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SOME NEW RESULTS ON COMMON COUPLED FIXED POINTS OF TWO HYBRID PAIRS OF MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. By using the concept of w-compatible mappings, we establish some new common coupled fixed point theorems for two hybrid pairs of mappings satisfying a symmetric type contractive condition in a partial metric space. We do not use the continuity of any mapping for finding the coupled coincidence and common coupled fixed points. We also provide some examples to support our new results.

Keywords. Common coupled fixed point; Coupled coincidence point; Compatible mapping pair; Partial metric space.

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1. INTRODUCTION

The concept of partial metric spaces was introduced by Matthews [1] in 1994. A partial metric space is a generalized metric space. In a partial metric space, the distance of a point to its self may not be zero. In [1], Matthews extended the Banach contraction principle from metric spaces to partial metric spaces. Based on the concept of partial metric spaces, several authors obtained some fixed point results for mappings satisfying different contractive conditions; see, e.g., [2, 3, 4, 5, 6, 7, 8, 9] and the references therein. Recently, Haghi, Rezapour and Shahzad [10] proved that some of fixed point theorems in partial metric spaces can be obtained from metric spaces.

The definition of partial metric spaces is given as follows.

Definition 1.1. [1] A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ such that, for all $x, y, z \in X$,

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,.x) \leq p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

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(p_4) $p(x,y) \leq p(x,z)+p(z,y)-p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark 1.2. It is clear that if $p(x,y)=0$, then one has from (p_1) and (p_2) that $x=y$. But if $x=y$, then $p(x,y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X, which has as base the family of open $p-$balls $\{B_p(x,\varepsilon): x\in X, \varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X: p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

It is remarkable that if p is a partial metric on X, then the function $d_p: X\times X\to\mathbb{R}^+$ given by

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y) \tag{1.1}$$

is a usual metric on X.

Definition 1.3. [1, 7] Let (X,p) be a partial metric space. Then

(i) a sequence $\{x_n\}$ in a partial metric space (X,p) is said to be convergent to a point $x\in X$ if and only if $p(x,x)=\lim_{n\to\infty} p(x,x_n)$;

(ii) a sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$;

(iii) a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty} p(x_n,x_m)$.

In 2006, Bhaskar and Lakshmikantham [11] introduced the concept of coupled fixed points and studied some nice coupled fixed point theorems. Later, Lakshmikantham and Ćirić [12] introduced the definition of a coupled coincidence point of mappings. For recent results on a coupled fixed point, we refer the reader to [13, 14, 15, 16, 17].

Definition 1.4. [11] Let X be a nonempty set. We call an element $(x,y)\in X\times X$ a coupled fixed point of the mapping $F: X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Definition 1.5. [12] An element $(x,y)\in X\times X$ is called

(i) a coupled coincidence point of the mapping $F: X\times X\to X$ and $g: X\to X$ if $F(x,y)=gx$ and $F(y,x)=gy$;

(ii) a common coupled fixed point of mappings $F: X\times X\to X$ and $g: X\to X$ if $F(x,y)=gx=x$ and $F(y,x)=gy=y$.

In 2010, Abbas, Khan and Radenovic [18] introduced the concept of w-compatible mappings as follows.

Definition 1.6. Let X be a nonempty set. We say that the mappings $F: X\times X\to X$ and $g: X\to X$ are w-compatible if $gF(x,y)=F(gx,gy)$ whenever $gx=F(x,y)$ and $gy=F(y,x)$.

Recently, Aydi [13] obtained the following.

Theorem 1.7. *Let (X,p) be a partial metric space. Suppose that the mapping $F: X\times X\to X$ satisfies the following contractive condition, for all $x,y,u,v\in X$,*

$$p(F(x,y),F(u,v))\leq kp(x,u)+lp(y,v), \tag{1.2}$$

where k,l are nonnegative constants with $k+l<1$. Then F has a unique coupled fixed point.

The purpose of this paper is to use the concept of w-compatible mappings to discuss some new common coupled fixed point problems for two hybrid pairs of mappings in the framework of partial metric spaces. We do not use the continuity of any mapping for finding the coupled coincidence and common coupled fixed points. The results presented in this paper extend and improve some known corresponding results in the literature.

We need the following useful lemmas to prove our main results.

Lemma 1.8. [2] *Let (X,p) be a partial metric space. Then the following statements hold*

(1) if $p(x,y)=0$, then $x=y$;

(2) if $x\neq y$, then $p(x,y)>0$.

Lemma 1.9. [1, 4, 7] *Let (X,p) be a partial metric space.*

(1) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).

(2) A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(x,x_n)=0 \Leftrightarrow \lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m)=p(x,x). \tag{1.3}$$

Lemma 1.10. [2, 3, 8] *Let (X,p) be a partial metric space and $x_n\to z$ with $p(z,z)=0$. Then $\lim_{n\to\infty} p(x_n,y)=p(z,y)$ for every $y\in X$.*

2. Main results

Theorem 2.1. *Let (X,p) be a partial metric space. Let $F,G:X\times X\to X$, $f,g:X\to X$ be four mappings. Suppose that there exist k_1,k_2,k_3,k_4 and k_5 in $[0,1)$ with*

$$k_1+k_2+k_3+2k_4+2k_5<1 \tag{2.1}$$

such that

$$\begin{aligned} & p(F(x,y),G(u,v))+p(F(y,x),G(v,u)) \\ \leq\ & k_1[p(fx,gu)+p(fy,gv)]+k_2[p(fx,F(x,y))+p(fy,F(y,x))] \\ & +k_3[p(gu,G(u,v))+p(gv,G(v,u))]+k_4[p(fx,G(u,v))+p(fy,G(v,u))] \\ & +k_5[p(gu,F(x,y))+p(gv,F(y,x))] \end{aligned} \tag{2.2}$$

for all $x,y,u,v\in X$. Also, suppose the following hypotheses:

(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset f(X)$,

(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,

(iii) (F,f) and (G,g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G, f and g has the form (u,u).

Proof. Let $x_0,y_0\in X$. From (i), there exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X such that

$$\begin{cases} F(x_{2n},y_{2n})=gx_{2n+1}=z_{2n}, \forall\, n\geq 0, \\ F(y_{2n},x_{2n})=gy_{2n+1}=w_{2n}, \forall\, n\geq 0, \\ G(x_{2n+1},y_{2n+1})=fx_{2n+2}=z_{2n+1}, \forall\, n\geq 0, \\ G(y_{2n+1},x_{2n+1})=fy_{2n+2}=w_{2n+1}, \forall\, n\geq 0. \end{cases} \tag{2.3}$$

It follows from (2.2), (p_2) and (p_4) that

$$\begin{aligned}
p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1}) &= p(F(x_{2n},y_{2n}),G(x_{2n+1},y_{2n+1}))+p(F(y_{2n},x_{2n}),G(y_{2n+1},x_{2n+1}))\\
&\leq k_1[p(fx_{2n},gx_{2n+1})+p(fy_{2n},gy_{2n+1})]+k_2[p(fx_{2n},F(x_{2n},y_{2n}))+p(fy_{2n},F(y_{2n},x_{2n}))]\\
&\quad +k_3[p(gx_{2n+1},G(x_{2n+1},y_{2n+1}))+p(gy_{2n+1},G(y_{2n+1},x_{2n+1}))]\\
&\quad +k_4[p(fx_{2n},G(x_{2n+1},y_{2n+1}))+p(fy_{2n},G(y_{2n+1},x_{2n+1}))]\\
&\quad +k_5[p(gx_{2n+1},F(x_{2n},y_{2n}))+p(gy_{2n+1},F(y_{2n},x_{2n}))]\\
&= k_1[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]+k_2[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]\\
&\quad +k_3[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})]+k_4[p(z_{2n-1},z_{2n+1})+p(w_{2n-1},w_{2n+1})]\\
&\quad +k_5[p(z_{2n},z_{2n})+p(w_{2n},w_{2n})]\\
&\leq (k_1+k_2)[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]+k_3[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})]\\
&\quad +k_4[p(z_{2n-1},z_{2n})+p(z_{2n},z_{2n+1})-p(z_{2n},z_{2n})+p(w_{2n-1},w_{2n})+p(w_{2n},w_{2n+1})\\
&\quad -p(w_{2n},w_{2n})]+k_5[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})]\\
&\leq (k_1+k_2+k_4)[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]\\
&\quad +(k_3+k_4+k_5)[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})],
\end{aligned}$$

which implies that

$$p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1}) \leq \frac{k_1+k_2+k_4}{1-k_3-k_4-k_5}[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]. \tag{2.4}$$

Similarly, we can obtain

$$\begin{aligned}
p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n}) &= p(F(x_{2n},y_{2n}),G(x_{2n-1},y_{2n-1}))+p(F(y_{2n},x_{2n}),G(y_{2n-1},x_{2n-1}))\\
&\leq k_1[p(fx_{2n},gx_{2n-1})+p(fy_{2n},gy_{2n-1})]+k_2[p(fx_{2n},F(x_{2n},y_{2n}))+p(fy_{2n},F(y_{2n},x_{2n}))]\\
&\quad +k_3[p(gx_{2n-1},G(x_{2n-1},y_{2n-1}))+p(gy_{2n-1},G(y_{2n-1},x_{2n-1}))]\\
&\quad +k_4[p(fx_{2n},G(x_{2n-1},y_{2n-1}))+p(fy_{2n},G(y_{2n-1},x_{2n-1}))]\\
&\quad +k_5[p(gx_{2n-1},F(x_{2n},y_{2n}))+p(gy_{2n-1},F(y_{2n},x_{2n}))]\\
&= k_1[p(z_{2n-1},z_{2n-2})+p(w_{2n-1},w_{2n-2})]+k_2[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]\\
&\quad +k_3[p(z_{2n-2},z_{2n-1})+p(w_{2n-2},w_{2n-1})]+k_4[p(z_{2n-1},z_{2n-1})+p(w_{2n-1},w_{2n-1})]\\
&\quad +k_5[p(z_{2n-2},z_{2n})+p(w_{2n-2},w_{2n})]\\
&\leq (k_1+k_3)[p(z_{2n-2},z_{2n-1})+p(w_{2n-2},w_{2n-1})]+k_2[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]\\
&\quad +k_4[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})]\\
&\quad +k_5[p(z_{2n-2},z_{2n-1})+p(z_{2n-1},z_{2n})-p(z_{2n-1},z_{2n-1})+p(w_{2n-2},w_{2n-1})+p(w_{2n-1},w_{2n})\\
&\quad -p(w_{2n-1},w_{2n-1})]\\
&\leq (k_1+k_3+k_5)[p(z_{2n-2},z_{2n-1})+p(w_{2n-2},w_{2n-1})]\\
&\quad +(k_2+k_4+k_5)[p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n})],
\end{aligned}$$

which implies that

$$p(z_{2n-1},z_{2n})+p(w_{2n-1},w_{2n}) \leq \frac{k_1+k_3+k_5}{1-k_2-k_4-k_5}[p(z_{2n-2},z_{2n-1})+p(w_{2n-2},w_{2n-1})]. \tag{2.5}$$

By combining (2.4) and (2.5), we obtain

$$p(z_n,z_{n+1})+p(w_n,w_{n+1}) \leq k[p(z_{n-1},z_n)+p(w_{n-1},w_n)], \tag{2.6}$$

where

$$k = \max\left\{\frac{k_1+k_2+k_4}{1-k_3-k_4-k_5}, \frac{k_1+k_3+k_5}{1-k_2-k_4-k_5}\right\}.$$

Obviously, $0 \leq k < 1$. Repetition of the above inequality (2.6) n-times, we get

$$p(z_n,z_{n+1})+p(w_n,w_{n+1}) \leq k^n[q_1(z_0,z_1)+q_1(w_0,w_1)]. \tag{2.7}$$

Next, we prove that $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in X.

In fact, for each $n,m \in \mathbb{N}$, $m > n$, we obtani from (p_4) and (2.7) that

$$\begin{aligned} p(z_n,z_m)+p(w_n,w_m) &\leq \sum_{i=n}^{m-1}[p(z_i,z_{i+1})+p(w_i,w_{i+1})] \\ &\leq \sum_{i=n}^{m-1} k^i[p(z_0,z_1)+p(w_0,w_1)] \\ &\leq \frac{k^n}{1-k}[p(z_0,z_1)+p(w_0,w_1)]. \end{aligned} \tag{2.8}$$

This implies that

$$\lim_{n,m\to\infty}[p(z_n,z_m)+p(w_n,w_m)] = 0.$$

It follows that

$$\lim_{n,m\to\infty} p(z_n,z_m) = 0 \text{ and } \lim_{n,m\to\infty} p(w_n,w_m) = 0. \tag{2.9}$$

Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in the partial metric space (X,p). By Lemma 1.9, $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in (X,d_p). Therefore, $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are Cauchy sequences in the subspace $(f(X),d_p)$. Suppose that $f(X)$ is complete. Since $\{z_{2n+1}\} \subset f(X)$ and $\{w_{2n+1}\} \subset f(X)$ are Cauchy sequences in $(f(X),d_p)$, it follows that the sequences $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are convergent in $(f(X),d_p)$. Hence, there exist $u,v \in f(X)$ such that

$$\lim_{n\to\infty} d_p(z_{2n+1},u) = 0 \text{ and } \lim_{n\to\infty} d_p(w_{2n+1},v) = 0.$$

Since $u,v \in f(X)$, there exist $s,t \in X$ such that $u = fs$ and $v = ft$. Since $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in X and $\{z_{2n+1}\} \to u$ and $\{w_{2n+1}\} \to v$, it follows that $\{z_{2n}\} \to u$ and $\{w_{2n}\} \to v$. From Lemma 1.9, we have

$$p(u,u) = \lim_{n\to\infty} p(z_{2n},u) = \lim_{n\to\infty} p(z_{2n+1},u) = \lim_{n,m\to\infty} p(z_n,z_m) \tag{2.10}$$

and

$$p(v,v) = \lim_{n\to\infty} p(w_{2n},v) = \lim_{n\to\infty} p(w_{2n+1},v) = \lim_{n,m\to\infty} p(w_n,w_m). \tag{2.11}$$

Combining (2.9), (2.10) with (2.11), we have that

$$p(u,u) = 0 = p(v,v). \tag{2.12}$$

By (p_4), we obtain

$$\begin{aligned} p(F(s,t),z_{2n+1}) &\leq p(F(s,t),u)+p(u,z_{2n+1})-p(u,u) \\ &= p(F(s,t),u)+p(u,z_{2n+1}) \\ &\leq p(F(s,t),z_{2n+1})+p(z_{2n+1},u)-p(z_{2n+1},z_{2n+1})+p(u,z_{2n+1}) \\ &\leq p(F(s,t),z_{2n+1})+p(z_{2n+1},u)+p(u,z_{2n+1}). \end{aligned}$$

Letting $n\to\infty$ in the above inequalities and using (2.10) and (2.12), we have

$$\lim_{n\to\infty} p(F(s,t),z_{2n+1}) \leq p(F(s,t),u) \leq \lim_{n\to\infty} p(F(s,t),z_{2n+1}).$$

That is,

$$\lim_{n\to\infty} p(F(s,t),z_{2n+1}) = p(F(s,t),u). \tag{2.13}$$

Similarly, using (2.11) and (2.12), we have

$$\lim_{n\to\infty} p(F(t,s),w_{2n+1}) = p(F(t,s),v). \tag{2.14}$$

Now we prove that $F(s,t)=u=fs$ and $F(t,s)=v=ft$. In fact, it follows from (2.2) and (2.3) that

$$\begin{aligned} &p(F(s,t),z_{2n+1})+p(F(t,s),w_{2n+1}) \\ &= p(F(s,t),G(x_{2n+1},y_{2n+1}))+p(F(t,s),G(y_{2n+1},x_{2n+1})) \\ &\leq k_1[p(fs,gx_{2n+1})+p(ft,gy_{2n+1})]+k_2[p(fs,F(s,t))+p(ft,F(t,s))] \\ &\quad +k_3[p(gx_{2n+1},G(x_{2n+1},y_{2n+1}))+p(gy_{2n+1},G(y_{2n+1},x_{2n+1}))] \\ &\quad +k_4[p(fs,G(x_{2n+1},y_{2n+1}))+p(ft,G(y_{2n+1},x_{2n+1}))] \\ &\quad +k_5[p(gx_{2n+1},F(s,t))+p(gy_{2n+1},F(t,s))] \\ &= k_1[p(u,z_{2n})+p(v,w_{2n})]+k_2[p(u,F(s,t))+p(v,F(t,s))] \\ &\quad +k_3[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})]+k_4[p(u,z_{2n+1})+p(v,w_{2n+1})] \\ &\quad +k_5[p(z_{2n},F(s,t))+p(w_{2n},F(t,s))] \\ &\leq k_1[p(u,z_{2n})+p(v,w_{2n})]+k_2[p(u,F(s,t))+p(v,F(t,s))] \\ &\quad +k_3[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})]+k_4[p(u,z_{2n+1})+p(v,w_{2n+1})] \\ &\quad +k_5[p(z_{2n},z_{2n+1})+p(z_{2n+1},F(s,t))-p(z_{2n+1},z_{2n+1}) \\ &\quad +p(w_{2n},w_{2n+1})+p(w_{2n+1},F(t,s))-p(w_{2n+1},w_{2n+1})] \\ &\leq k_1[p(u,z_{2n})+p(v,w_{2n})]+k_2[p(u,F(s,t))+p(v,F(t,s))] \\ &\quad +k_3[p(z_{2n},z_{2n+1})+p(w_{2n},w_{2n+1})]+k_4[p(u,z_{2n+1})+p(v,w_{2n+1})] \\ &\quad +k_5[p(z_{2n},z_{2n+1})+p(z_{2n+1},F(s,t))+p(w_{2n},w_{2n+1})+p(w_{2n+1},F(t,s))]. \end{aligned}$$

Letting $n\to\infty$ in the above inequality and using (2.9)-(2.14), we arrive at

$$p(F(s,t),u)+p(F(t,s),v) \leq (k_2+k_5)[p(F(s,t),u)+p(F(t,s),v)]. \tag{2.15}$$

By (2.1), we have that $k_2+k_5<1$. Hence, it follows from (2.15) that $p(F(s,t),u)=p(F(t,s),v)=0$. By Lemma 1.8, we get $F(s,t)=u=fs$ and $F(t,s)=v=ft$. Hence, (s,t) is a coincidence point of mappings F and f. Since (F,f) is w-compatible, we have $fu = f(F(s,t)) = F(fs,ft) = F(u,v)$ and $fv = f(F(t,s)) = F(ft,fs) = F(v,u)$. Suppose that $fu \neq u$ or $fv \neq v$. Hence

$$d_p(fu,z_{2n}) = 2p(fu,z_{2n}) - p(fu,fu) - p(z_{2n},z_{2n}).$$

Letting $n \to \infty$, we get

$$d_p(fu,u) = 2\lim_{n\to\infty} p(fu,z_{2n}) - p(fu,fu),$$

which implies that

$$2p(fu,u) - p(fu,fu) - p(u,u) = d_p(fu,u) = 2\lim_{n\to\infty} p(fu,z_{2n}) - p(fu,fu).$$

By using (2.12), we obtain

$$p(fu,u) = \lim_{n\to\infty} p(fu,z_{2n}). \tag{2.16}$$

Similarly, we have

$$p(fv,v) = \lim_{n\to\infty} p(fv,w_{2n}). \tag{2.17}$$

Thus

$$\lim_{n\to\infty}[p(fu,z_{2n}) + p(fv,w_{2n})] = p(fu,u) + p(fv,v) > 0. \tag{2.18}$$

From (2.2), (2.3), (p_2) and (p_4), we get

$$\begin{aligned}
&p(fu,u) + p(fv,v) \\
&\quad\leq p(fu,z_{2n+1}) + p(z_{2n+1},u) - p(z_{2n+1},z_{2n+1}) \\
&\qquad + p(fv,w_{2n+1}) + p(w_{2n+1},v) - p(w_{2n+1},w_{2n+1}) \\
&\quad\leq p(fu,z_{2n+1}) + p(z_{2n+1},u) + p(fv,w_{2n+1}) + p(w_{2n+1},v) \\
&\quad= p(F(u,v),G(x_{2n+1},y_{2n+1})) + p(F(v,u),G(y_{2n+1},x_{2n+1})) + p(z_{2n+1},u) + p(w_{2n+1},v) \\
&\quad\leq k_1[p(fu,gx_{2n+1}) + p(fv,gy_{2n+1})] + k_2[p(fu,F(u,v)) + p(fv,F(v,u))] \\
&\qquad + k_3[p(gx_{2n+1},G(x_{2n+1},y_{2n+1})) + p(gy_{2n+1},G(y_{2n+1},x_{2n+1}))] \\
&\qquad + k_4[p(fu,G(x_{2n+1},y_{2n+1})) + p(fv,G(y_{2n+1},x_{2n+1}))] \\
&\qquad + k_5[p(gx_{2n+1},F(u,v)) + p(gy_{2n+1},F(v,u))] + p(z_{2n+1},u) + p(w_{2n+1},v) \\
&\quad= k_1[p(fu,z_{2n}) + p(fv,w_{2n})] + k_2[p(fu,fu) + p(fv,fv))] \\
&\qquad + k_3[p(z_{2n},z_{2n+1}) + p(w_{2n},w_{2n+1})] + k_4[p(fu,z_{2n+1}) + p(fv,w_{2n+1})] \\
&\qquad + k_5[p(z_{2n},fu) + p(w_{2n},fv))] + p(z_{2n+1},u) + p(w_{2n+1},v) \\
&\quad\leq k_1[p(fu,z_{2n}) + p(fv,w_{2n})] + k_2[p(fu,u) + p(fv,v))] \\
&\qquad + k_3[p(z_{2n},z_{2n+1}) + p(w_{2n},w_{2n+1})] + k_4[p(fu,z_{2n+1}) + p(fv,w_{2n+1})] \\
&\qquad + k_5[p(fu,z_{2n}) + p(fv,w_{2n})] + p(z_{2n+1},u) + p(w_{2n+1},v).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (2.9)-(2.12) and (2.16)-(2.18), we obtain

$$p(fu,u) + p(fv,v) \leq (k_1 + k_2 + k_4 + k_5))[p(fu,u) + p(fv,v)] < p(fu,u) + p(fv,v).$$

It is a contradiction. Hence $fu = u$ and $fv = v$. Thus

$$F(u,v) = fu = u \text{ and } F(v,u) = fv = v. \tag{2.19}$$

Since $F(X\times X)\subset gX$, there exist $a,b\in X$ such that $u=F(u,v)=ga$ and $v=F(v,u)=gb$. Hence, it follows from (2.2) that

$$\begin{aligned}
&p(u,G(a,b))+p(v,G(b,a))\\
&\quad= p(F(u,v),G(a,b))+p(F(v,u),G(b,a))\\
&\quad\le k_1[p(fu,ga)+p(fv,gb)]+k_2[p(fu,F(u,v))+p(fv,F(v,u))]\\
&\qquad +k_3[p(ga,G(a,b))+p(gb,G(b,a))]+k_4[p(fu,G(a,b))+p(fv,G(b,a))]\\
&\qquad +k_5[p(ga,F(u,v))+p(gb,F(v,u))]\\
&\quad= k_1[p(u,u)+p(v,v)]+k_2[p(u,u))+p(v,v))]\\
&\qquad +k_3[p(u,G(a,b))+p(v,G(b,a))]+k_4[p(u,G(a,b))+p(v,G(b,a))]\\
&\qquad +k_5[p(u,u))+p(v,v))]\\
&\quad= (k_3+k_4)[p(u,G(a,b))+p(v,G(b,a))],
\end{aligned}$$

which implies that $p(u,G(a,b))+p(v,G(b,a))=0$. So, $G(a,b)=u=ga$ and $G(b,a)=v=gb$. Since the pair (G,g) is w-compatible, we have $gu=g(G(a,b))=G(ga,gb)=G(u,v)$ and $gv=g(G(b,a))=G(gb,ga)=G(v,u)$. Suppose $gu\neq u$ or $gv\neq v$. It follows from (2.2) that

$$\begin{aligned}
&p(u,gu)+p(v,gv)\\
&\quad= p(F(u,v),G(u,v))+p(F(v,u),G(v,u))\\
&\quad\le k_1[p(fu,gu)+p(fv,gv)]+k_2[p(fu,F(u,v))+p(fv,F(v,u))]\\
&\qquad +k_3[p(gu,G(u,v))+p(gv,G(v,u))]+k_4[p(fu,G(u,v))+p(fv,G(v,u))]\\
&\qquad +k_5[p(gu,F(u,v))+p(gv,F(v,u))]\\
&\quad= k_1[p(u,gu)+p(v,gv)]+k_2[p(u,u))+p(v,v))]\\
&\qquad +k_3[p(gu,gu)+p(gv,gv)]+k_4[p(u,gu))+p(v,gv))]\\
&\qquad +k_5[p(gu,u))+p(gv,v))]\\
&\quad\le k_1[p(u,gu)+p(v,gv)]+k_2[p(u,u))+p(v,v))]\\
&\qquad +k_3[p(u,gu)+p(v,gv)]+k_4[p(u,gu))+p(v,gv))]\\
&\qquad +k_5[p(u,gu))+p(v,gv))]\\
&\quad= (k_1+k_3+k_4+k_5)[p(u,gu)+p(v,gv)]\\
&\quad< p(u,gu)+p(v,gv),
\end{aligned}$$

which is a contradiction. It follows that $gu=u$ and $gv=v$. Hence

$$G(u,v)=gu=u \text{ and } G(v,u)=gv=v. \tag{2.20}$$

From (2.19) and (2.20), it follows that (u,v) is a common coupled fixed point of F, G, f and g.

Next, we show that (u,v) is unique common coupled fixed point of F, G, f and g. Suppose that (u^*,v^*) be another common coupled fixed point of F, G, f and g. Using (2.2) and (p_2), we obtain

$$\begin{aligned}
& p(u,u^*)+p(v,v^*) \\
&= p(F(u,v),G(u^*,v^*))+p(F(v,u),G(v^*,u^*)) \\
&\le k_1[p(fu,gu^*)+p(fv,gv^*)]+k_2[p(fu,F(u,v))+p(fv,F(v,u))] \\
&\quad +k_3[p(gu^*,G(u^*,v^*))+p(gv^*,G(v^*,u^*))]+k_4[p(fu,G(u^*,v^*))+p(fv,G(v^*,u^*))] \\
&\quad +k_5[p(gu^*,F(u,v))+p(gv^*,F(v,u))] \\
&= k_1[p(u,u^*)+p(v,v^*)]+k_2[p(u,u)+p(v,v)]+k_3[p(u^*,u^*)+p(v^*,v^*)] \\
&\quad +k_4[p(u,u^*)+p(v,v^*)]+k_5[p(u^*,u)+p(v^*,v)] \\
&\le k_1[p(u,u^*)+p(v,v^*)]+k_2[p(u,u^*)+p(v,v^*)]+k_3[p(u,u^*)+p(v,v^*)] \\
&\quad +k_4[p(u,u^*)+p(v,v^*)]+k_5[p(u,u^*)+p(v,v^*)] \\
&= (k_1+k_2+k_3+k_4+k_5)[p(u,u^*)+p(v,v^*)]. \qquad (2.21)
\end{aligned}$$

From (2.2) and (2.21), we have $p(u,u^*)=p(v,v^*)=0$. By Lemma 1.8, we get $u^*=u$ and $v^*=v$, which implies that the uniqueness of the common coupled fixed point of F, G, f and g.

Next, we show that $u=v$. In fact, we find from (2.2) and (2.12) that

$$\begin{aligned}
& p(u,v)+p(v,u) \\
&= p(F(u,v),G(v,u))+p(F(v,u),G(u,v)) \\
&\le k_1[p(fu,gv)+p(fv,gu)]+k_2[p(fu,F(u,v))+p(fv,F(v,u))] \\
&\quad +k_3[p(gv,G(v,u))+p(gu,G(u,v))]+k_4[p(fu,G(v,u))+p(fv,G(u,v))] \\
&\quad +k_5[p(gv,F(u,v))+p(gu,F(v,u))] \\
&= k_1[p(u,v)+p(v,u)]+k_2[p(u,u)+p(v,v)]+k_3[p(v,v)+p(u,u)] \\
&\quad +k_4[p(u,v)+p(v,u)]+k_5[p(v,u)+p(u,v)] \\
&= (k_1+k_4+k_5)[p(u,v)+p(v,u)].
\end{aligned}$$

From (2.1), we have $p(u,v)=p(v,u)=0$. By Lemma 1.8, we get that $u=v$. Thus $u=fu=F(u,u)=G(u,u)=gu$, that is, the common coupled fixed point of F, G, f and g has the form (u,u). If $g(X)$ is a complete subspace of X, we obtain the desired conclusion immediately. □

Corollary 2.2. *Let (X,p) be a partial metric space. Let $F,G:X\times X\to X$, $f,g:X\to X$ be four mappings. Suppose that there exist $k\in[0,1)$ such that*

$$p(F(x,y),G(u,v))+p(F(y,x),G(v,u))\le k[p(fx,gu)+p(fy,gv)] \qquad (2.22)$$

for all $x,y,u,v\in X$. Also, suppose the following hypotheses:

(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset f(X)$,

(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,

(iii) (F,f) and (G,g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G, f and g have the form (u,u).

Corollary 2.3. *Let (X,p) be a partial metric space. Let $F,G:X\times X\to X$, $g:X\to X$ be three mappings. Suppose that there exist $k\in[0,1)$ such that*

$$p(F(x,y),G(u,v))+p(F(y,x),G(v,u))\leq k[p(gu,F(x,y))+p(gv,F(y,x))] \tag{2.23}$$

for all $x,y,u,v\in X$. Also, suppose the following hypotheses:
(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset g(X)$,
(ii) $g(X)$ is a complete subspace of X,
(iii) (F,g) and (G,g) are w-compatible.

Then F, G and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G and g have the form (u,u).

Theorem 2.4. *Let (X,p) be a partial metric space. Let $F,G:X\times X\to X$, $f,g:X\to X$ be four mappings. Suppose that there exist $a_i\in[0,1)$ $(i=1,2,3,\cdots,10)$ with*

$$a_1+a_2+a_3+a_4+a_5+a_6+2(a_7+a_8+a_9+a_{10})<1 \tag{2.24}$$

such that

$$\begin{aligned}
&p(F(x,y),G(u,v))\\
&\quad\leq\quad a_1p(fx,gu)+a_2p(fy,gv)+a_3p(fx,F(x,y))+a_4p(fy,F(y,x))\\
&\qquad\quad +a_5p(gu,G(u,v))+a_6p(gv,G(v,u))+a_7p(fx,G(u,v))+a_8p(fy,G(v,u))\\
&\qquad\quad +a_9p(gu,F(x,y))+a_{10}p(gv,F(y,x))
\end{aligned} \tag{2.25}$$

for all $x,y,u,v\in X$. Also, suppose the following hypotheses:
(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset f(X)$,
(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,
(iii) (F,f) and (G,g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G, f and g have the form (u,u).

Proof. Fix $x,y,u,v\in X$. It follows from (2.25) that

$$\begin{aligned}
&p(F(x,y),G(u,v))\\
&\quad\leq\quad a_1p(fx,gu)+a_2p(fy,gv)+a_3p(fx,F(x,y))+a_4p(fy,F(y,x))\\
&\qquad\quad +a_5p(gu,G(u,v))+a_6p(gv,G(v,u))+a_7p(fx,G(u,v))+a_8p(fy,G(v,u))\\
&\qquad\quad +a_9p(gu,F(x,y))+a_{10}p(gv,F(y,x))
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
&p(F(y,x),G(v,u))\\
&\quad\leq\quad a_1p(fy,gv)+a_2p(fx,gu)+a_3p(fy,F(y,x))+a_4p(fx,F(x,y))\\
&\qquad\quad +a_5p(gv,G(v,u))+a_6p(gu,G(u,v))+a_7p(fy,G(v,u))+a_8p(fx,G(u,c))\\
&\qquad\quad +a_9p(gv,F(y,x))+a_{10}p(gu,F(x,y)).
\end{aligned} \tag{2.27}$$

From (2.26) and (2.27), we get

$$\begin{aligned} &p(F(x,y),G(u,v))+p(F(y,x),G(v,u)) \\ &\quad\leq (a_1+a_2)[p(fx,gu)+p(fy,gv)]+(a_3+a_4)[p(fx,F(x,y))+p(fy,F(y,x))] \\ &\qquad +(a_5+a_6)[p(gu,G(u,v))+p(gv,G(v,u))]+(a_7+a_8)[p(fx,G(u,v))+p(fy,G(v,u))] \\ &\qquad +(a_9+a_{10})[p(gu,F(x,y))+p(gv,F(y,x))]. \end{aligned} \tag{2.28}$$

Therefore, the result follows from Theorem 2.4 immediately. □

Remark 2.5. Theorem 2.4 improves and extends Theorem 1.7 from one mapping to two hybrid pairs of mappings, and the contractive condition (1.2) is replaced by the new contractive condition defined by (2.25).

Remark 2.6. Corollary 2.3 of Aydi and Abbas [14] is a particular case of Theorem 2.4 by taking $a_4 = a_6 = a_8 = a_{10} = 0$, $F = G$ and $f = g = I$, the identity on X.

Corollary 2.7. *Let (X,p) be a partial metric space. Let $F,G : X\times X \to X$, $f : X \to X$ be three mappings. Suppose that there exist $k,l \in [0,1)$ with $k+l < 1$ such that*

$$p(F(x,y),G(u,v)) \leq kp(fx,gu)+lp(fy,gv) \tag{2.29}$$

for all $x,y,u,v \in X$. Also, suppose the following hypotheses:
(i) $F(X\times X) \subset f(X)$ and $G(X\times X) \subset f(X)$,
(ii) $f(X)$ is a complete subspace of X,
(iii) (F, f) and (G, f) are w-compatible.

Then F, G and f have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G and f have the form (u,u).

Remark 2.8. Corollary 2.7 generalizes and extends the corresponding results in Aydi [13, Theorem 2.1] from one mapping to two hybrid pairs of mappings. In fact, if $F = G$ and $f = g = I$, Corollary 2.7 is reduced to Theorem 2.1 of Aydi [13].

Corollary 2.9. *Let (X,p) be a partial metric space. Let $F,G : X\times X \to X$, $f,g : X \to X$ be four mappings. Suppose that there exist $k,l \in [0,1)$ with $k+l < 1$ such that*

$$p(F(x,y),G(u,v)) \leq kp(fx,F(x,y))+lp(gu,G(u,v)) \tag{2.30}$$

for all $x,y,u,v \in X$. Also, suppose the following hypotheses:
(i) $F(X\times X) \subset g(X)$ and $G(X\times X) \subset f(X)$,
(ii) $f(X)$ or $g(X)$ is a complete subspace of X.
(iii) (F, f) and (G, g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G, f and g have the form (u,u).

Remark 2.10. (1) If $F = G$ and $f = g = I$ in Corollary 2.9, then Corollary 2.9 is reduced to Theorem 2.4 of Aydi [13].

(2) If we take $F = G$, $f = g = I$ and $k = l$, then Corollary 2.9 is reduced to Corollary 2.6 of Aydi [13].

Corollary 2.11. *Let (X,p) be a partial metric space. Let $F,G: X\times X\to X,\ f,g: X\to X$ be four mappings. Suppose that there exist $k,l\in[0,1)$ with $k+l<\frac{1}{2}$ such that*

$$p(F(x,y),G(u,v))\leq kp(fx,G(u,v))+lp(gu,F(x,y)) \tag{2.31}$$

for all $x,y,u,v\in X$. Also, suppose the following hypotheses:
(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset f(X)$,
(ii) $f(X)$ or $g(X)$ is a complete subspace of X,
(iii) (F,f) and (G,g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G, f and g have the form (u,u).

Remark 2.12. (1) If $F=G$ and $f=g=I$ in Corollary 2.11, Corollary 2.11 is reduced to Theorem 2.5 of Aydi [13].

(2) If $F=G$, $f=g=I$ and $k=l$ in Corollary 2.11, Corollary 2.11 is reduced to Corollary 2.7 of Aydi [13].

Corollary 2.13. *Let (X,p) be a partial metric space. Let $F,G: X\times X\to X,\ f,g: X\to X$ be four mappings. Suppose that there exist $k,l\in[0,1)$ with $k+l<\frac{1}{2}$ such that*

$$p(F(x,y),G(u,v))\leq kp(fy,G(v,u))+lp(gv,F(y,x)) \tag{2.32}$$

holds for all $x,y,u,v\in X$. Also, suppose the following hypotheses:
(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset f(X)$,
(ii) either $f(X)$ or $g(X)$ is a complete subspace of X,
(iii) (F,f) and (G,g) are w-compatible.

Then F, G, f and g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F, G, f and g have the form (u,u).

Let $f=g=I$ in Theorem 2.4 and Corollaries 2.7-2.13, we have the following results.

Corollary 2.14. *Let (X,p) be a complete partial metric space. Let $F,G: X\times X\to X$ be two mappings. Suppose that there exist $a_i\in[0,1)$ $(i=1,2,3,\cdots,10)$ with*
$a_1+a_2+a_3+a_4+a_5+a_6+2(a_7+a_8+a_9+a_{10})<1$ such that

$$\begin{aligned}&p(F(x,y),G(u,v))\\ &\quad\leq a_1p(x,u)+a_2p(y,v)+a_3p(x,F(x,y))+a_4p(y,F(y,x))\\ &\qquad+a_5p(u,G(u,v))+a_6p(v,G(v,u))+a_7p(x,G(u,v))+a_8p(y,G(v,u))\\ &\qquad+a_9p(u,F(x,y))+a_{10}p(v,F(y,x))\end{aligned} \tag{2.33}$$

for all $x,y,u,v\in X$. Then F and G have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F and G have the form (u,u).

Remark 2.15. Corollary 2.14 improves and extends the corresponding results in Aydi [13, Theorem 2.1] from one self-mappings to two self-mappings, and the contractive condition is replaced by the new contractive condition defined by (2.33).

Corollary 2.16. *Let (X,p) be a complete partial metric space. Let $F,G : X\times X \to X$ be two mappings. Suppose that there exist $k,l \in [0,1)$ with $k+l<1$ such that*

$$p(F(x,y),G(u,v)) \le kp(x,u)+lp(y,v) \tag{2.34}$$

for all $x,y,u,v \in X$. Then F and G have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F and G have the form (u,u).

Remark 2.17. (1) If $F=G$ in Corollary 2.16, then Corollary 2.16 is reduced to Theorem 2.1 of Aydi [13].

(1) If $F=G$ and $k=l$ in Corollary 2.16, then Corollary 2.16 is reduced to Corollary 2.2 of Aydi [13].

Corollary 2.18. *Let (X,p) be a complete partial metric space. Let $F,G : X\times X \to X$ be two mappings. Suppose that there exist $k,l \in [0,1)$ with $k+l<1$ such that*

$$p(F(x,y),G(u,v)) \le kp(x,F(x,y))+lp(u,G(u,v)) \tag{2.35}$$

for all $x,y,u,v \in X$. Then F and G have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F and G have the form (u,u).

Remark 2.19. (1) If $F=G$ in Corollary 2.18, then Corollary 2.18 is reduced to Theorem 2.4 of Aydi [13].

(2) If $F=G$ and $k=l$ in Corollary 2.18, then Corollary 2.18 is reduced to Corollary 2.6 of Aydi [13].

Corollary 2.20. *Let (X,p) be a complete partial metric space. Let $F,G : X\times X \to X$ be two mappings. Suppose that there exist $k,l \in [0,1)$ with $k+l<\frac{1}{2}$ such that*

$$p(F(x,y),G(u,v)) \le kp(x,G(u,v))+lp(u,F(x,y)) \tag{2.36}$$

for all $x,y,u,v \in X$. Then F and G have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F and G have the form (u,u).

Remark 2.21. (1) If $F=G$ in Corollary 2.20, then Corollary 2.20 is reduced to Theorem 2.5 of Aydi [13].

(1) If $F=G$ and $k=l$ in Corollary 2.20, then Corollary 2.20 is reduced to Corollary 2.7 of Aydi [13].

Now, we give some examples to support our main results.

Example 2.22. Let $X=[0,1]$ be a partial metric space with $p(x,y)=\max\{x,y\}$, and let the mappings $F,G : X\times X \to X$ and $f,g : X\times X \to X$ be defined by

$$F(x,y)=\frac{x^2+y^2}{6},\ \ G(x,y)=\frac{x+y}{12},\ \ fx=x^2 \ \text{ and } \ gx=\frac{x}{2}$$

for all $x,y\in X$. Then

(i) $F(X\times X)\subset g(X)$ and $G(X\times X)\subset f(X)$,

(ii) either $f(X)$ is a complete subspace of X,

(iii) (F,f) and (G,g) are w-compatible,

(iv) for any $x,y,u,v\in X$, we have

$$p(F(x,y),G(u,v))+p(F(y,x),G(v,u)) \le \frac{1}{3}[p(fx,gu)+p(fy,gv)].$$

Proof. (i), (ii) and (iii) are obvious. Next we show that (iv). In fact, for $x,y,u,v\in X$, we have

$$\begin{aligned}
&p(F(x,y),G(u,v))+p(F(y,x),G(v,u))\\
=&\ 2\max\left\{\frac{x^2+y^2}{6},\frac{u+v}{12}\right\}\\
\leq&\ 2\left(\max\left\{\frac{x^2}{6},\frac{u}{12}\right\}+\max\left\{\frac{y^2}{6},\frac{v}{12}\right\}\right)\\
=&\ \frac{1}{3}\left(\max\left\{x^2,\frac{u}{2}\right\}+\max\left\{y^2,\frac{v}{2}\right\}\right)\\
=&\ \frac{1}{3}[p(fx,gu)+p(fy,gv)].
\end{aligned}$$

Thus, F, G, f and g satisfy all the hypotheses of Corollary 2.2. So, F, G, f and g have a unique common coupled fixed point. Here $(0,0)$ is the unique common coupled fixed point of F, G, f and g. □

Example 2.23. Let $X=R^+$ be a partial metric space with $p(x,y)=\max\{x,y\}$. Let $F,G:X\times X\to X$ and $f,g:X\times X\to X$ be mapping defined by

$$F(x,y)=\frac{2x+3y}{72},\ \ G(x,y)=\frac{3x+4y}{72},\ \ fx=\frac{x}{6}\ \text{ and }\ gx=\frac{x}{2},\quad \forall x,y\in X.$$

We find that all the hypotheses of Corollary 2.7 are satisfied. Clearly, $F(X\times X)\subset g(X)$, $G(X\times X)\subset f(X)$, $f(X)$ is a complete subspace of X, and the pairs (F,f) and (G,g) are w-compatible. For all $x,y,u,v\in X$, we have

$$\begin{aligned}
p(F(x,y),G(u,v)) &= \max\left\{\frac{2x+3y}{72},\frac{3u+4v}{72}\right\}\\
&= \max\left\{\frac{x}{36}+\frac{y}{24},\frac{u}{24}+\frac{v}{18}\right\}\\
&\leq \max\left\{\frac{x}{36},\frac{u}{24}\right\}+\max\left\{\frac{y}{24},\frac{v}{18}\right\}\\
&\leq \max\left\{\frac{x}{36},\frac{u}{12}\right\}+\max\left\{\frac{y}{24},\frac{v}{8}\right\}\\
&= \frac{1}{6}\max\left\{\frac{x}{6},\frac{u}{2}\right\}+\frac{1}{4}\max\left\{\frac{y}{6},\frac{v}{2}\right\}\\
&= \frac{1}{6}p(fx,gu)+\frac{1}{4}p(fy,gv).
\end{aligned}$$

Thus, F, G, f and g satisfy all the hypotheses of Corollary 2.7. So, F, G, f and g have a unique common coupled fixed point. Here $(0,0)$ is the unique common coupled fixed point of F, G, f and g.

Example 2.24. Let $X=\{0,1,2\}$ be endowed with the partial metric p given by $p(x,y)=\max\{x,y\}$ for all $x,y\in X$. It is clear that (X,p) is a complete partial metric. Define the mappings $F,G:X\times X\to X$ and $f,g:X\to X$ by

TABLE 1. The definition of maps f and g on X

x	$f(x)$	$g(x)$
0	0	0
1	2	2
2	1	2

and

TABLE 2. The definition of maps F and G on $X \times X$

(x,y)	$F(x,y)$	$G(x,y)$
(0,0)	0	0
(0,1)	0	0
(0,2)	0	0
(1,0)	0	0
(1,1)	0	0
(1,2)	0	0
(2,0)	0	1
(2,1)	0	1
(2,2)	0	1

Clearly, $f(X) = X$ is complete, $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$. It is easy to show that (F, f) and (G, g) are w-weakly compatible. To check contractive condition (2.30) for all $x, y \in X$, $k = \frac{1}{3}$ and $l = \frac{1}{2}$, we consider the following cases:

Case (I) $(x,y) \in X \times X$ and $(u,v) \in \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$. Then $p(F(x,y),G(u,v)) = p(0,0) = 0$. Hence (2.30) is satisfied.

Case (II) $(x,y) \in X \times X$ and $(u,v) \in \{(2,0),(2,1),(2,2)\}$. Then

$$\begin{aligned} kp(fx,F(x,y)) + lp(gu,G(u,v)) &\geq lp(gu,G(u,v)) \\ &= lp(2,1) = 2l = 1. \end{aligned}$$

Thus

$$p(F(x,y),G(u,v)) = p(0,1) = 1 \leq kp(fx,F(x,y)) + lp(gu,G(u,v)).$$

Hence, all of the conditions of Corollary 2.9 are satisfied. Moreover, $(0,0)$ is the unique common coupled fixed point of F, G, f and g.

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