpha}h_i(t)$ $(i = 1,\ 2)$, $t \mapsto t^{1-\alpha}b_i(t)$ $(i = 1,\ 2)$ are continuous on $[0,1]$, $0 < k_2 < 1$, $A = \max_{t\in[0,1]} \left|a_1(t)t^{1-\alpha}\right|$ and $B = \max_{t\in[0,1]} \left|b_2(t)t^{1-\alpha}\right|$.

Theorem 6.6. *Problem (6.3) admits at least one solution a solution (u,v).*

Proof. We put $(\underline{u},\underline{v}) = (0,0)$ and $(\overline{u},\overline{v}) = \left(Lt^{\alpha-1}(1+t), Lt^{\alpha-1}(1+t)\right)$, where L is a positive constant. $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower-upper solutions of type 3 for problem (6.3) if

$$\begin{cases} D^\alpha \underline{u}(t) \le a_1(t)t^{1-\alpha}\underline{u} + a_2(t)t^{(1-\alpha)k_2}\underline{v}^{k_2} + h_1(t),\ t \in (0,1], \\ D^\alpha \overline{u}(t) \ge a_1(t)t^{1-\alpha}\overline{u} + a_2(t)t^{(1-\alpha)k_2}\overline{v}^{k_2} + h_1(t),\ t \in (0,1], \\ D^\alpha \underline{v}(t) \le -b_1(t)t^{1-\alpha}\overline{u} + b_2(t)t^{1-\alpha}\underline{v} + h_2(t),\ t \in (0,1], \\ D^\alpha \overline{v}(t) \ge -b_1(t)t^{1-\alpha}\underline{u} + b_2(t)t^{1-\alpha}\overline{v} + h_2(t),\ t \in (0,1], \end{cases}$$

and

$$\begin{cases} t^{1-\alpha}\underline{u}(t)|_{t=0} \le \dfrac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(A)} \int_0^1 \dfrac{s^{1-\alpha}}{10}\underline{u}(s)ds, \\ t^{1-\alpha}\overline{u}(t)|_{t=0} \ge \dfrac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(A)} \int_0^1 \dfrac{s^{1-\alpha}}{10}\overline{u}(s)ds, \\ t^{1-\alpha}\underline{v}(t)|_{t=0} \le \dfrac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(B)} \int_0^1 \dfrac{s^{1-\alpha}}{10}\underline{v}(s)ds, \\ t^{1-\alpha}\overline{v}(t)|_{t=0} \ge \dfrac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(B)} \int_0^1 \dfrac{s^{1-\alpha}}{10}\overline{v}(s)ds, \end{cases}$$

that is,

$$\begin{cases} 0 \le h_1(t),\ t \in (0,1], \\ L\Gamma(\alpha+1) \ge a_1(t)L(1+t) + a_2(t)L^{k_2}(1+t)^{k_2} + h_1(t),\ t \in (0,1], \\ 0 \le -b_1(t)L(1+t) + h_2(t),\ t \in (0,1], \\ L\Gamma(\alpha+1) \ge b_2(t)L(1+t) + h_2(t),\ t \in (0,1],\ t \in (0,1]. \end{cases}$$

If we suppose that the following constants

$$\widetilde{C_5} = \sup_{t\in(0,1]} \frac{h_1(t)}{\Gamma(\alpha+1) - a_1(t)(1+t) - a_2(t)(1+t)^{k_2}},$$

$$\widehat{C_5} = \sup_{t\in(0,1]} \frac{h_2(t)}{\Gamma(\alpha+1) - b_2(t)(1+t)},$$

and

$$C_6 = \inf_{t\in(0,1]} \frac{h_2(t)}{b_1(t)(1+t)},$$

are well defined and

$$C_5 := \max\left(1, \widetilde{C_5}, \widehat{C_5}\right) \le C_6,$$

and if we choose $C_5 \le L \le C_6$, then $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is a pair of lower -upper solutions of type 3 for problem (6.3). Consequently, we find from Theorem 5.4 that this problem admits at least a solution (u,v) such that $t^{1-\alpha}\underline{u}(t) \le t^{1-\alpha}u(t) \le t^{1-\alpha}\overline{u}(t)$ and $t^{1-\alpha}\underline{v}(t) \le t^{1-\alpha}v(t) \le t^{1-\alpha}\overline{v}(t)$ in $[0,1]$. □

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