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BOUNDED PERTURBATION RESILIENCE OF A VISCOSITY ITERATIVE METHOD FOR SPLIT FEASIBILITY PROBLEMS

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Abstract. In this article, we propose a viscosity approximation method to solve a split feasibility problem. The bounded perturbation resilience of the method is investigated in Hilbert spaces. As tools, averaged mappings and resolvents of maximal monotone operators are technically maneuvered to facilitate the proofs of the main results. Under mild conditions, we prove that our algorithms strongly converge to a solution of the split feasibility problem, which is also the unique solution of a variational inequality problem. Furthermore, we also show the convergence and effectiveness of the algorithms by a numerical example.

Keywords. Split feasibility problem; Maximal monotone operator; Averaged mapping; Variational inequality problem.

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1. INTRODUCTION

Let H_1 and H_2 be two real Hilbert spaces with inner product$\langle , \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H_1 and let P_C the metric projection from H_1 onto C. Let Q be a nonempty closed convex subsets of H_2 and let P_Q be the metric projection from H_2 onto Q. Recall that the split feasibility problem (SFP) is to find a point x^* satisfying the conditions

$$x^* \in C \ \ and \ \ Ax^* \in Q, \tag{1.1}$$

where A is a bounded linear operator from H_1 to H_2. Assume that $C \cap A^{-1}Q$ is nonempty (i.e., problem (1.1) has a solution). It is not hard to see that $x^* \in C \cap A^{-1}Q$ is equivalent to

$$x^* = P_C(I - \lambda A^*(I - P_Q)A)x^*, \tag{1.2}$$

where $\lambda > 0$ is some positive real number and A^* is the adjoint operator of A.

In 1994, the SFP was first introduced by Censor and Elfving [1]. They used their algorithm to solve the SFP in finite-dimensional Euclidean spaces. In 2002, Byrne [2] improved Censor and Elfving's

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algorithm in infinite-dimensional spaces and presented a new method called CQ algorithm for solving SFP (1.1):

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n, \tag{1.3}$$

In 2007, Censor, Motova and Segal [3] studied a multiple-sets split-feasibility problem. In 2010, Moudafi [4] proposed an iterative method to solve split common fixed point problems of quasi-nonexpansive mappings. In 2014, combining the Moudafi's method with the Halpern iterative method, Kraikaew and Saejung [5] proposed a new iterative algorithm that does not involve projection operators to solve a split common fixed point problem (SCFP). More precisely, they gave the following algorithm:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)U(x_n + \gamma A^*(T - I)Ax_n), \quad n \geq 0. \end{cases}$$

Under the reasonable conditions, they proved that $\{x_n\}$ is strongly convergent. If we take U and T as the projection operators P_C and P_Q, respectively, then $Fix(U) = C$ and $Fix(T) = Q$. Hence the SCFP immediately reduces to the SFP and the above algorithm can solve the SFP. In 2015, Takahashi, Xu and Yao [6] proposed the following algorithm:

$$x_{n+1} = J^B_{\lambda_n}(x_n - \tau_n A^*(I - T)Ax_n), \tag{1.4}$$

$J^B_{\lambda_n}$ is the resolvent operator of maximal monotone operator B and T is a nonexpansive mapping on H_2. They proved that the sequence generated by the above iterative process converges weakly to a point $x^* \in B^{-1}0 \cap A^{-1}Fix(T)$ in the framework of Hilbert spaces. We also describe that this problem is to find a point $x^* \in H_1$ such that

$$0 \in Bx^* \ \ and \ \ Ax^* \in Fix(T). \tag{1.5}$$

Especially, if $B = \partial I_C = N_C$ and $T = P_Q$, where

$$I_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \bar{\in} C \end{cases}$$

is the indicator function of closed convex subset C and

$$N_C x = \{u \in H : \ \langle u, x - y \rangle \geq 0, \forall y \in C\}$$

is the normal cone to C at $x \in C$. Then problem (1.5) reduces to SFP (1.1). We denote the solution set of problem (1.5) by $S = B^{-1}0 \cap A^{-1}Fix(T)$.

The following lemma is very useful in constructing iterative algorithms.

Lemma 1.1. [6] *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J^B_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. Let $\lambda, \tau > 0$. Then the following equality holds:*

$$Fix(J^B_\lambda(I - \tau A^*(I - T)A)) = (A^*(I - T)A + B)^{-1}0 = B^{-1}0 \cap A^{-1}Fix(T). \tag{1.6}$$

For the SFP and its extensions, many authors have studied them via fixed-point methods and weak-strong convergence theorems of solutions have been established in Hilbert or Banach spaces; see [7, 8, 9, 10, 11, 12, 13, 14, 15] and the references therein. Recently, bounded perturbation resilience of iterative methods have also been extensively studied; see [16, 17, 18, 19, 20] and the references therein.

This problem received much attention due to its applications in convex feasibility problems [21], inverse problems of radiation therapy [22] and image reconstruction [23], and so on.

Let $\mathbf{P}$ denote an algorithm operator, if the iteration $x_{n+1} = \mathbf{P}x_n$ is replaced by $x_{n+1} = \mathbf{P}(x_n + \beta_n v_n)$, where β_n is a sequence of nonnegative real numbers v_n is a sequence in H such that

$$\sum_{n=0}^{\infty} \beta_n < \infty \ \ and \ \ \|v_n\| \le M. \tag{1.7}$$

If the algorithm is still convergent, then algorithm $\mathbf{P}$ is bounded perturbation resilient; see [16] and the references therein.

In 2016, Jin, Censor and Jiang [19] introduced the projected scaled gradient (PSG) method with bounded perturbations for solving the following minimization problem:

$$\min_{x\in C} f(x), \tag{1.8}$$

where f is a continuous differentiable, convex function. Their method generates a sequence $\{x_n\}$ by the iterative scheme:

$$x_{n+1} = P_C(x_n - \gamma_n D(x_n)\nabla f(x_n) + e(x_n)), \ \ n \ge 0, \tag{1.9}$$

where $D(x_n)$ is a diagonal scaling matrix. Under suitable conditions, they obtained a convergence theorem.

Recently, Xu [20] projected the superiorization techniques for the relaxed PSG. The iterative algorithm is defined as following:

$$x_{n+1} = (1-\tau_n)x_n + \tau_n P_C(x_n - \gamma_n D(x_n)\nabla f(x_n) + e(x_n)), \ \ n \ge 0, \tag{1.10}$$

where $\{\tau_n\}$ is a sequence in $[0,1]$. The weak convergence was proved in [20].

Very recently, Guo and Cui [24] presented the following modified proximal gradient algorithm with perturbations for solving non-smooth composite convex optimization problem $min_{x\in H}(f(x)+g(x))$:

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n) prox_{\lambda_n g}(x_n - \lambda_n \nabla f(x_n) + e(x_n)), \ \ n \ge 0, \tag{1.11}$$

where h is contractive. They obtained strong convergence and bounded resilience of the above method.

In this paper, motivated by the results in [6, 16, 20, 24], we propose a viscosity approximation method for solving problem (1.5) and prove our iterative method is bounded perturbation resilient. We prove the convergence point of the iterative method which is also the unique solution of some variational inequality problem. a numerical example is also given to demonstrate the effectiveness of our iterative schemes. Our method for problem (1.5) is as follows:

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n) J^B_{\lambda_n}(x_n - \tau_n A^*(I-T)Ax_n + e(x_n)). \tag{1.12}$$

We give the bounded perturbation of (1.12) yields a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n h(y_n) + (1-\alpha_n) J^B_{\lambda_n}(y_n - \tau_n A^*(I-T)Ay_n + e(y_n)), \end{cases} \tag{1.13}$$

We also discuss the convergence of the viscosity method and show it is bounded perturbation resilient.

2. PRELIMINARIES

Let $\{x_n\}$ be a sequence in a real Hilbert space H. We adopt the following notations:
(1) Denote $\{x_n\}$ converging weakly to x by $x_n \rightharpoonup x$ and $\{x_n\}$ converging strongly to x by $x_n \to x$.
(2) Use $Fix(T)$ to denote the set of fixed points of mapping T; that is, $Fix(T) = \{x \in H : Tx = x\}$.
(3) Denote the weak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$.

We also need the following definitions.

Definition 2.1. A mapping $F : H \to H$ is said to be
(i) Lipschizian if there exists a positive constant L such that

$$\|Fx - Fy\| \leq L\|x - y\|, \ \ \forall x, y \in H.$$

In particular, if $L = 1$, we say that F is nonexpansive, namely,

$$\|Fx - Fy\| \leq \|x - y\|, \ \forall x, y \in H;$$

if $L \in [0, 1)$, we say that F is contractive.
(ii) α-averaged mapping (α-av for short) if

$$F = (1 - \alpha)I + \alpha T,$$

where $\alpha \in [0, 1)$ and $T : H \to H$ is nonexpansive.

Definition 2.2. A mapping $B : H \to H$ is said to be
(i) monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \ \ \forall x, y \in H.$$

(ii) η-strongly monotone if there exists a positive constant η such that

$$\langle Bx - By, x - y\rangle \geq \eta\|x - y\|^2, \ \ \forall x, y \in H.$$

(iii) α-inverse strongly monotone(for short α-ism) if there exists a positive constant α such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \ \ \forall x, y \in H.$$

In particular, if $\alpha = 1$, we say that B is firmly nonexpansive, namely,

$$\langle Bx - By, x - y\rangle \geq \|Bx - By\|^2, \ \ \forall x, y \in H.$$

Definition 2.3. Let $B : H \to H$ be a monotone mapping. Then B is maximal monotone if there exists no monotone operator $A : H \to 2^H$ such that $graA$ properly contains $graB$, i.e., for every $(x, u) \in H \times H$,

$$(x, u) \in graB \ \Leftrightarrow \forall\ (y, v) \in graB, \ \langle x - y, u - v\rangle \geq 0.$$

The following two lemmas are trivial.

Lemma 2.4. *Let H be a real Hilbert space. There holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle x + y, y\rangle, \ \forall x, y \in H.$$

Lemma 2.5. *Let $h : H \to H$ be a ρ-contraction with $\rho \in (0, 1)$ and let $T : H \to H$ be a nonexpansive mapping. Then*

(i) *$I-h$ is $(1-\rho)$-strongly monotone:*

$$\langle (I-h)x-(I-h)y, x-y\rangle \geq (1-\rho)\|x-y\|^2, \quad \forall x,y \in H.$$

(ii) *$I-T$ is monotone:*

$$\langle (I-T)x-(I-T)y, x-y\rangle \geq 0, \quad \forall x,y \in H.$$

Proposition 2.6. [25]

(i) *If $T_1, T_2, \cdots, T_n$ are averaged mappings, then $T_nT_{n-1}\cdots T_1$ is averaged. In particular, if T_i is α_i-av, i=1,2, where $\alpha_i \in (0,1)$, then T_2T_1 is $(\alpha_2+\alpha_1-\alpha_2\alpha_1)$-av.*

(ii) *If $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1 \ldots T_N).$$

(iii) *A mapping T is nonexpansive if and only if $I-T$ is $\frac{1}{2}$-ism.*

(iv) *If T is ν-ism, then, for $\tau > 0$, τT is $\frac{\nu}{\tau}$-ism.*

(v) *T is averaged if and only if $I-T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $0<\alpha<1$, T is α-averaged if and only if $I-T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.7. [6] *Assume that H_1 and H_2 are Hilbert spaces. Let $B: H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $A: H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T: H_2 \to H_2$ be a nonexpansive mapping. Then*

(i) *$A^*(I-T)A$ is $\frac{1}{2\|A\|^2}$-ism.*

(ii) *For $0 < \tau < \frac{1}{\|A\|^2}$,*
$I-\tau A^*(I-T)A$ is $\tau\|A\|^2$-averaged and $J_\lambda^B(I-\tau A^*(I-T)A)$ is $\frac{1+\tau\|A\|^2}{2}$-averaged.

Lemma 2.8. [26] *Let H be a real Hilbert space, and let $T: H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particulary, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.9. [27] *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \; n \geq 0,$$

$$s_{n+1} \leq s_n - \eta_n + \varphi_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\varphi_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} \varphi_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence $(n_k) \subset (n)$.*

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.10. [28] *Let B be a maximal monotone operator. Let $J_\lambda^B = (I+\lambda B)^{-1}$ and $J_\mu^B = (I+\mu B)^{-1}$, where $\lambda > 0$ and $\mu > 0$ are two real numbers, be the resolvent operators of B. Then*

$$J_\lambda^B x = J_\mu^B\left(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda^B x\right), \quad \forall x \in H.$$

3. Main results

Let C be a nonempty closed and convex subset of a Hilbert space H. Let $N : C \to C$ be a nonexpansive mapping and let $h : H \to H$ be a ρ-contractive mapping. In 2000, Moudafi [29] proposed the following viscosity approximation method:

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)Nx_n,$$

where $\{\alpha_n\}$ is some sequence in $(0,1)$. He proved that the viscosity algorithm converges strongly to a fixed point x^* of nonexpansive mapping N and x^* also uniquely solves the following variational inequality

$$\langle (I-h)x^*, \tilde{x}-x^* \rangle \geq 0, \quad \forall \tilde{x} \in Fix(N). \tag{3.1}$$

Recently, some authors extended the above results to the framework of Banach spaces; see [30, 31, 32] and the references therein.

In this section, we present a viscosity iterative algorithm for solving problem (1.5). Rewrite iteration (1.12) as

$$\begin{aligned} x_{n+1} &= \alpha_n h(x_n) + (1-\alpha_n)J^B_{\lambda_n}(x_n - \tau_n A^*(I-T)Ax_n + e(x_n)) \\ &= \alpha_n h(x_n) + (1-\alpha_n)(J^B_{\lambda_n}(x_n - \tau_n A^*(I-T)Ax_n) + \tilde{e}_n), \end{aligned} \tag{3.2}$$

Since $J^B_{\lambda_n}$ is nonexpansive, we find that

$$\begin{aligned} \|\tilde{e}_n\| &= \|J^B_{\lambda_n}(x_n - \tau_n A^*(I-T)Ax_n + e(x_n)) - J^B_{\lambda_n}(x_n - \tau_n A^*(I-T)Ax_n)\| \\ &\leq \|e(x_n)\|. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let H_1, H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $B : H_1 \to 2^{H_1}$ is a maximal monotone operator and $T : H_2 \to H_2$ is a nonexpansive mapping. Assume that $S = B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. Let h be a ρ-contractive on H_1 with $0 \leq \rho < 1$. Choose $x_0 \in H_1$ arbitrarily and define a sequence $\{x_n\}$ in the following manner:*

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)J^B_{\lambda_n}(x_n - \tau_n A^*(I-T)Ax_n + e(x_n)). \tag{3.4}$$

If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \infty$;
(iii) $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(x_n)\| < \infty$,

then $\{x_n\}$ converges strongly to $x^* \in B^{-1}0 \cap A^{-1}Fix(T)$, which is also the unique solution of variational inequality problem (3.1).

Proof. Set $V_n := J^B_{\lambda_n}(I - \tau_n A^*(I-T)A)$. From Proposition 2.7, it is easy to obtain that $J^B_{\lambda_n}(I - \tau_n A^*(I - T)A)$ is $\frac{1+\tau_n L}{2}$-av as $0 < \tau_n < 1/L$.

Next, we split the proof into 3 steps.

Step 1. Show that $\{x_n\}$ is bounded.

For any $z \in S$, we have

$$\begin{aligned} \|x_{n+1}-z\| &= \|\alpha_n h(x_n)+(1-\alpha_n)(V_nx_n+\tilde{e}_n)-z\| \\ &= \|\alpha_n(h(x_n)-z)+(1-\alpha_n)(V_nx_n-z)+(1-\alpha_n)\tilde{e}_n\| \\ &\le \alpha_n\|h(x_n)-h(z)\|+\alpha_n\|h(z)-z\|+(1-\alpha_n)\|V_nx_n-z\|+\|\tilde{e}_n\| \\ &\le \alpha_n\rho\|x_n-z\|+\alpha_n\|h(z)-z\|+(1-\alpha_n)\|x_n-z\|+\|\tilde{e}_n\| \\ &= (1-\alpha_n(1-\rho))\|x_n-z\|+\alpha_n(1-\rho)\frac{\|h(z)-z\|+\|\tilde{e}_n\|/\alpha_n}{1-\rho}. \end{aligned} \tag{3.5}$$

From conditions (i), (iv) and $\alpha_n > 0$, we find that $\{\|\tilde{e}_n\|/\alpha_n\}$ is bounded. Thus there exists some $M_1 > 0$ such that $\sup\{\|h(z)-z\|+\|\tilde{e}_n\|/\alpha_n\} \le M_1$ for all $n \ge 0$. An induction argument shows that

$$\|x_n-z\| \le \max\{\|x_0-z\|,\ \frac{M_1}{1-\rho}\},$$

which implies that sequence $\{x_n\}$ is bounded, so are $\{h(x_n)\}$, $\{V_nx_n\}$ and $\{A^*(I-T)Ax_n\}$.

Step 2. Show that for any sequence $(n_k) \subset (n)$, $\lim_{k\to\infty}\|x_{n_k}-V_{n_k}x_{n_k}\| = 0$.

Fixing $z \in S$, we have

$$\begin{aligned} &\|x_{n+1}-z\|^2 \\ &= \|\alpha_n h(x_n)+(1-\alpha_n)(V_nx_n+\tilde{e}_n)-z\|^2 \\ &\le \|\alpha_n h(x_n)+(1-\alpha_n)V_nx_n-z\|^2+2\langle\alpha_n h(x_n)+(1-\alpha_n)V_nx_n-z,(1-\alpha_n)\tilde{e}_n\rangle+\|\tilde{e}_n\|^2 \\ &\le \alpha_n^2\|h(x_n)-z\|^2+(1-\alpha_n)^2\|V_nx_n-z\|^2+2\alpha_n(1-\alpha_n)\langle h(x_n)-z,V_nx_n-z\rangle \\ &\quad +(2\alpha_n\|h(x_n)-z\|+2(1-\alpha_n)\|x_n-z\|+\|\tilde{e}_n\|)\|\tilde{e}_n\| \\ &\le 2\alpha_n^2(\|h(x_n)-h(z)\|^2+\|h(z)-z\|^2)+(1-\alpha_n)^2\|V_nx_n-z\|^2 \\ &\quad +2\alpha_n(1-\alpha_n)\langle h(x_n)-z,V_nx_n-z\rangle+M_2\|\tilde{e}_n\| \\ &\le 2\alpha_n^2(\|h(x_n)-h(z)\|^2+\|h(z)-z\|^2)+(1-\alpha_n)^2\|V_nx_n-z\|^2 \\ &\quad +2\alpha_n(1-\alpha_n)(\|h(x_n)-h(z)\|\|x_n-z\|+\langle h(z)-z,V_nx_n-z\rangle)+M_2\|\tilde{e}_n\| \\ &\le (1-\alpha_n(2-\alpha_n(1+2\rho^2)-2(1-\alpha_n)\rho))\|x_n-z\|^2+2\alpha_n(1-\alpha_n)\langle h(z)-z,V_nx_n-z\rangle \\ &\quad +2\alpha_n^2\|h(z)-z\|^2+M_2\|\tilde{e}_n\|, \end{aligned} \tag{3.6}$$

where

$$M_2 := \sup_{n\in\mathbb{N}}\{2\alpha_n\|h(x_n)-z\|+2(1-\alpha_n)\|x_n-z\|+\|\tilde{e}_n\|\}.$$

Note that

$$V_n = J_{\lambda_n}^B(I-\tau_nA^*(I-T)A) = (1-w_n)I+w_nU_n, \tag{3.7}$$

where $w_n = \frac{1+\tau_nL}{2}$, and U_n is nonexpansive. By condition (iii), we get that

$$\frac{1}{2} < \liminf_{n\to\infty} w_n \le \limsup_{n\to\infty} w_n < 1.$$

Since $z \in S$, we have $V_n z = z$. Furthermore, one has $(1-w_n)z + w_n U_n z = z$. It is easy to get $U_n z = z$. Thus, we find from (3.2) and (3.6) that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|\alpha_n h(x_n) + (1-\alpha_n)(V_n x_n + \tilde{e}_n) - z\|^2 \\
&\leq \|\alpha_n h(x_n) + (1-\alpha_n)V_n x_n - z\|^2 + M_2\|\tilde{e}_n\| \\
&= \|V_n x_n - z + \alpha_n(h(x_n) - V_n x_n)\|^2 + M_2\|\tilde{e}_n\| \\
&= \|V_n x_n - z\|^2 + \alpha_n{}^2\|h(x_n) - V_n x_n\|^2 + 2\alpha_n\langle V_n x_n - z, h(x_n) - V_n x_n\rangle + M_2\|\tilde{e}_n\| \\
&= \|(1-w_n)x_n + w_n U_n x_n - z\|^2 + \alpha_n{}^2\|h(x_n) - V_n x_n\|^2 \\
&\quad + 2\alpha_n\langle V_n x_n - z, h(x_n) - V_n x_n\rangle + M_2\|\tilde{e}_n\| \\
&= (1-w_n)\|x_n - z\|^2 + w_n\|U_n x_n - U_n z\|^2 - w_n(1-w_n)\|U_n x_n - x_n\|^2 \\
&\quad + \alpha_n{}^2\|h(x_n) - V_n x_n\|^2 + 2\alpha_n\langle V_n x_n - z, h(x_n) - V_n x_n\rangle + M_2\|\tilde{e}_n\| \\
&\leq \|x_n - z\|^2 - w_n(1-w_n)\|U_n x_n - x_n\|^2 + \alpha_n{}^2\|h(x_n) - V_n x_n\|^2 \\
&\quad + 2\alpha_n\langle V_n x_n - z, h(x_n) - V_n x_n\rangle + M_2\|\tilde{e}_n\|. \qquad (3.8)
\end{aligned}$$

Furthermore, we set

$s_n = \|x_n - z\|^2, \qquad \gamma_n = \alpha_n(2 - \alpha_n(1+2\rho^2) - 2(1-\alpha_n)\rho),$

$\delta_n = \frac{1}{2-\alpha_n(1+2\rho^2)-2(1-\alpha_n)\rho}[2\alpha_n\|h(z)-z\|^2 + M_2\frac{\|\tilde{e}_n\|}{\alpha_n} + 2(1-\alpha_n)\langle h(z)-z, V_n x_n - z\rangle],$

$\eta_n = w_n(1-w_n)\|U_n x_n - x_n\|^2$, and

$\varphi_n = \alpha_n{}^2\|h(x_n) - V_n x_n\|^2 + 2\alpha_n\langle V_n x_n - z, h(x_n) - V_n x_n\rangle + M_2\|\tilde{e}_n\|.$

Note that $\gamma_n \to 0$, $\Sigma_{n=0}^{\infty}\gamma_n = \infty$ $(\lim_{n\to\infty}(2-\alpha_n(1+2\rho^2) - 2(1-\alpha_n)\rho) = 2(1-\rho) > 0)$ and $\varphi_n \to 0$ $(\alpha_n \to 0)$. From Lemma 2.9, it suffices to verify that $\eta_{n_k} \to 0$ $(k\to\infty)$ implies that $\limsup_{k\to\infty}\delta_{n_k} \leq 0$ for any subsequence $(n_k) \subset (n)$. Indeed, $\eta_{n_k} \to 0$ $(k\to\infty)$ implies that $\|U_{n_k}x_{n_k} - x_{n_k}\| \to 0$ $(k\to\infty)$ due to condition (iii). It follows from (3.7) that

$$\|x_{n_k} - V_{n_k}x_{n_k}\| = w_{n_k}\|x_{n_k} - U_{n_k}x_{n_k}\| \to 0. \qquad (3.9)$$

Step 3. Show that

$$\omega_w(x_{n_k}) \subset S, \qquad (3.10)$$

where $\omega_w(x_{n_k})$ is the set of all weak cluster points of $\{x_{n_k}\}$.

Take $\tilde{x} \in \omega_w\{x_{n_k}\}$ and assume that $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_{n_k}\}$ weakly converging to $\tilde{x}$. Without loss of generality, we still use $\{x_{n_k}\}$ to denote $\{x_{n_{k_j}}\}$. Assume $\tau_{n_k} \to \tau$. Then $0 < \tau < \frac{1}{L}$. Similarly, we take a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ by condition (ii), and assume $\lambda_{n_k} \to \lambda$. Letting $V = J_\lambda^B(I - \tau A^*(I - T)A)$, we see that V is nonexpansive. Set

$$t_k = x_{n_k} - \tau_{n_k}A^*(I-T)Ax_{n_k}, \;\; z_k = x_{n_k} - \tau A^*(I-T)Ax_{n_k}.$$

Using the resolvent identity, we deduce that

$$\begin{aligned}
&\|V_{n_k}x_{n_k} - Vx_{n_k}\| \\
&= \|J^B_{\lambda_{n_k}}(x_{n_k} - \tau_{n_k}A^*(I-T)Ax_{n_k}) - J^B_\lambda(x_{n_k} - \tau A^*(I-T)Ax_{n_k})\| \\
&= \|J^B_\lambda(\frac{\lambda}{\lambda_{n_k}}t_k + (1-\frac{\lambda}{\lambda_{n_k}})J^B_{\lambda_{n_k}}t_k) - J^B_\lambda(z_k)\| \\
&\le \|\frac{\lambda}{\lambda_{n_k}}t_k + (1-\frac{\lambda}{\lambda_{n_k}})J^B_{\lambda_{n_k}}t_k - z_k\| \\
&\le \frac{\lambda}{\lambda_{n_k}}\|t_k - z_k\| + (1-\frac{\lambda}{\lambda_{n_k}})\|J^B_{\lambda_{n_k}}t_k - z_k\| \\
&= \frac{\lambda}{\lambda_{n_k}}|\tau_{n_k} - \tau|\|A^*(I-T)Ax_{n_k}\| + (1-\frac{\lambda}{\lambda_{n_k}})\|J^B_{\lambda_{n_k}}t_k - z_k\|. \qquad (3.11)
\end{aligned}$$

Since $\lambda_{n_k} \to \lambda$ and $\tau_{n_k} \to \tau$ as $k \to \infty$, we immediately derive from the last relation that $\|V_{n_k}x_{n_k} - Vx_{n_k}\| \to 0$. As a result, we find

$$\|x_{n_k} - Vx_{n_k}\| \le \|x_{n_k} - V_{n_k}x_{n_k}\| + \|V_{n_k}x_{n_k} - Vx_{n_k}\| \to 0. \qquad (3.12)$$

Using Lemma 2.8, we get $\omega_w(x_{n_k}) \subset Fix(V)$. It follows from Lemma 1.1 that $\omega_w(x_{n_k}) \subset S$. We also have

$$\begin{aligned}
&\limsup_{k\to\infty}\langle h(x^*) - x^*, V_{n_k}x_{n_k} - x^*\rangle \\
=&\limsup_{k\to\infty}\langle h(x^*) - x^*, x_{n_k} - x^*\rangle + \lim_{k\to\infty}\langle h(x^*) - x^*, V_{n_k}x_{n_k} - x_{n_k}\rangle, \qquad (3.13)
\end{aligned}$$

and

$$\limsup_{k\to\infty}\langle h(x^*) - x^*, x_{n_k} - x^*\rangle = \langle h(x^*) - x^*, \tilde{x} - x^*\rangle, \quad \forall \tilde{x} \in S. \qquad (3.14)$$

It is easy to get from (3.9) the second item of (3.13) tends to zero. Also, since x^* is the unique solution of variational inequality problem (3.1), we obtain that

$$\langle h(x^*) - x^*, \tilde{x} - x^*\rangle \le 0.$$

Hence $\limsup_{k\to\infty}\delta_{n_k} \le 0$. □

The bounded perturbation of (3.4) by the following iterative process:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n h(y_n) + (1-\alpha_n)J^B_{\lambda_n}((I - \tau_n A^*(I-T)A)y_n + e(y_n)). \end{cases} \qquad (3.15)$$

Similarly, we put

$$\tilde{e}_n = J^B_{\lambda_n}((I - \tau_n A^*(I-T)A)y_n + e(y_n)) - J^B_{\lambda_n}(I - \tau_n A^*(I-T)A(y_n)). \qquad (3.16)$$

Theorem 3.2. *Assume the sequences $\{\beta_n\}$ and $\{v_n\}$ satisfy condition (1.7). Let H_1, H_2 be two real Hilbert spaces and let A be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $B: H_1 \to 2^{H_1}$ is a maximal monotone operator and $T: H_2 \to H_2$ is a nonexpansive mapping. Assume that $S = B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. Let h be a ρ-contractive mapping on H_1 with $0 \le \rho < 1$. Choose $x_0 \in H_1$ arbitrarily and define the sequence $\{x_n\}$ by (3.15). If the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < \infty$;

(iii) $0 < \liminf_{n\to\infty} \tau_n \le \limsup_{n\to\infty} \tau_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(y_n)\| < \infty$,

then $\{x_n\}$ *converges strongly to* x^*, *where* x^* *is a solution of problem (1.5), which is also the unique solution of variational inequality problem (3.1).*

Proof. We can rewrite (3.15) as

$$x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)(J^B_{\lambda_n}(I - \tau_n A^*(I-T)A)(x_n) + \tilde{e}_n) + \bar{e}_n, \tag{3.17}$$

where

$$\bar{e}_n = \alpha_n(h(y_n) - h(x_n)) + (1-\alpha_n)(J^B_{\lambda_n}(I - \tau_n A^*(I-T)A)y_n - J^B_{\lambda_n}(I - \tau_n A^*(I-T)A)x_n), \tag{3.18}$$

In fact, by Proposition 2.7 (i), we see that $A^*(I-T)A$ is $\frac{1}{2L}$-ism. It is not hard to see that it is $2L$-Lipschitz. Thus,

$$\begin{aligned}\|\bar{e}_n\| &\le \alpha_n \|h(y_n) - h(x_n)\| + (1-\alpha_n)\|y_n - x_n - \tau_n(A^*(I-T)Ay_n - A^*(I-T)Ax_n)\| \\ &\le \alpha_n \rho \|y_n - x_n\| + (1-\alpha_n)(\|y_n - x_n\| + 2\tau_n L \|y_n - x_n)\| \\ &\le (\alpha_n \rho + (1-\alpha_n)(1+2\tau_n L))\beta_n \|v_n\|.\end{aligned} \tag{3.19}$$

From condition (1.7), it turns out $\sum_{n=0}^{\infty} \|\bar{e}_n\| < \infty$. Consequently, we find from Theorem 3.1 that algorithm (3.4) is bounded perturbation resilient. □

4. Numerical Results

In this section, we consider the following numerical example to demonstrate the effectiveness, realization, and convergence of Theorem 3.1.

Let $H_1 = H_2 = \mathbb{R}^2$. Define $h(x) = \frac{1}{10}x$. Take $B : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$B = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$T = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}.$$

Observe that B is a positive linear operator. Then it is maximal monotone. T is a rotation operator. Then it is nonexpansive. So, we obtain the resolvent mapping $J^B_\lambda = (I + \lambda B)^{-1}$. It follows that

$$J^B_\lambda = \frac{1}{(8\lambda+1)(2\lambda+1)} \begin{pmatrix} 2\lambda+1 & 0 \\ 0 & 8\lambda+1 \end{pmatrix}.$$

Generate a 2×2 random matrix A, and compute the Lipschitz constant $L = \|A^T A\|$, where A^T represents the transpose of A. Take $\lambda_n = 0.5$, $\tau_n = \tau = \frac{1}{100*L}$ and $\alpha_n = \frac{1}{n+1}$.

According to the iterative process of Theorem 3.1, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \frac{1}{n+1} * \frac{1}{10} x_n + (1 - \frac{1}{n+1}) J^B_{\lambda_n}(x_n - \tau_n A^T (I-T)Ax_n).$$

As $n \to \infty$, we have $\{x_n\} \to x^*$. Taking random initial guess x_0, we obtain the numerical experiment results in Table 1.

TABLE 1. $x_0 = rand(2,1)$.

$\tau = \frac{1}{100*L}$	n(iterative number)	$time(s)$	x_n	$err(\|x_{n+1} - x_n\|)$
0.0069	20	0.000067	$10^{-6} * [0.0000\ 0.1210]^T$	$6.2812 * 10^{-8}$
0.0070	22	0.000073	$10^{-7} * [-0.0002\ 0.1638]^T$	$8.4813 * 10^{-9}$
0.0173	27	0.000087	$10^{-9} * [-0.0002\ 0.9495]^T$	$4.8840 * 10^{-10}$

Next, we consider the algorithm with bounded perturbation resilience. Choose the bounded sequence $\{v_n\}$ and the summable nonnegative real sequence $\{\beta_n\}$ as follows:

$$v_n = \begin{cases} -\frac{d_n}{\|d_n\|}, & if\ 0 \neq d_n \in B(x_n), \\ 0, & if\ 0 \in B(x_n). \end{cases}$$

where $B(x_n) = (8x_n(1)\ \ 2x_n(2))^T$, $x_n(i),\ \ i = 1,2$ denotes the ith element of x_n, and $\beta_n = c^n$, for some $c \in (0,1)$. Setting $c = 0.5$, the numerical results can be seen in Table 2.

TABLE 2. $x_0 = rand(2,1)$.

$\tau = \frac{1}{100*L}$	n(iterative number)	$time(s)$	x_n	$err(\|x_{n+1} - x_n\|)$
0.0173	19	0.002022	$10^{-6} * [0.6864\ 0.0000]^T$	$5.5268 * 10^{-7}$
0.0056	22	0.000125	$10^{-7} * [-0.8573\ 0.0001]^T$	$6.8973 * 10^{-8}$
0.0046	25	0.002163	$10^{-7} * [-0.1070\ 0.0000]^T$	$8.6055 * 10^{-9}$

As we have seen, the error of the solution becomes smaller as the increasing iterative numbers. And, sequence $\{x_n\}$ converges to $(0,0)$ which is the solution of the example. Of course, it is also the unique solution of the variational inequality $\langle (I-h)x^*, x - x^* \rangle \geq 0$.

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