J. Nonlinear Funct. Anal. 2019 (2019), Article ID 11 https://doi.org/10.23952/jnfa.2019.11

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

SPLIT EQUALITY FIXED POINT PROBLEMS OF QUASI-NONEXPANSIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H_1, H_2, H_3 be three Hilbert spaces. Let $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ be two quasi-nonexpansive operators. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded and linear operators. The split equality fixed point problem of quasi-nonexpansive operators is to find $x \in H_1$ and $y \in H_2$ such that $x = T_1x$, $y = T_2y$ and $Ax = By$. In this paper, we introduce an iterative algorithm to solve the split equality fixed point problem. We show that the proposed algorithm is strongly convergent without any compactness imposed on the operators.

Keywords. Split equality problem; Split feasibility problem; Split common fixed point problem.

2010 Mathematics Subject Classification. 47H05, 47J25, 65K15.

1. Introduction

Throughout this paper, we always assume that H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induce norm $\|\cdot\|$. Let C be a closed and convex subset of H and let I be the identity operator on H. Let $\{x_n\}$ be a sequence in H. We use $x_n \to x$, where x is an element in H, to denote that $\{x_n\}$ converges strongly to $x \in H$ and use $x_n \rightharpoonup x$ to denote that $\{x_n\}$ converges weakly to $x \in H$. Let T be an operator on H. In this paper, we use $F(T)$ to denote the fixed-point set of operator T, i.e., $F(T) = \{x \in H : x = Tx\}$. In this paper, we use P_C to denote the orthogonal projection from H onto C

$$P_C(x) = \arg\min_{y\in C} \|x-y\|, \quad \forall x \in H.$$

Recall that an operator $T : H \mapsto H$ is said to be nonexpansive iff $\|Tx-Ty\| \le \|x-y\|$, $\forall x,y \in H$. Recall that T is said to be quasi-nonexpansive iff $F(T) \neq \emptyset$ and $\|Tx-y\| \le \|x-y\|$, $\forall x \in H, y \in F(T)$. Recall that T is said to be firmly nonexpansive iff

$$\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2, \quad \forall x,y \in H.$$

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Received September 7, 2018; Accepted February 22, 2019.

This shows $2T-I$ is a nonexpansive operator. Recall that T is said to be firmly quasi-nonexpansive iff $F(T)\neq\emptyset$ and

$$\langle x-y, Tx-y\rangle \geq \|Tx-y\|^2, \quad \forall x\in H, y\in F(T).$$

This shows $2T-I$ is a nonexpansive operator. For such an operator, we also say that it is directed. Recall that T is said to be strictly pseudocontractive iff there there exists a real constant $\kappa\in[0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+\kappa\|x-y-(Tx-Ty)\|, \quad \forall x,y\in H.$$

Recall that T is said to be strictly quasi-pseudocontractive iff $F(T)\neq\emptyset$ and there there exists a real constant $\kappa\in[0,1)$ such that

$$\|Tx-y\|^2 \leq \|x-y\|^2+\kappa\|x-Tx\|, \quad \forall x\in H, y\in F(T).$$

Recall that T is said to be pseudocontractive iff

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+\|x-y-(Tx-Ty)\|, \quad \forall x,y\in H.$$

Recall that T is said to be quasi-pseudocontractive iff $F(T)\neq\emptyset$

$$\|Tx-y\|^2 \leq \|x-y\|^2+\|x-Tx\|, \quad \forall x\in H, y\in F(T).$$

Recall that T is said to be demi-contractive iff $F(T)\neq\emptyset$ and there there exists a real constant $\kappa<1$ such that

$$\|Tx-y\|^2 \leq \|x-y\|^2+\kappa\|x-Tx\|, \quad \forall x\in H, y\in F(T).$$

Recall that an operator $T: H\to H$ is said to be demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx=0$. Recall that an operator $T: H\to H$ is said to be semi-compact if for any bounded sequence $\{x_n\}$ with $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and some element x^* such that $x_{n_j}\to x^*$ as $j\to\infty$.

Split feasibility problems are mathematical models of inverse problems arising from image restoration, signal processing and radiation therapy treatment. In 1994, Censor and Elfving [1], in the framework of finite dimensional spaces, first studied the following split feasibility problem: find a point $x\in H_1$ such that

$$x\in C, \quad Ax\in Q, \tag{1.1}$$

where C,Q are closed and convex sets, and A is bounded linear operator. They also investigated several iterative methods to solve problem (1.1).

Let H_1 and H_2 be two Hilbert spaces. Let C be a nonempty closed convex subset of H_1 and let Q be a nonempty closed convex subset of H_2. In 2002, Byrne [2], based on orthogonal projections and fixed-point methods, introduced the the following known CQ-algorithm:

$$x_{n+1}=P_C(I-\gamma A^*(I-P_Q)A)x_n,$$

where P_C is the metric projection from H_1 onto C and P_Q is the metric projection from H_2 onto Q, γ is some positive real number, A is a bounded linear operator and A^* is the joint joint operator of A.

In 2010, Xu [3] studied, in the frame of infinite dimensional Hilbert spaces, the split feasibility problem. It was proved that split feasibility problem (1.1) is equivalent to a fixed point problem, that is,

$$x=P_C(x-\gamma A^*(I-P_Q)Ax). \tag{1.2}$$

Based on fixed-point methods, Xu obtained the weak convergence of a Mann's algorithm in infinite dimensional Hilbert spaces. Recently, many weak and strong convergence theorems were established via Mann-type algorithms and different regularization methods; see [4, 5, 6, 7, 8, 9] and the references therein.

Let H_1 and H_2 be two real Hilbert spaces. Let $T_1 : H_1 \to H_1$ be a nonlinear operator with $F(T_1) \neq \emptyset$ and let $T_2 : H_2 \to H_2$ be a nonlinear operator with $F(T_2) \neq \emptyset$. Recently, many authors further studied the split feasibility problem with nonlinear operator constraints, that is, split common fixed point problem. This problem consists of finding $x \in H_1$ such that

$$x \in F(T_1), \quad Ax \in F(T_2), \tag{1.3}$$

where $A : H_1 \to H_2$ is a bounded linear operator. To solve problem (1.3), Censor and Segal [10] proposed the following iterative method: for any initial guess $x_1 \in H_1$, define $\{x_n\}$ recursively by

$$x_{n+1} = T_1(x_n - \lambda A^*(I - T_2)Ax_n),$$

where T_1 and T_2 are directed operators. The further generalization of this algorithm was studied by Moudafi [11]. Under suitable conditions, he proved that sequence $\{x_n\}$ converges weakly to a point of problem (1.3).

Recently, Moudafi [12], and Moudafi and Al-Shemas [13] introduced the following split equality feasibility problem (SEFP):

$$find\ x \in C,\ y \in Q \quad such\ that\ Ax = By, \tag{1.4}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, C is a nonempty closed and convex subset of H_1, Q is a nonempty closed and convex subset of H_2. If $B = I$ and $H_3 = H_2$, then problem (1.4) is reduced to problem 1.1. Recent, many authors studied problems (1.1) and (1.4) with regularization methods; see [14, 15, 16, 17, 18] and the references therein.

Recently, Moudafi [19] introduced the following split equality fixed point problem:

$$find\ x \in F(T_1),\ y \in F(T_2)\ such\ that\ Ax = By, \tag{1.5}$$

where H_1, H_2, H_3, A, B are the same as in problem (1.4), and $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are two nonlinear operators with $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$. For solving problem (1.5), Moudafi [19] introduced the following iterative scheme:

$$\begin{cases} x_{n+1} = T_1(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T_2(y_n + \beta_n B^*(Ax_{n+1} - By_n)), \end{cases}$$

where T_1 and T_2 are firmly quasi-nonexpansive operators. Convergence theorems of solutions were established in the framework of Hilbert spaces.

In 2015, Chang, Wang and Qin [20] introduced the following iterative scheme:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi)I + \xi T_1((1-\eta)I + \eta T_1))u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \gamma_n y_n + (1-\alpha_n)((1-\xi)I + \xi S((1-\eta)I + \eta T_2))v_n, \end{cases}$$

where T_1 and T_2 are quasi-pseudocontractive operators. Under suitable conditions imposed on the control sequences, they obtained weak convergence results without the semi-compactness on operators T_1 and T_2.

In this paper, we introduce a new iterative scheme to solve problem (1.5) with two quasi-nonexpansive operators T_1 and T_2:

$$\begin{cases} u_n^1 = x_n - \lambda A^*(Ax_n - By_n), \\ u_n^2 = y_n + \lambda B^*(Ax_n - By_n), \\ v_n^1 = \alpha_n u_n^1 + (1-\alpha_n)T_1 u_n^1, \\ v_n^2 = \alpha_n u_n^2 + (1-\alpha_n)T_2 u_n^2, \\ S_{n+1}^1 = \{l^1 \in S_n^1 : \|v_n^1 - l^1\| \le \|u_n^1 - l^1\| \le \|x_n - l^1\|\}, \\ S_{n+1}^2 = \{l^2 \in S_n^2 : \|v_n^2 - l^2\| \le \|u_n^2 - l^2\| \le \|y_n - l^2\|\}, \\ x_{n+1} = P_{S_{n+1}^1} x_1, \\ y_{n+1} = P_{S_{n+1}^2} y_1, \end{cases} \tag{1.6}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, A^* is the adjoint operator of A, B^* is the adjoint operator of B, T_1 is a quasi-nonexpansive operator and T_2 is a quasi-nonexpansive operator, $\{\alpha_n\}$ is a real sequence in $(0,1)$ and λ is some positive real number. In Section 3, we will show the sequence generated in (1.6) is strongly convergent without semi-compact assumptions imposed on both T_1 and T_2.

2. Preliminaries

In this short section, we list some necessary definitions and lemmas. These are important to prove our main results.

Let H be a real Hilbert space and let C be a convex closed set in H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\| P_C x - x \| = \inf\{\| y - x \| : y \in C\}$. P_C is called the nearest point (metric) projection from H onto C. For any $x \in H$, $z = P_C x$ iff

$$\langle x - z, y - z\rangle \le 0, \forall y \in C.$$

It is known that P_C has the following properties.

For any $x, y \in H$ and $z \in C$, it holds

(i) $\| P_C x - P_C y \|^2 \le \langle P_C x - P_C y, x - y\rangle$;

(ii) $\|P_C x - z\|^2 + \|P_C x - x\|^2 \le \|x - z\|^2$.

The following is useful and known equality in the framework of Hilbert spaces

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 - \lambda(1-\lambda)\|x - y\|^2 + (1-\lambda)\|y\|^2,$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Let C be a closed and convex subset of a Hilbert space H, and let $T : C \to C$ be an operator such that $F(T) \ne \emptyset$. Recall that T is said to be demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$. This property is known as the Demiclosed Principle.

We know that every nonexpansive mapping has the demiclosed principle.

Lemma 2.1. [21] *(Demiclosed Principle) Let C be a closed and convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive operator such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$.*

3. Main results

Set $H = H_1 \times H_2$, $T = (T_1, T_2)$ and $G = (A, -B) : H \to H_3$. Denote by G^* the adjoint operator of G. Then the equivalent form of problem (1.5) is as follows

$$find\ w \in F(T)\ such\ that\ Gw = 0. \tag{3.1}$$

Set $u_n = (u_n^1, u_n^2)$, $v_n = (v_n^1, v_n^2)$, $l = (l^1, l^2)$, $S_n = S_n^1 \times S_n^2$ and $w_n = (x_n, y_n)$. Then iterative scheme (1.6) can be reformed as follows

$$\begin{cases} u_n = w_n - \lambda G^* G w_n, \\ v_n = \alpha_n u_n + (1-\alpha_n) T u_n, \\ S_{n+1} = \{l \in S_n : \|v_n - l\| \le \|u_n - l\| \le \|w_n - l\|\}, \\ w_{n+1} = P_{S_{n+1}} w_1. \end{cases} \tag{3.2}$$

Now, we are ready to present our main result.

Theorem 3.1. *Let H_1, H_2 and H_3 be three Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let A^* is the adjoint operator of A and let B^* be the adjoint operator of B. Let $T_1 : H_1 \to H_2$ be a quasi-nonexpansive operator such that $F(T_1) \neq \emptyset$ and let $T_2 : H_2 \to H_3$ be a quasi-nonexpansive operator such that $F(T_2) \neq \emptyset$. Let T be a quasi-nonexpansive operator satisfying the demiclosed principle. Let λ be a real number in $(0, \frac{2}{\|G\|^2})$ and let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Let Γ be the solution set of problem (3.1). Assume that $\Gamma = \{w \in H : Tw = w, Gw = 0\} \neq \emptyset$. Choose an initial element $w_1 \in H$ arbitrarily and take $S_1 = H$. Let $\{w_n\}$ be a sequence generated in (3.2). Then $\{w_n\}$ converges strongly to some element $w^* \in \Gamma$.*

Proof. First, we show that S_n is closed and convex for all $n \geq 1$. It is obvious that $S_1 = H$ is closed and convex. We assume that S_n is closed and convex. For any $l \in S_n$, one has

$$\|v_n - l\|^2 \le \|u_n - l\|^2 \Leftrightarrow 2\langle u_n - v_n, l\rangle \le \|u_n\|^2 - \|v_n\|^2,$$

and

$$\|u_n - l\|^2 \le \|w_n - l\|^2 \Leftrightarrow 2\langle w_n - u_n, l\rangle \le \|w_n\|^2 - \|u_n\|^2.$$

This shows that S_{n+1} is closed and convex. By the mathematical induction, S_n is closed and convex for all $n \geq 1$.

Next, we show that $\Gamma \subset S_n$ for all $n \geq 1$. Fixing $w \in \Gamma$, we have $Tw = w$ and $Gw = 0$. By the definition of $\{u_n\}$ in (3.2), we have

$$\begin{aligned}
\|u_n - w\|^2 &= \|w_n - \lambda G^* G w_n - w\|^2 \\
&= \|w_n - w\|^2 + \lambda^2 \|G^* G w_n\|^2 - 2\lambda \langle w_n - w, G^* G w_n \rangle \\
&= \|w_n - w\|^2 + \lambda^2 \|G^* G w_n\|^2 - 2\lambda \langle G w_n - Gw, G w_n \rangle \\
&\leq \|w_n - w\|^2 + \lambda^2 \|G\|^2 \|G w_n\|^2 - 2\lambda \|G w_n\|^2 \\
&= \|w_n - w\|^2 - \lambda (2 - \lambda \|G\|^2) \|G w_n\|^2.
\end{aligned} \tag{3.3}$$

On the other hand, from (3.2) the definition of $\{v_n\}$, we have

$$\begin{aligned}
\|v_n - w\| &= \|\alpha_n u_n + (1 - \alpha_n) T u_n - w\| \\
&\leq \alpha_n \|u_n - w\| + (1 - \alpha_n) \|T u_n - w\| \\
&\leq \alpha_n \|u_n - w\| + (1 - \alpha_n) \|u_n - w\| \\
&= \|u_n - w\|.
\end{aligned} \tag{3.4}$$

As a result, we conclude from (3.3) and (3.4) that $w \in S_n$. Since $w \in \Gamma$ is taken arbitrarily, we have $\Gamma \subset S_n$ for all $n \geq 1$.

Since $\Gamma \subset S_{n+1} \subset S_n$ and

$$w_{n+1} = P_{S_{n+1}}(w_1) \in S_{n+1} \subset S_n,$$

we obtain from the property of projection operators that

$$\|w_{n+1} - w_1\| \leq \|w - w_1\|, \quad \forall n \geq 1, \ w \in \Gamma. \tag{3.5}$$

This shows that $\{w_n\}$ is a bounded sequence. It follows that

$$\begin{aligned}
\|w_{n+1} - w_n\|^2 + \|w_1 - w_n\|^2 &= \|w_{n+1} - P_{S_n} w_1\|^2 + \|w_1 - P_{S_n} w_1\|^2 \\
&\leq \|w_{n+1} - w_1\|^2,
\end{aligned}$$

that is,

$$\|w_{n+1} - w_1\|^2 - \|w_n - w_1\|^2 \geq \|w_{n+1} - w_n\|^2 \geq 0.$$

It means that $\{\|w_n - w_1\|\}$ is a monotone sequence. From the boundedness of $\{w_n\}$, we obtain that $\lim_{n\to\infty} \|w_n - w_1\|$ exists. Take positive integers m, n arbitrarily, without loss of generality, with $m \geq n$. By $w_m = P_{S_m} w_1 \in S_n$, we know that

$$\|w_m - w_n\|^2 + \|w_n - w_1\|^2 = \|w_m - P_{S_n} w_1\|^2 + \|w_1 - P_{S_n} w_1\|^2 \leq \|w_m - w_1\|^2. \tag{3.6}$$

Due to the fact that $\lim_{n\to\infty} \|w_n - w_1\|$ exists, we conclude from (3.6) that $\lim_{n\to\infty} \|w_m - w_n\| = 0$. Thus we prove that $\{w_n\}$ is a Cauchy sequence.

Next, we show that $\lim_{n\to\infty} \|G w_n\| = 0$ and $\lim_{n\to\infty} \|w_n - T w_n\| = 0$. Observe

$$w_{n+1} = P_{S_{n+1}} w_1 \in S_{n+1} \subset S_n.$$

Using the definition of $\{S_n\}$ in (3.2), we have

$$\|u_n - w_n\| \leq \|u_n - w_{n+1}\| + \|w_{n+1} - w_n\| \leq 2\|w_{n+1} - w_n\| \to 0, \tag{3.7}$$

$$\|v_n - w_n\| \leq \|v_n - w_{n+1}\| + \|w_{n+1} - w_n\| \leq 2\|w_{n+1} - w_n\| \to 0, \tag{3.8}$$

and

$$\|u_n - v_n\| \le \|u_n - w_n\| + \|v_n - w_n\| \to 0. \tag{3.9}$$

From (3.3), we see that

$$\begin{aligned}\lambda(2-\lambda\|G\|^2)\|Gw_n\|^2 &\le \|w_n - w\|^2 - \|u_n - w\|^2\\ &= (\|w_n - w\| + \|u_n - w\|)(\|w_n - w\| - \|u_n - w\|)\\ &\le (\|w_n - w\| + \|u_n - w\|)\|w_n - u_n\|.\end{aligned} \tag{3.10}$$

Since $\{w_n\}$ is a bounded sequence, we see that $\{u_n\}$ is also bounded. Hence, using the assumption on parameter λ, it follows from (3.7) and (3.10) that

$$\lim_{n\to\infty} \|Gw_n\| = 0. \tag{3.11}$$

Using the definition of $\{v_n\}$ in (3.2), we also have

$$\begin{aligned}\|v_n - w\|^2 &= \|\alpha_n u_n + (1-\alpha_n)Tu_n - w\|^2\\ &= \|\alpha_n(u_n - w) + (1-\alpha_n)(Tu_n - w)\|^2\\ &= \alpha_n\|u_n - w\|^2 + (1-\alpha_n)\|Tu_n - w\|^2 - \alpha_n(1-\alpha_n)\|u_n - Tu_n\|^2\\ &\le \|u_n - w\|^2 - \alpha_n(1-\alpha_n)\|u_n - Tu_n\|^2,\end{aligned}$$

which shows that

$$\begin{aligned}\alpha_n(1-\alpha_n)\|u_n - Tu_n\|^2 &\le \|u_n - w\|^2 - \|v_n - w\|^2\\ &= (\|u_n - w\| + \|v_n - w\|)(\|u_n - w\| - \|v_n - w\|)\\ &\le (\|u_n - w\| + \|v_n - w\|)\|u_n - v_n\|.\end{aligned} \tag{3.12}$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, one finds from (3.9) and (3.12) that

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0 \tag{3.13}$$

and

$$\begin{aligned}\|w_n - Tw_n\| &\le \|w_n - u_n\| + \|u_n - Tu_n\| + \|Tu_n - Tw_n\|\\ &\le (1+\|T\|)\|w_n - u_n\| + \|u_n - Tu_n\|.\end{aligned}$$

By (3.7) and (3.13), we arrive at

$$\lim_{n\to\infty} \|w_n - Tw_n\| = 0. \tag{3.14}$$

Finally, we show that $\{w_n\}$ converges strongly to some element in Γ. Since $\{w_n\}$ is a Cauchy sequence, we may assume $w_n \to w^*$. From (3.14) and the demiclosed principle, we obtain that $w^* \in F(T)$. By the Fréchet-Riesz representation theorem and (3.11), we arrive at

$$\|Gw^*\|^2 = \langle Gw^*, Gw^*\rangle = \langle w^*, G^*Gw^*\rangle = \lim_{n\to\infty}\langle w_n, G^*Gw^*\rangle = \lim_{n\to\infty}\langle Gw_n, Gw^*\rangle = 0,$$

that is,

$$Gw^* = 0. \tag{3.15}$$

Combining $w^* \in F(T)$ with $Gw^* = 0$, we conclude that $w^* \in \Gamma$. It means that $\{w_n\}$ converges strongly to some element $w^* \in \Gamma$. The proof is complete. □

If $B = I$, $H_3 = H_2$ in problem (1.5), it is reduced to problem (1.3). From Theorem 3.1, we have the following result immediately.

Corollary 3.2. *Let H_1, H_2 be two Hilbert spaces. Let C be a nonempty closed and convex subset of H_1 and let Q be a nonempty closed and convex subset of H_2. Let $A : H_1 \to H_3$ be a bounded linear operator and let A^* is the adjoint operator of A. Let $T_1 : H_1 \to H_2$ be a quasi-nonexpansive operator such that $F(T_1) \neq \emptyset$ and let $T_2 : H_2 \to H_2$ be a quasi-nonexpansive operator such that $F(T_2) \neq \emptyset$. Let T be a quasi-nonexpansive operator satisfying the demiclosed principle. Let λ be a real number in $(0, \frac{2}{\|G\|^2})$ and let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Let Γ be the solution set of problem (1.3). Assume that $\Gamma = \{w \in H : Tw = w, Gw = 0\} \neq \emptyset$. Let $\tilde{G} = (A, -I)$ and let $\tilde{G}^* = \begin{pmatrix} A^* \\ -I \end{pmatrix}$. Choose an initial element $w_1 \in H$ arbitrarily and take $S_1 = H$. Let $\{w_n\}$ be a sequence generated in the following iterative process*

$$\begin{cases} u_n = w_n - \lambda \tilde{G}^* \tilde{G} w_n, \\ v_n = \alpha_n u_n + (1-\alpha_n) T u_n, \\ S_{n+1} = \{l \in S_n : \|v_n - l\| \leq \|u_n - l\| \leq \|w_n - l\|\}, \\ w_{n+1} = P_{S_{n+1}} w_1. \end{cases} \tag{3.16}$$

Then $\{w_n\}$ converges strongly to some element $w^* = (x^*, y^*) \in F(T)$, that is, $x^* = T_1 x^*, Ax^* = y^* = T_2 y^*$.

4. Applications

In this section, we consider the split equality equilibrium problem which consists of finding $x^* \in C, y^* \in Q$ such that

$$\begin{cases} (i)\ T_1(x^*, x) \geq 0, \forall\, x \in C, \\ (ii)\ T_2(y^*, y) \geq 0, \forall\, y \in Q, \\ (iii)\ Ax^* = By^*, \end{cases} \tag{4.1}$$

where $T_1 : C \times C \to R,\ T_2 : Q \times Q \to R$ are two equilibrium functions which satisfy:
(i) $T_1(x,x) = 0,\ T_2(y,y) = 0,\ \forall x \in C, y \in Q$;
(ii) $T_1(x_1,x_2) + T_1(x_2,x_1) \leq 0,\ T_2(y_1,y_2) + T_2(y_2,y_1) \leq 0,\ \forall x_1,x_2 \in C,\ y_1,y_2 \in Q$;
(iii) $\limsup_{t\downarrow 0} T_1(tx_3 + (1-t)x_1, x_2) \leq T_1(x_1,x_2)$, $\limsup_{t\downarrow 0} T_2(ty_3 + (1-t)y_1, y_2) \leq T_2(y_1,y_2)$, $\forall x_1,x_2,x_3 \in C,\ y_1,y_2,y_3 \in Q$;
(iv) $m \to T_1(x,m)$ is convex and lower semi-continuous, $\forall x \in C$,
$n \to T_2(y,n)$ is convex and lower semi-continuous, $\forall y \in Q$.

For any given $\rho > 0$ and $x \in H_1, y \in H_2$, define the resolvents of T_1, T_2 by $R_{\rho,T_1}, R_{\rho,T_2}$, respectively, by

$$R_{\rho,T_1}(x) = \{a \in C : T_1(a,b) + \frac{1}{\rho}\langle b-a, a-x\rangle \geq 0,\ \forall\, b \in C\},$$

and

$$R_{\rho,T_2}(y) = \{c \in Q : T_2(c,d) + \frac{1}{\rho}\langle d-c, c-y\rangle \geq 0,\ \forall\, d \in Q\}.$$

We use $VI(T_1,C), VI(T_2,Q)$ to denote the solution sets of (4.1)(i), (4.1)(ii), respectively.

Lemma 4.1. [22] *$R_{\rho,T_1}, R_{\rho,T_2}$ are single-valued. $F(R_{\rho,T_1}) = VI(T_1,C)$, $F(R_{\rho,T_2}) = VI(T_2,Q)$. $R_{\rho,T_1}, R_{\rho,T_2}$ are firmly nonexpansive.*

From the above lemma, one sees that R_{ρ,T_1} and R_{ρ,T_2} are quasi-nonexpansive operators satisfying the demiclosed principle. And problem (4.1) is equivalent to

$$finding\ x^* \in F(R_{\rho,T_1}),\ y^* \in F(R_{\rho,T_2})\ such\ that\ Ax^* = By^*. \tag{4.2}$$

Let $S = C \times Q, T = (T_1, T_2), R_{\rho,T} = (R_{\rho,T_1}, R_{\rho,T_2}), G = (A, -B)$. Then problem (4.2) can be reformed as

$$finding\ w^* \in F(R_{\rho,T})\ such\ \ that\ Gw^* = 0. \tag{4.3}$$

From Theorem 3.1, we have the following result.

Theorem 4.2. *Let $\Gamma = \{w \in S : R_{\rho,T}w = w, Gw = 0\}$ be the solution set of problem (4.3). Let λ be a real number in $(0, \frac{2}{\|G\|^2})$ and let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Choose an initial element $w_1 \in S$ arbitrarily and take $S_1 = S$. Let $\{x_n\}$ be a sequence generated in the following process*

$$\begin{cases} u_n = w_n - \lambda G^* G w_n, \\ v_n = \alpha_n u_n + (1-\alpha_n) R_{\rho,T} u_n, \\ S_{n+1} = \{l \in S_n : \|v_n - l\| \le \|u_n - l\| \le \|w_n - l\|\}, \\ w_{n+1} = P_{S_{n+1}} w_1. \end{cases} \tag{4.4}$$

If $\Gamma (\neq \emptyset)$, then $\{x_n\}$ converges strongly to some element $w^* \in \Gamma$.

Proof. From the proofs presented in Theorem 3.1, we can obtain our desired conclusion immediately. □

If $B = I$ and $H_3 = H_2$ in problem (4.3), it is reduced to e problem (1.3): split common fixed point problem of the operators R_{ρ,T_1} and R_{ρ,T_2}.

Corollary 4.3. *Let $\tilde{G} = (A, -I), \tilde{G}^* = \begin{pmatrix} A^* \\ -I \end{pmatrix}$. Choose $w_1 \in S$ arbitrarily and take $S_1 = S$, define a sequence $\{w_n\}$ by*

$$\begin{cases} u_n = w_n - \lambda \tilde{G}^* \tilde{G} w_n, \\ v_n = \alpha_n u_n + (1-\alpha_n) R_{\rho,T} u_n, \\ S_{n+1} = \{l \in S_n : \|v_n - l\| \le \|u_n - l\| \le \|w_n - l\|\}, \\ w_{n+1} = P_{S_{n+1}} w_1, \end{cases} \tag{4.5}$$

where $\lambda \in (0, \frac{2}{\|\tilde{G}\|^2})$ and $\{\alpha_n\} \subset (0,1)$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. If $F(R_{\rho,T}) \neq \emptyset$, then $\{w_n\}$ converges strongly to some element $w^* = (x^*, y^*) \in F(R_{\rho,T})$, i.e.,

$$\begin{cases} T_1(x^*, x) \ge 0, \forall\, x \in C, \\ T_2(y^*, y) \ge 0, \forall\, y \in Q, \\ Ax^* = y^*. \end{cases}$$

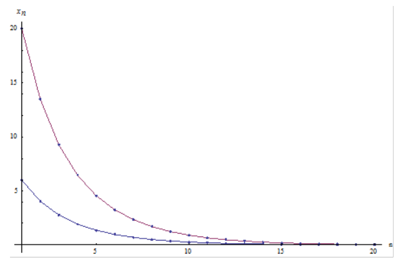

FIGURE 1.

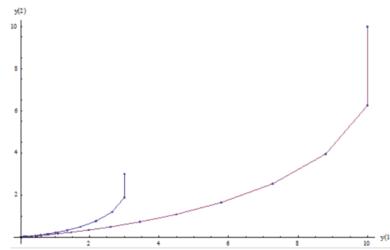

FIGURE 2.

5. NUMERICAL EXPERIMENTS

In this section, we provide a numerical example to illustrate the effectiveness of our algorithm. The whole program was written in Wolfram Mathematica (version 9.0). All the numerical results are carried out on a personal Lenovo computer with Intel(R) Core(TM)i5-6600 CPU @ 3.30GHz and RAM 8.00 GB.

We consider the split feasibility problem(1.1) with $H_1 = R, H_2 = R \times R$, $C = (-\infty, 0], Q = [0, \infty) \times (-\infty, 0]$, $A = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T_1 = P_C, T_2 = P_Q$. Take $\lambda = \frac{1}{\|G\|^2}, \alpha_n = \frac{1}{3}$, two sets of initial points $x_1 = 6, y_1 = (3,3)$ and $x_1 = 20, y_1 = (10,10)$. Obviously, $x^* = 0, y^* = (0,0)$ is a solution of this problem.

n	x_n	y_n
1	6.00000	(3.00000, 3.00000)
2	4.04082	(3.00000, 1.87755)
3	2.77551	(2.64015, 1.18673)
4	1.93769	(2.18009, 0.75912)
5	1.37057	(1.73514, 0.49222)
6	0.97944	(1.34948, 0.32382)
7	0.70549	(1.03365, 0.21620)
8	0.51124	(0.78352, 0.14646)
9	0.37217	(0.58960, 0.10057)
10	0.27186	(0.44137, 0.06992)
11	0.19910	(0.32917, 0.04915)
12	0.14609	(0.24482, 0.03487)
13	0.10734	(0.18172, 0.02494)
14	0.07896	(0.13468, 0.01796)
15	0.05813	(0.09971, 0.01299)
16	0.04282	(0.07376, 0.00944)
17	0.03155	(0.05453, 0.00689)
18	0.02326	(0.04029, 0.00504)
19	0.01715	(0.02976, 0.00369)
20	0.01265	(0.02198, 0.00271)

TABLE 1.

n	x_n	y_n
1	20.00000	(10.00000, 10.00000)
2	13.46940	(10.00000, 6.25850)
3	9.25170	(8.80050, 3.95576)
4	6.45898	(7.26696, 2.53041)
5	4.56858	(5.78380, 1.64073)
6	3.26479	(4.49827, 1.07939)
7	2.35162	(3.44550, 0.72068)
8	1.70413	(2.61174, 0.48819)
9	1.24056	(1.96533, 0.33524)
10	0.90620	(1.47123, 0.23308)
11	0.66365	(1.09722, 0.16383)
12	0.48695	(0.81606, 0.11624)
13	0.35781	(0.60572, 0.08314)
14	0.26320	(0.44893, 0.05985)
15	0.19376	(0.33236, 0.04331)
16	0.14272	(0.24586, 0.03148)
17	0.10517	(0.18176, 0.02296)
18	0.07753	(0.13431, 0.01679)
19	0.05717	(0.09921, 0.01230)
20	0.04216	(0.07326, 0.00903)

TABLE 2.

From Figure $1-2$ and Table $1-2$, we see that the sequences $\{x_n\}$ and $\{y_n\}$ generated by our algorithm converge to x^* and y^*, respectively.

6. Conclusions

In this paper, we proposed a new iterative method to solve the split equality fixed point problem of quasi-nonexpansive operators and obtained a strong convergence result without any semi-compact assumption imposed on operators. Our results improved and unified many recent results such as split common fixed point problems, split equality problems and split feasibility problems and so on.

Acknowledgements
The third author was supported by the National Natural Science Foundation under Grant No. 11226125 and Grant No. 11301379.

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