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CONSTRUCTION TECHNIQUES OF PROJECTION SETS IN HYBRID METHODS FOR INFINITE WEAKLY RELATIVELY NONEXPANSIVE MAPPINGS WITH APPLICATIONS

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Abstract. In this paper, new projection sets in hybrid iterative schemes are constructed for approximating common fixed points of two infinite families of weakly relatively nonexpansive mappings in a real uniformly convex and uniformly smooth Banach space. Some applications of the main results are demonstrated.

Keywords. Common fixed point; Lyapunov functional; Metric projection; Weakly relatively nonexpansive mapping.

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1. INTRODUCTION AND PRELIMINARIES

In this paper, we suppose that X is a real Banach space and X^* is its the dual space. We suppose that C is the nonempty closed and convex subset of X. $\langle x, f\rangle$ denotes the value of $f \in X^*$ at $x \in X$. We use $x_n \to x$ (or $x_n \rightharpoonup x$) to denote $\{x_n\}$ converges strongly (or weakly) to x, respectively.

A Banach space X is said to be uniformly convex [1] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$. The function $\eta_X : [0,+\infty) \to [0,+\infty)$ is said to be the modulus of smoothness of X [1] if

$$\eta_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| \le t\}.$$

A Banach space X is said to be uniformly smooth [1] if $\lim_{t\to 0} \frac{\eta_X(t)}{t} \to 0$, as $t \to 0$. A Banach space X is said to have Property (H): if for any sequence $\{x_n\} \subset X$, which satisfies both $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$. The uniformly convex and uniformly smooth Banach space has Property (H).

The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ x \in X.$$

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It is known that if X is a real uniformly convex and uniformly smooth Banach space, then the normalized duality mapping J is single-valued, surjective and $J(\varepsilon x) = \varepsilon J(x)$ for $x \in X$, $\varepsilon \in (-\infty, +\infty)$. Moreover, J^{-1} is also the normalized duality mapping from X^* into X and both J and J^{-1} are uniformly continuous on each bounded subset of X or X^*, respectively [2]. Recall that the Lyapunov functional $\phi : X \times X \to (0, +\infty)$ is defined as follows [3]:

$$\phi(x,y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2,\ \forall x,y \in X,\ j(y) \in J(y).$$

Let $S : C \to C$ be a single-valued mapping.

(1) If $Sp = p$, then p is called a fixed point of S. The set of fixed points of S is denoted by $F(S)$;

(2) if there exists a sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup p \in C$ such that $x_n - Sx_n \to 0$, as $n \to \infty$, then p is called an asymptotic fixed point of S [4]. The set of asymptotic fixed points of S is denoted by $\widehat{F}(S)$;

(3) if there exists a sequence $\{x_n\} \subset C$ with $x_n \to p \in C$ such that $x_n - Sx_n \to 0$, as $n \to \infty$, then p is called a strong asymptotic fixed point of S [4]. The set of strong asymptotic fixed points of S is denoted by $\widetilde{F}(S)$;

(4) S is called strongly relatively nonexpansive [4] if $\widehat{F}(S) = F(S) \neq \emptyset$ and $\phi(p, Sx) \leq \phi(p,x)$ for $x \in C$ and $p \in F(S)$;

(5) S is called weakly relatively nonexpansive [4] if $\widetilde{F}(S) = F(S) \neq \emptyset$ and $\phi(p, Sx) \leq \phi(p,x)$ for $x \in C$ and $p \in F(S)$.

It is obvious that strongly relatively nonexpansive mappings are weakly relatively non-expansive mappings. If X is a real reflexive, strictly convex and smooth Banach space and C is a nonempty closed and convex subset of X, then, for all $x \in X$, there exists a unique point $x_0 \in C$ such

$$\phi(x_0, x) = \inf\{\phi(y,x) : y \in C\}.$$

In this case, we can define the generalized projection mapping $\Pi_C : X \to C$ by $\Pi_C x = x_0$, for all $x \in X$ [3]. Weakly (or strongly) relatively nonexpansive mappings are important nonlinear mappings. Recently, much attention has been paid to fixed points of weakly (or strongly) relatively nonexpansive mappings in both Hilbert spaces and Banach spaces (see, e.g., [4, 5, 6, 7, 8, 9, 10, 11] and the references therein).

In 2005, Matsushita and Takahashi [5] presented the following hybrid iterative scheme to approximate fixed points of a strongly relatively nonexpansive mapping T in a real uniformly convex and uniformly smooth Banach space X:

$$\begin{cases} u_1 \in C, \\ v_n = J^{-1}[\beta_n J u_1 + (1-\beta_n) JTu_n], \\ C_n = \{p \in C : \phi(p, v_n) \leq \phi(p, u_n)\}, \\ Q_n = \{p \in C : \langle p - u_n, Ju_1 - Ju_n\rangle \leq 0\}, \\ u_{n+1} = \Pi_{C_n \bigcap Q_n}(u_1),\ n \in N, \end{cases} \tag{1.1}$$

where $\{\beta_n\}$ is a real sequence in $(0,1)$. They proved that $\{u_n\}$ generated by (1.1) is strongly convergent without compact assumptions on operator T and space X.

In 2009, Wei, Cho and Zhou [6] presented the following hybrid iterative scheme for two strongly relatively nonexpansive mappings T and S in a real uniformly convex and uniformly smooth Banach space X:

$$\begin{cases} u_1 \in C, \\ v_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)JTu_n], \\ w_n = J^{-1}[\beta_n Ju_n + (1-\alpha_n)JSv_n], \\ C_n = \{p \in C : \phi(p,w_n) \leq \beta_n \phi(p,u_n) + (1-\beta_n)\phi(p,v_n) \leq \phi(p,u_n)\}, \\ Q_n = \{p \in C : \langle p-u_n, Ju_1 - Ju_n\rangle \leq 0\}, \\ u_{n+1} = \Pi_{C_n \bigcap Q_n}(u_1),\ n \in N, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$. They proved that $\{u_n\}$ generated by (1.2) converges strongly to $\Pi_{F(T)\bigcap F(S)}(u_1)$ under some conditions.

In 2010, Su, Xu and Zhang [7] presented the following hybrid iterative scheme for two infinite families of weakly relatively nonexpansive mappings $\{T_n\}$ and $\{S_n\}$ in a real uniformly convex and uniformly smooth Banach space X:

$$\begin{cases} u_0 \in C, \\ v_n = J^{-1}[\beta_n^{(1)} Ju_n + \beta_n^{(2)} JT_n u_n + \beta_n^{(3)} JS_n u_n], \\ w_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)Jv_n], \\ C_n = \{p \in C_{n-1}\bigcap Q_{n-1} : \phi(p,w_n) \leq \phi(p,u_n)\}, \\ C_0 = \{p \in C : \phi(p,w_0) \leq \phi(p,u_0)\}, \\ Q_n = \{p \in C_{n-1}\bigcap Q_{n-1} : \langle p-u_n, Ju_0 - Ju_n\rangle \leq 0\}, \\ Q_0 = C, \\ u_{n+1} = \Pi_{C_n \bigcap Q_n}(u_0),\ n \in N\bigcup\{0\}, \end{cases} \tag{1.3}$$

$\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are four real number sequences in $(0,1)$. They proved that $\{u_n\}$ generated by (1.3) converges strongly to $\Pi_{(\bigcap_{n=1}^{\infty} F(T_n))\bigcap(\bigcap_{n=1}^{\infty} F(S_n))}(u_0)$ under some conditions.

In 2012, Zhang, Su and Cheng [8] removed the projection set Q_n in algorithms (1.1), (1.2) and (1.3), and introduced the following hybrid iterative scheme for a multi-valued weakly relatively nonexpansive mapping T in a real uniformly convex and uniformly smooth Banach space X:

$$\begin{cases} u_0 \in C, \\ v_n = J^{-1}[\beta_n^{(1)} Ju_0 + \beta_n^{(2)} Ju_n + \beta_n^{(3)} Jw_n], \\ w_n \in Tu_n, \\ C_n = \{p \in C_{n-1} : \phi(p,v_n) \leq (1-\alpha_n)\phi(p,u_n) + \alpha_n \phi(p,u_0)\}, \\ C_0 = C, \\ u_{n+1} = \Pi_{C_n}(u_0),\ n \in N\bigcup\{0\}, \end{cases} \tag{1.4}$$

where $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are three real number sequences in $(0,1)$. They proved that $\{u_n\}$ generated by (1.4) converges strongly to $\Pi_{F(T)}(u_0)$ under some conditions.

If X is a real reflexive and strictly convex Banach space and C is a nonempty closed and convex subset of X, then, for each $x \in X$, there exists a unique point $x_0 \in C$ such that $\|x - x_0\| = \inf\{\|x - y\| : y \in C\}$. In this case, we can define the metric projection mapping $P_C : X \to C$ by $P_C x = x_0$, for all $x \in X$; see [3].

Suppose A is a multi-valued mapping from X into X^*. Recall that A is said to be monotone [12] if, for all $v_i \in Au_i$, $i = 1, 2$, $\langle u_1 - u_2, v_1 - v_2 \rangle \geq 0$. A monotone mapping A is said to be maximal monotone if $R(J + \lambda A) = X^*$, for $\lambda > 0$, A point $x \in D(A)$ is called a zero of A if $Ax = 0$. The set of zeros of A is denoted by $N(A)$.

In 2018, Wei and Agarwal [13] constructed new projection sets and investigated the following hybrid iterative scheme for approximating a common point which lies in the zero set of infinite maximal monotone operators A_i and in the fixed point set of infinite weakly relatively nonexpansive mappings B_i:

$$\begin{cases} x_1 \in X, r_{1,i} \in (0, +\infty), \ i \in N, \\ y_{n,i} = (J + r_{n,i} A_i)^{-1} J(x_n + e_n), \ i \in N, \\ z_{n,i} = J^{-1}[\alpha_n J x_n + (1-\alpha_n) J B_i y_{n,i}], \ i \in N, \\ V_1 = X = W_1, \\ V_{n+1,i} = \{p \in X : \langle y_{n,i} - p, J(x_n + e_n) - J y_{n,i} \rangle \geq 0\}, \ i \in N, \\ V_{n+1} = (\bigcap_{i=1}^{\infty} V_{n+1,i}) \bigcap V_n, \\ W_{n+1,i} = \{p \in V_{n+1,i} : \phi(p, z_{n,i}) \leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, y_{n,i})\}, \ i \in N, \\ W_{n+1} = (\bigcap_{i=1}^{\infty} W_{n+1,i}) \bigcap W_n, \\ U_{n+1} = \{p \in W_{n+1} : \|x_1 - p\|^2 \leq \|P_{W_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in U_{n+1}, \ n \in N, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$ and $\{r_{n,i}\}$ is positive real number sequence for each i. They proved that $\{x_n\}$ generated by (1.5) converges strongly to $P_{\bigcap_{n=1}^{\infty} W_n}(x_1) \in (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} F(B_i))$ under some conditions.

In [14], Wei and Agarwal further studied the following iterative process

$$\begin{cases} x_1 \in X, e_1 \in X, \\ y_n = J^{-1}[\alpha_n J x_n + (1-\alpha_n) \sum_{i=1}^{\infty} a_{n,i} J (J + r_{n,i} A_i)^{-1} J(x_n + e_n)], \\ z_n = J^{-1}[\beta_n J x_n + (1-\beta_n) \sum_{i=1}^{\infty} b_i J B_i y_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{v \in U_n : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1-\alpha_n) \phi(v, x_n + e_n), \\ \ \phi(v, z_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) \phi(v, y_n)\}, \\ V_{n+1} = \{v \in U_{n+1} : \|x_1 - v\|^2 \leq \|P_{U_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \ n \in N, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$. They proved that $\{x_n\}$ generated by (1.6) converges strongly to $P_{\bigcap_{n=1}^{\infty} U_n}(x_1) \in (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} F(B_i))$ under some conditions.

Compared to traditional hybrid iterative schemes (e.g., (1.1), (1.2), (1.3) and (1.4)), the main different ideas in (1.5) and (1.6) are the iterative item x_{n+1}, which can be chosen arbitrarily in U_{n+1} or V_{n+1}, for each $n \in N$. This may provide different choices for different uses. Can we borrow the ideas presented in [13] and [14] and construct new projection sets in hybrid iterative schemes for two infinite families of weakly relatively nonexpansive mappings? In this paper, we will provide a positive answer to this questions.

We also need the following tools to prove our main results.

Lemma 1.1. [4] *Suppose that X is a uniformly convex and uniformly smooth Banach space and C is a nonempty closed and convex subset of X. If $S : C \to C$ is weakly relatively nonexpansive, then $F(S)$ is a closed and convex subset of X.*

Lemma 1.2. [13] *Let X be a real uniformly smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$, as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Let $\{C_n\}$ be a sequence of nonempty closed and convex subsets of X. The strong lower limit of $\{C_n\}$, s-$\liminf C_n$, is defined as the set of all $x \in X$ such that there exists $x_n \in C_n$ for almost all n and it tends to x as $n \to \infty$ in the norm, the weak upper limit of $\{C_n\}$, w-$\limsup C_n$ is defined as the set of all $x \in X$ such that there exists a subsequence $\{C_{n_m}\}$ of $\{C_n\}$ and $x_{n_m} \in C_{n_m}$ for every n_m and it tends to x as $n_m \to \infty$ in the weak topology, and the limit of $\{C_n\}$, $\lim C_n$ is the common value when s-$\liminf C_n = w$-$\limsup C_n$; see [15].

Lemma 1.3. [15] *Let $\{C_n\}$ be a decreasing sequence of closed and convex subsets of X, i.e. $C_n \subset C_m$ as $n \geq m$. Then $\{C_n\}$ converges in X and $\lim C_n = \bigcap_{n=1}^{\infty} C_n$.*

Lemma 1.4. [16] *Suppose that X is a real uniformly convex Banach space. If $limC_n$ exists and is not empty, then $\{P_{C_n}x\}$ converges weakly to $P_{limC_n}x$ for every $x \in X$. Moreover, if X has Property (H), the convergence is in norm.*

Lemma 1.5. [17] *Let X be a real uniformly convex Banach and $r \in (0, +\infty)$. Then there exists a continuous, strictly increasing and convex function $\eta : [0, 2r] \to [0, +\infty)$ with $\eta(0) = 0$ such that*

$$\|\sum_{i=1}^{\infty} k_i x_i\|^2 \leq \sum_{i=1}^{\infty} k_i \|x_i\|^2 - k_1 k_m \eta(\|x_1 - x_m\|),$$

for all $\{x_n\}_{n=1}^{\infty} \subset \{x \in X : \|x\| \leq r\}$, $\{k_n\}_{n=1}^{\infty} \subset (0,1)$ with $\sum_{n=1}^{\infty} k_n = 1$ and $m \in N$.

Lemma 1.6. [12] *Let $A : X \to 2^{X^*}$ be a maximal monotone operator. Then*

(1) *$N(A)$ is a closed and convex subset of X;*

(2) *if $x_n \to x$ and $y_n \in Ax_n$ with $y_n \rightharpoonup y$, or $x_n \rightharpoonup x$ and $y_n \in Ax_n$ with $y_n \to y$, then $x \in D(A)$ and $y \in Ax$.*

2. MAIN RESULTS

In this section, our discussion is based on the following conditions:

(I_1) X is a real uniformly convex and uniformly smooth Banach space and $J : X \to X^*$ is the normalized duality mapping;

(I_2) $S_i, T_i : X \to X$ are weakly relatively nonexpansive mappings, for each $i \in N$, and

$$(\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i)) \neq \emptyset;$$

(I_3) $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $[0,1)$;

(I_4) $\{\lambda_n\}$ is a real number sequence in $(0, +\infty)$ with $\lambda_n \to 0$ as $n \to \infty$;

(I_5) $\{a_i\}$ and $\{b_i\}$ are two real number sequences in $(0,1)$ and $\sum_{i=1}^{\infty} a_i = 1 = \sum_{i=1}^{\infty} b_i$.

Theorem 2.1. *Let $\{u_n\}$ be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, \\ v_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i JT_i u_n], \\ w_n = J^{-1}[\beta_n Ju_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i JS_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p,v_n) \le \phi(p,u_n), \phi(p,w_n) \le \beta_n \phi(p,u_n) + (1-\beta_n)\phi(p,v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1},\ n \in N. \end{cases} \tag{2.1}$$

If $0 \le \sup_n \alpha_n < 1$ *and* $0 \le \sup_n \beta_n < 1$, *then* $u_n \to P_{\bigcap_{m=1}^{\infty} U_m}(u_1) \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$, *as* $n \to \infty$.

Proof. We split the proof into eight steps.

Step 1. Prove that $(\bigcap_{i=1}^{\infty} F(T_i) \bigcap (\bigcap_{i=1}^{\infty} F(S_i)) \subset U_n$ for $n \in N$.

Fix $q \in (\bigcap_{i=1}^{\infty} F(T_i) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$. If $n = 1$, it is obvious that $q \in U_1 = X$. It follows from the definitions of the Lyapunov functional and weakly relatively nonexpansive mappings that

$$\begin{aligned} \phi(q,v_1) &= \|q\|^2 - 2\langle q, \alpha_1 Ju_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_i JT_i u_1\rangle \\ &\quad + \|\alpha_1 Ju_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_i JT_i u_1\|^2 \\ &\le \alpha_1 \phi(q,u_1) + (1-\alpha_1)\sum_{i=1}^{\infty} a_i \phi(q, T_i u_1) \\ &\le \alpha_1 \phi(q,u_1) + (1-\alpha_1)\phi(q,u_1) = \phi(q,u_1) \end{aligned}$$

and

$$\begin{aligned} \phi(q,w_1) &\le \|q\|^2 - 2\beta_1 \langle q, Ju_1\rangle - 2(1-\beta_1)\sum_{i=1}^{\infty} b_i \langle q, JS_i v_1\rangle \\ &\quad + \beta_1 \|u_1\|^2 + (1-\beta_1)\sum_{i=1}^{\infty} b_i \|S_i v_1\|^2 \\ &= \beta_1 \phi(q,u_1) + (1-\beta_1)\sum_{i=1}^{\infty} b_i \phi(q, S_i v_1) \le \beta_1 \phi(q,u_1) + (1-\beta_1)\phi(q,v_1). \end{aligned}$$

Therefore, $q \in U_2$. Suppose the result is true for $n = k+1$. If $n = k+2$, then

$$\begin{aligned} \phi(q,v_{k+1}) &= \|q\|^2 - 2\langle q, \alpha_{k+1} Ju_{k+1} + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i JT_i u_{k+1}\rangle \\ &\quad + \|\alpha_{k+1} Ju_{k+1} + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i JT_i u_{k+1}\|^2 \\ &\le \alpha_{k+1}\phi(q,u_{k+1}) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i \phi(q, T_i u_{k+1}) \\ &\le \alpha_{k+1}\phi(q,u_{k+1}) + (1-\alpha_{k+1})\phi(q,u_{k+1}) = \phi(q,u_{k+1}). \end{aligned}$$

Moreover,

$$\begin{aligned}\phi(q,w_{k+1}) &\le \|q\|^2 - 2\beta_{k+1}\langle q, Ju_{k+1}\rangle - 2(1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\langle q, JS_i v_{k+1}\rangle \\ &\quad + \beta_{k+1}\|u_{k+1}\|^2 + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\|S_i v_{k+1}\|^2 \\ &\le \beta_{k+1}\phi(q,u_{k+1}) + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\phi(q,S_i v_{k+1}) \\ &\le \beta_{k+1}\phi(q,u_{k+1}) + (1-\beta_{k+1})\phi(q,v_{k+1}).\end{aligned}$$

Therefore, $q \in U_{k+2}$. By induction,

$$\emptyset \ne (\bigcap_{i=1}^{\infty} F(T_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \subset U_n.$$

Step 2. Prove that U_n is a closed and convex subset of X, for each $n \in N$.

If $n = 1$, the result is trivial. For $n \in N \setminus \{1\}$, we have the facts that $\phi(p,v_n) \le \phi(p,u_n)$ is equivalent to

$$2\langle p, Ju_n - Jv_n\rangle \le \|u_n\|^2 - \|v_n\|^2$$

and

$$\phi(p,w_n) \le \beta_n\phi(p,u_n) + (1-\beta_n)\phi(p,v_n)$$

is equivalent to

$$2\beta_n\langle p, Ju_n\rangle + 2(1-\beta_n)\langle p, Jv_n\rangle - 2\langle p, Jw_n\rangle \le \beta_n\|u_n\|^2 + (1-\beta_n)\|v_n\|^2 - \|w_n\|^2.$$

Then we can easily know that U_n is closed and convex, for each $n \in N$.

Step 3. Prove that V_n is a nonempty subset of X, for $n \in N$, which ensures that $\{u_n\}$ is well-defined.

In fact, if $n = 1$, the result is trivial. If $n \in N \setminus \{1\}$, then we see from the definition of metric projection that

$$\|P_{U_{n+1}}(u_1) - u_1\| = \inf_{y \in U_{n+1}} \|y - u_1\|.$$

For λ_{n+1}, there exists $\xi_{n+1} \in U_{n+1}$ such that

$$\|u_1 - \xi_{n+1}\|^2 \le (\inf_{y \in U_{n+1}} \|u_1 - y\|)^2 + \lambda_{n+1} = \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}.$$

This ensures that $V_{n+1} \ne \emptyset$ for $n \in N$.

Step 4. Prove that $P_{U_{n+1}}(u_1) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$.

It follows from Steps 1 and 2 and Lemma 1.3 that $\lim U_n$ exists and $lim U_n = \bigcap_{n=1}^{\infty} U_n \ne \emptyset$. Since X has Property (H), then Lemma 1.4 ensures that $P_{U_{n+1}}(u_1) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$.

Step 5. Prove that both $\{u_n\}$ and $\{P_{U_{n+1}}(u_1)\}$ are bounded.

It immediately follows from Step 4 that $\{P_{U_{n+1}}(u_1)\}$ is bounded. Since $u_{n+1} \in V_{n+1}$, one sees that

$$\|u_1 - u_{n+1}\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}.$$

Since $\lambda_n \to 0$ and $\{P_{U_{n+1}}(u_1)\}$ is bounded, it is easy to see that $\{u_n\}$ is also bounded.

Step 6. Prove that $u_{n+1} - P_{U_{n+1}}(u_1) \to 0$, as $n \to \infty$.

Since $u_{n+1} \in V_{n+1} \subset U_{n+1}$ and U_n is a convex subset of X, for $\forall t \in (0,1)$, one has

$$tP_{U_{n+1}}(u_1) + (1-t)u_{n+1} \in U_{n+1}.$$

It follows from the definition of metric projection that

$$\|P_{U_{n+1}}(u_1)-u_1\| \leq \|tP_{U_{n+1}}(u_1)+(1-t)u_{n+1}-u_1\|.$$

Lemma 1.5 ensures that

$$\begin{aligned}\|P_{U_{n+1}}(u_1)-u_1\|^2 &\leq \|tP_{U_{n+1}}(u_1)+(1-t)u_{n+1}-u_1\|^2 \\ &\leq t\|P_{U_{n+1}}(u_1)-u_1\|^2+(1-t)\|u_{n+1}-u_1\|^2-t(1-t)\eta(\|P_{U_{n+1}}(u_1)-u_{n+1}\|).\end{aligned}$$

Thus,

$$t\eta(\|P_{U_{n+1}}(u_1)-u_{n+1}\|) \leq \|u_{n+1}-u_1\|^2-\|P_{U_{n+1}}(u_1)-u_1\|^2 \leq \lambda_{n+1}.$$

Letting $t\to 1$ and $n\to\infty$, we know that $P_{U_{n+1}}(u_1)-u_{n+1}\to 0$ as $n\to\infty$.

Step 7. Prove that $u_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$, $v_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$ and $w_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$, as $n\to\infty$.

In fact, it follows from Step 4 and Step 6 that $u_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$ as $n\to\infty$. Then $\phi(u_{n+1},u_n)\to 0$ as $n\to\infty$. Since $u_{n+1}\in V_{n+1}\subset U_{n+1}$, one has $\phi(u_{n+1},v_n)\leq\phi(u_{n+1},u_n)\to 0$. Using Lemma 1.2, we obtain that $u_{n+1}-v_n\to 0$, which ensures that $v_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$ as $n\to\infty$. Since $u_{n+1}\in V_{n+1}\subset U_{n+1}$, we have

$$\phi(u_{n+1},w_n)\leq\beta_n\phi(u_{n+1},u_n)+(1-\beta_n)\phi(u_{n+1},v_n)\to 0,$$

as $n\to\infty$. Thus $u_{n+1}-w_n\to 0$, which implies that $w_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$ as $n\to\infty$.

Step 8. Prove that $P_{\bigcap_{n=1}^\infty U_n}(u_1)\in(\bigcap_{i=1}^\infty F(T_i))\bigcap(\bigcap_{i=1}^\infty F(S_i))$.

First, we show that $P_{\bigcap_{n=1}^\infty U_n}(u_1)\in F(T_1)$. For $\forall q\in(\bigcap_{i=1}^\infty F(T_i))\bigcap(\bigcap_{i=1}^\infty F(S_i))$, we conclude from Lemma 1.5 that

$$\begin{aligned}\phi(q,J^{-1}(\sum_{i=1}^\infty a_iJT_iu_n)) &= \|q\|^2-2\langle q,\sum_{i=1}^\infty a_iJT_iu_n\rangle+\|\sum_{i=1}^\infty a_iJT_iu_n\|^2 \\ &\leq \|q\|^2-2\sum_{i=1}^\infty a_i\langle q,JT_iu_n\rangle+\sum_{i=1}^\infty a_i\|T_iu_n\|^2-a_1a_m\eta(\|JT_1u_n-JT_mu_n\|) \\ &= \sum_{i=1}^\infty a_i\phi(q,T_iu_n)-a_1a_m\eta(\|JT_1u_n-JT_mu_n\|).\end{aligned}$$

Then

$$\begin{aligned}a_1a_m\eta(\|JT_1u_n-JT_mu_n\|) &\leq \sum_{i=1}^\infty a_i\phi(q,T_iu_n)-\phi(q,J^{-1}(\sum_{i=1}^\infty a_iJT_iu_n)) \\ &\leq \phi(q,u_n)-\phi(q,J^{-1}(\sum_{i=1}^\infty a_iJT_iu_n)) \\ &= \|u_n\|^2-2\langle q,Ju_n\rangle+2\sum_{i=1}^\infty a_i\langle q,JT_iu_n\rangle-\|\sum_{i=1}^\infty a_iJT_iu_n\|^2.\end{aligned}\tag{2.2}$$

Since

$$v_n=J^{-1}[\alpha_nJu_n+(1-\alpha_n)\sum_{i=1}^\infty a_iJT_iu_n],$$

one has

$$Jv_n-Ju_n=(1-\alpha_n)(\sum_{i=1}^\infty a_iJT_iu_n-Ju_n).$$

Note that both J and J^{-1} are uniformly continuous on each bounded subset of X and X^*, respectively. From $u_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$, $v_n \to P_{\bigcap_{n=1}^\infty U_n}(u_1)$ and $0\leq\sup_n\alpha_n<1$, one has $\sum_{i=1}^\infty a_iJT_iu_n-Ju_n\to 0$, which implies that

$$J^{-1}(\sum_{i=1}^{\infty} a_iJT_iu_n) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1), \tag{2.3}$$

as $n \to \infty$. Moreover, from (2.2), we also know that $JT_1u_n - JT_mu_n \to 0$ as $n \to \infty$ for $m \neq 1$. Note that $(\|q\| - \|T_iu_n\|)^2 \le \phi(q, T_iu_n) \le \phi(q, u_n) \le (\|q\| + \|u_n\|)^2$, for $\forall q \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$, $\{T_iu_n\}$ is bounded for $\forall i \in N$ since $\{u_n\}$ is bounded. We may assume that $M = \sup\{\|T_iu_n\| : i, n \in N\}$. Since $\sum_{i=1}^{\infty} a_i = 1$, for $\forall \delta > 0$, there exists sufficiently large integer N_0 such that

$$\sum_{i=N_0+1}^{\infty} a_i < \frac{\delta}{4M}.$$

From the fact that $JT_1u_n - JT_mu_n \to 0$, as $n \to \infty$, for $\forall m \in \{1, 2, \cdots, N_0\}$, we see that there exists sufficiently large integer M_0 such that $\|JT_1u_n - JT_mu_n\| < \frac{\delta}{2}$ for all $n \ge M_0$ and $m \in \{2, \cdots, N_0\}$. If $n \ge M_0$,

$$\begin{aligned}
\|JT_1u_n - \sum_{i=1}^{\infty} a_iJT_iu_n\| &\le \sum_{i=2}^{N_0} a_i\|JT_1u_n - JT_iu_n\| + \sum_{i=N_0+1}^{\infty} a_i\|JT_1u_n - JT_iu_n\| \\
&< (\sum_{i=2}^{N_0} a_i)\frac{\delta}{2} + (\sum_{i=N_0+1}^{\infty} a_i)2M \\
&< \frac{\delta}{2} + \frac{\delta}{2} \\
&= \delta.
\end{aligned}$$

Therefore, $JT_1u_n - \sum_{i=1}^{\infty} a_iJT_iu_n \to 0$ as $n \to \infty$. Then (2.3) implies that $T_1u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$. Combining with the fact that $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, and by using Lemma 1.1, $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(T_1)$. Repeating the above process, we can also prove that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(T_m)$, $\forall m \in N$. Therefore,

$$P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} F(T_i).$$

Next, we show that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(S_1)$. For $\forall q \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$, we have

$$b_1b_m\eta(\|JS_1v_n - JS_mv_n\|) \le \|v_n\|^2 - 2\langle q, Jv_n\rangle + 2\sum_{i=1}^{\infty} b_i\langle q, JS_iv_n\rangle - \|\sum_{i=1}^{\infty} b_iJS_iv_n\|^2. \tag{2.4}$$

Since $w_n = J^{-1}[\beta_nJu_n + (1-\beta_n)\sum_{i=1}^{\infty} b_iJS_iv_n]$, one has

$$Jw_n - Ju_n = (1-\beta_n)(\sum_{i=1}^{\infty} b_iJS_iv_n - Ju_n).$$

From the facts that $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $w_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ and $0 \le sup_n\beta_n < 1$, we have $\sum_{i=1}^{\infty} b_iJS_iv_n - Ju_n \to 0$, which implies that

$$J^{-1}(\sum_{i=1}^{\infty} b_iJS_iv_n) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1), \tag{2.5}$$

as $n \to \infty$. Coming back to (2.4), $JS_1v_n - JS_mv_n \to 0$ as $n \to \infty$ for $m \neq 1$. In the same way, we can show that $\{S_iv_n\}$ is bounded for $\forall i \in N$. We may assume that $M' = \sup\{\|S_iv_n\| : i, n \in N\}$. Since $\sum_{i=1}^{\infty} b_i = 1$, for $\forall \delta > 0$, there exists sufficiently large integer N_0 such that

$$\sum_{i=N_0+1}^{\infty} b_i < \frac{\delta}{4M'}.$$

Since $JS_1v_n - JS_mv_n \to 0$ as $n \to \infty$, for $\forall m \in \{1,2,\cdots,N_0\}$, we find that there exists sufficiently large integer M_0 such that $\|JS_1v_n - JS_mv_n\| < \frac{\delta}{2}$ for all $n \geq M_0$ and $m \in \{2,\cdots,N_0\}$. If $n \geq M_0$, then

$$\begin{aligned}
\|JS_1v_n - \sum_{i=1}^{\infty} b_iJS_iv_n\| &\leq \sum_{i=2}^{N_0} b_i\|JS_1v_n - JS_iv_n\| + \sum_{i=N_0+1}^{\infty} b_i\|JS_1v_n - JS_iv_n\| \\
&< (\sum_{i=2}^{N_0} b_i)\frac{\delta}{2} + (\sum_{i=N_0+1}^{\infty} b_i)2M' \\
&< \frac{\delta}{2} + \frac{\delta}{2} \\
&= \delta.
\end{aligned}$$

Therefore, $JS_1v_n - \sum_{i=1}^{\infty} b_iJS_iv_n \to 0$. From (2.5), one implies that $S_1v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$. Combining with the fact that $v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, we have $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(S_1)$. Repeating the above process again, we can also prove that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(S_m)$, $\forall m \in N$. Therefore, $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} F(S_i)$. This completes the proof. □

Remark 2.2. From algorithm (2.1), we see that we have infinite choices of iterative sequence $\{u_n\}$, which is the main difference compared with the traditional hybrid method (e.g. (1.1), (1.2), (1.3) and (1.4)) on this topic.

Theorem 2.3. *Let $\{u_n\}$ be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases}
u_1 \in X, \\
v_n = J^{-1}[\alpha_n Ju_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_iJT_iu_n], \\
w_n = J^{-1}[\beta_n Ju_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_iJS_iv_n], \\
U_1 = X = V_1, \\
U_{n+1} = \{p \in U_n : \phi(p,v_n) \leq \alpha_n\phi(p,u_1) + (1-\alpha_n)\phi(p,u_n), \phi(p,w_n) \leq \beta_n\phi(p,u_1) + (1-\beta_n)\phi(p,v_n)\}, \\
V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\
u_{n+1} \in V_{n+1},\ n \in N.
\end{cases} \tag{2.6}$$

If $\alpha_n \to 0$ and $\beta_n \to 0$. Then $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} F(T_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i))$ as $n \to \infty$.

Proof. The proof is also split into eight steps. Steps 3,4, 5 and 6 are the same with Theorem 2.1. Next, we only give the proof for Steps 1, 2, 7 and 8.

Step 1. Prove that $(\bigcap_{i=1}^{\infty} F(T_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \subset U_n$ for $n \in N$.

Fix $q \in (\bigcap_{i=1}^{\infty} F(T_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i))$. If $n = 1$, $q \in U_1 = X$ is obvious. In view of the convexity of $\|\cdot\|^2$ and the definition of weakly relatively nonexpansive mappings, we have

$$\begin{aligned}
\phi(q,v_1) &= \|q\|^2 - 2\langle q, \alpha_1 Ju_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_iJT_iu_1\rangle + \|\alpha_1 Ju_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_iJT_iu_1\|^2 \\
&\leq \|q\|^2 - 2\alpha_1\langle q, Ju_1\rangle - 2(1-\alpha_1)\sum_{i=1}^{\infty} a_i\langle q, JT_iu_1\rangle + \alpha_1\|u_1\|^2 + (1-\alpha_1)\sum_{i=1}^{\infty} a_i\|T_iu_1\|^2 \\
&= \alpha_1\phi(q,u_1) + (1-\alpha_1)\sum_{i=1}^{\infty} a_i\phi(q,T_iu_1) \\
&\leq \alpha_1\phi(q,u_1) + (1-\alpha_1)\phi(q,u_1)
\end{aligned}$$

and

$$\begin{aligned}\phi(q,w_1) &\le \|q\|^2 - 2\beta_1\langle q, Ju_1\rangle - 2(1-\beta_1)\sum_{i=1}^{\infty} b_i\langle q, JS_iv_1\rangle \\ &\quad + \beta_1\|u_1\|^2 + (1-\beta_1)\sum_{i=1}^{\infty} b_i\|S_iv_1\|^2 \\ &= \beta_1\phi(q,u_1) + (1-\beta_1)\sum_{i=1}^{\infty} b_i\phi(q,S_iv_1) \le \beta_1\phi(q,u_1) + (1-\beta_1)\phi(q,v_1).\end{aligned}$$

Thus $q \in U_2$. Suppose the result is true for $n = k+1$. If $n = k+2$, then

$$\begin{aligned}\phi(q,v_{k+1}) &= \|q\|^2 - 2\langle q, \alpha_{k+1}Ju_1 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_iJT_iu_{k+1}\rangle \\ &\quad + \|\alpha_{k+1}Ju_1 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_iJT_iv_{k+1}\|^2 \\ &\le \|q\|^2 - 2\alpha_{k+1}\langle q, Ju_1\rangle - 2(1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i\langle q, JT_iu_{k+1}\rangle \\ &\quad + \alpha_{k+1}\|u_1\|^2 + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i\|T_iu_{k+1}\|^2 \\ &= \alpha_{k+1}\phi(q,u_1) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i\phi(q,T_iu_{k+1}) \\ &\le \alpha_{k+1}\phi(q,u_1) + (1-\alpha_{k+1})\phi(q,u_{k+1}).\end{aligned}$$

Moreover,

$$\begin{aligned}\phi(q,w_{k+1}) &\le \|q\|^2 - 2\beta_{k+1}\langle q, Ju_1\rangle - 2(1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\langle q, JS_iv_{k+1}\rangle \\ &\quad + \beta_{k+1}\|u_1\|^2 + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\|S_iv_{k+1}\|^2 \\ &= \beta_{k+1}\phi(q,u_1) + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_i\phi(q,S_iv_{k+1}) \\ &\le \beta_{k+1}\phi(q,u_1) + (1-\beta_{k+1})\phi(q,v_{k+1}).\end{aligned}$$

Then $q \in U_{k+2}$. Therefore,

$$\emptyset \ne (\bigcap_{i=1}^{\infty} F(T_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \subset U_n.$$

Step 2. Prove that U_n is a closed and convex subset of X.

Notice that

$$\begin{aligned}\phi(p,v_n) &\le \alpha_n\phi(p,u_1) + (1-\alpha_n)\phi(p,u_n) \\ &\iff 2\alpha_n\langle p, Ju_1\rangle + 2(1-\alpha_n)\langle p, Ju_n\rangle - 2\langle p, Jv_n\rangle \\ &\le \alpha_n\|u_1\|^2 + (1-\alpha_n)\|u_n\|^2 - \|v_n\|^2\end{aligned}$$

and

$$\begin{aligned}\phi(p,w_n) &\le \beta_n\phi(p,u_1) + (1-\beta_n)\phi(p,v_n) \\ &\iff 2\beta_n\langle p, Ju_1\rangle + 2(1-\beta_n)\langle p, Jv_n\rangle - 2\langle p, Jw_n\rangle \\ &\le \beta_n\|u_1\|^2 + (1-\beta_n)\|v_n\|^2 - \|w_n\|^2.\end{aligned}$$

Thus U_n is closed and convex for $n \in N$.

Step 7. Prove that $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ and $w_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$.

Following from the results of Step 4 and Step 6, $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$. It follows that $u_{n+1} - u_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} 0 \le \phi(u_{n+1}, u_n) &= \|u_{n+1}\|^2 - 2\langle u_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &= (\|u_{n+1}\|^2 - \|u_n\|^2) + 2\langle u_n - u_{n+1}, Ju_n\rangle \\ &\le (\|u_{n+1}\|^2 - \|u_n\|^2) + 2\|u_n - u_{n+1}\|\|u_n\| \to 0, \end{aligned}$$

as $n \to \infty$. Since $u_{n+1} \in V_{n+1} \subset U_{n+1}$, and $\alpha_n \to 0$, we have $0 \le \phi(u_{n+1}, v_n) \le \alpha_n \phi(u_{n+1}, u_1) + (1 - \alpha_n)\phi(u_{n+1}, u_n) \to 0$ as $n \to \infty$. It follows from Lemma 1.2 that $u_{n+1} - v_n \to 0$. So, $v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$. Since $u_{n+1} \in V_{n+1} \subset U_{n+1}$ and $\beta_n \to 0$, we have

$$0 \le \phi(u_{n+1}, w_n) \le \beta_n \phi(u_{n+1}, u_1) + (1 - \beta_n)\phi(u_{n+1}, v_n) \to 0,$$

as $n \to \infty$. Lemma 1.2 implies that $u_{n+1} - w_n \to 0$. Hence, $w_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$.

Step 8. Prove that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$.

First, we show that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(T_1)$. For $\forall q \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$, by using Lemma 1.5, we know that (2.2) in Theorem 2.1 is still true. Since $v_n = J^{-1}[\alpha_n Ju_1 + (1 - \alpha_n)\sum_{i=1}^{\infty} a_i JT_i u_n]$, we have

$$Jv_n - Ju_n = \alpha_n (Ju_1 - Ju_n) + (1 - \alpha_n)(\sum_{i=1}^{\infty} a_i JT_i u_n - Ju_n).$$

Note that both J and J^{-1} are uniformly continuous on each bounded subset of X and X^*, respectively. From $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ and $\alpha_n \to 0$, we have $\sum_{i=1}^{\infty} a_i JT_i u_n - Ju_n \to 0$, which implies that (2.3) is still true. Copy the corresponding part of Step 8 in Theorem 2.1, we have $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(T_1)$. Repeating the process above, we have $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(T_m)$, $\forall m \in N$. Therefore, $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} F(T_i)$. Next, we show that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(S_1)$. For $\forall q \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$, (2.4) is still true. Since $w_n = J^{-1}[\beta_n Ju_1 + (1 - \beta_n)\sum_{i=1}^{\infty} b_i JS_i v_n]$, we have

$$Jw_n - Ju_n = \beta_n (Ju_1 - Ju_n) + (1 - \beta_n)(\sum_{i=1}^{\infty} b_i JS_i v_n - Ju_n).$$

From the facts that $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $w_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ and $\beta_n \to 0$, we have $\sum_{i=1}^{\infty} b_i JS_i v_n - Ju_n \to 0$, which implies that (2.5) is still true. Copy the corresponding part of Step 8 in Theorem 2.1, we have $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(S_1)$. Repeating the process above, we can also prove that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in F(S_m)$, $\forall m \in N$. Therefore, $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} F(S_i)$. This completes the proof. □

From Theorem 2.1 and Theorem 2.3, we can obtain the following results.

Theorem 2.4. *Let $\{u_n\}$ be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, \\ v_n = J^{-1}[\alpha_n Ju_n + (1 - \alpha_n)\sum_{i=1}^{\infty} a_i JT_i u_n], \\ w_n = J^{-1}[\beta_n Ju_1 + (1 - \beta_n)\sum_{i=1}^{\infty} b_i JS_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \le \phi(p, u_n), \phi(p, w_n) \le \beta_n \phi(p, u_1) + (1 - \beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1}, \ n \in N. \end{cases} \tag{2.7}$$

If $0 \le \sup_n \alpha_n < 1$ *and* $\beta_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$ *as* $n \to \infty$.

Theorem 2.5. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, \\ v_n = J^{-1}[\alpha_n J u_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J T_i u_n], \\ w_n = J^{-1}[\beta_n J u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i J S_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \le \alpha_n \phi(p, u_1) + (1-\alpha_n)\phi(p, u_n), \phi(p, w_n) \le \beta_n \phi(p, u_n) + (1-\beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1}, \ n \in N. \end{cases} \tag{2.8}$$

If $0 \le \sup_n \beta_n < 1$ *and* $\alpha_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} F(T_i)) \bigcap (\bigcap_{i=1}^{\infty} F(S_i))$ *as* $n \to \infty$.

3. Applications

In this section, our discussion is still based on conditions (I_1), (I_3), (I_4) and (I_5) in Section 2. In addition, we assume that $A_i, B_i : X \to X^*$ are maximal monotone mappings and $S_i : X \to X$ is weakly relatively for each $i \in N$.

Lemma 3.1. [14] *Let* X *be a real uniformly smooth and uniformly convex Banach space and let* $A : X \to 2^{X^*}$ *be a maximal monotone mapping with* $N(A) \ne \emptyset$. *Then, for* $\forall x \in X, \forall y \in N(A)$ *and* $r > 0$, $\phi(y, (J+rA)^{-1}Jx) + \phi((J+rA)^{-1}Jx, x) \le \phi(y, x)$.

Lemma 3.2. *If* $N(A) \ne \emptyset$, *then under the assumptions of Lemma 3.1, one has that* $(J+rA)^{-1}J : X \to X$ *is strongly relatively nonexpansive, and* $F((J+rA)^{-1}J) = N(A)$ *for* $\forall r > 0$.

Proof. The result of $F((J+rA)^{-1}J) = N(A)$ $(\forall r > 0)$ follows from [18]. From Lemma 3.1, we know that $\phi(p, (J+rA)^{-1}Jx) \le \phi(p, x)$ for $x \in X$ and $p \in N(A)$. It is obvious that $F((J+rA)^{-1}J) \subset \widehat{F}((J+rA)^{-1}J)$. We next show that $\widehat{F}((J+rA)^{-1}J) \subset F((J+rA)^{-1}J)$. In fact, $\forall p \in \widehat{F}((J+rA)^{-1}J)$, there exists $\{x_n\} \subset X$ such that $x_n \rightharpoonup p \in X$ and $x_n - (J+rA)^{-1}Jx_n \to 0$ as $n \to \infty$. Denote $u_n = (J+rA)^{-1}Jx_n$. Then $x_n - u_n \to 0$, which implies that $u_n \rightharpoonup p$ as $n \to \infty$. Rewriting u_n, we have $Ju_n + rAu_n = Jx_n$, which implies from Lemma 1.1 that $Au_n \to 0$ for $r > 0$ as $n \to \infty$. It follows from Lemma 1.6 that $p \in N(A) = F((J+rA)^{-1}J)$. Therefore, $F((J+rA)^{-1}J) = \widehat{F}((J+rA)^{-1}J)$ for $r > 0$. Hence $(J+rA)^{-1}J : X \to X$ is strongly relatively nonexpansive, which is of course weakly relatively nonexpansive. This completes the proof. □

From Lemma 3.2 and Theorems 2.1, 2.3, 2.4 and 2.5, we can get the following results.

Theorem 3.3. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, \\ v_n = J^{-1}[\alpha_n J u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J+rA_i)^{-1} J u_n], \\ w_n = J^{-1}[\beta_n J u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i J S_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \leq \phi(p, u_n), \phi(p, w_n) \leq \beta_n \phi(p, u_n) + (1-\beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1},\ n \in N. \end{cases} \tag{3.1}$$

If $0 \leq \sup_n \alpha_n < 1$ *and* $0 \leq \sup_n \beta_n < 1$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i))$ *as* $n \to \infty$.

Theorem 3.4. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, \\ v_n = J^{-1}[\alpha_n J u_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J+rA_i)^{-1} J u_n], \\ w_n = J^{-1}[\beta_n J u_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_i J S_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \leq \alpha_n \phi(p, u_1) + (1-\alpha_n)\phi(p, u_n), \phi(p, w_n) \leq \beta_n \phi(p, u_1) + (1-\beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1},\ n \in N. \end{cases} \tag{3.2}$$

If $\alpha_n \to 0$ *and* $\beta_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i))$ *as* $n \to \infty$.

Theorem 3.5. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, \\ v_n = J^{-1}[\alpha_n J u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J+rA_i)^{-1} J u_n], \\ w_n = J^{-1}[\beta_n J u_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_i J S_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \leq \phi(p, u_n), \phi(p, w_n) \leq \beta_n \phi(p, u_1) + (1-\beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1},\ n \in N. \end{cases} \tag{3.3}$$

If $0 \leq \sup_n \alpha_n < 1$ *and* $\beta_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i))$ *as* $n \to \infty$.

Theorem 3.6. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, \\ v_n = J^{-1}[\alpha_n Ju_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J+rA_i)^{-1}Ju_n], \\ w_n = J^{-1}[\beta_n Ju_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i JS_i v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p,v_n) \le \alpha_n\phi(p,u_1) + (1-\alpha_n)\phi(p,u_n), \phi(p,w_n) \le \beta_n\phi(p,u_n) + (1-\beta_n)\phi(p,v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1}, \ n \in N. \end{cases} \tag{3.4}$$

If $0 \le \sup_n \beta_n < 1$ *and* $\alpha_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} F(S_i))$ *as* $n \to \infty$.

Using Theorems 3.3, 3.4, 3.5 and 3.6 and replacing S_i by $(J+sB_i)^{-1}J$, we can obtain the following results.

Theorem 3.7. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, s > 0, \\ v_n = J^{-1}[\alpha_n Ju_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J+rA_i)^{-1}Ju_n], \\ w_n = J^{-1}[\beta_n Ju_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i J(J+sB_i)^{-1}Jv_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p,v_n) \le \phi(p,u_n), \phi(p,w_n) \le \beta_n\phi(p,u_n) + (1-\beta_n)\phi(p,v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1}, \ n \in N. \end{cases} \tag{3.5}$$

If $0 \le \sup_n \alpha_n < 1$ *and* $0 \le \sup_n \beta_n < 1$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$ *as* $n \to \infty$.

Theorem 3.8. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, s > 0, \\ v_n = J^{-1}[\alpha_n Ju_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J+rA_i)^{-1}Ju_n], \\ w_n = J^{-1}[\beta_n Ju_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_i J(J+sB_i)^{-1}Jv_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p,v_n) \le \alpha_n\phi(p,u_1) + (1-\alpha_n)\phi(p,u_n), \phi(p,w_n) \le \beta_n\phi(p,u_1) + (1-\beta_n)\phi(p,v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1}, \ n \in N. \end{cases} \tag{3.6}$$

If $\alpha_n \to 0$ *and* $\beta_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$ *as* $n \to \infty$.

Theorem 3.9. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, s > 0, \\ v_n = J^{-1}[\alpha_n J u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J + rA_i)^{-1} J u_n], \\ w_n = J^{-1}[\beta_n J u_1 + (1-\beta_n)\sum_{i=1}^{\infty} b_i J(J + sB_i)^{-1} J v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \le \phi(p, u_n), \phi(p, w_n) \le \beta_n \phi(p, u_1) + (1-\beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1},\ n \in N. \end{cases} \tag{3.7}$$

If $0 \le \sup_n \alpha_n < 1$ *and* $\beta_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$ *as* $n \to \infty$.

Theorem 3.10. *Suppose* $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence generated by the following hybrid iterative scheme*

$$\begin{cases} u_1 \in X, r > 0, s > 0, \\ v_n = J^{-1}[\alpha_n J u_1 + (1-\alpha_n)\sum_{i=1}^{\infty} a_i J(J + rA_i)^{-1} J u_n], \\ w_n = J^{-1}[\beta_n J u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_i J(J + sB_i)^{-1} J v_n], \\ U_1 = X = V_1, \\ U_{n+1} = \{p \in U_n : \phi(p, v_n) \le \alpha_n \phi(p, u_1) + (1-\alpha_n)\phi(p, u_n), \phi(p, w_n) \le \beta_n \phi(p, u_n) + (1-\beta_n)\phi(p, v_n)\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \le \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1},\ n \in N. \end{cases} \tag{3.8}$$

If $0 \le \sup_n \beta_n < 1$ *and* $\alpha_n \to 0$, *then* $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$ *as* $n \to \infty$.

Remark 3.11. From theorems 3.3, 3.4, 3.5 and 3.6, we see that Theorems 2.1, 2.3, 2.4 and 2.5 are extensions of the corresponding results in [13] and [14] on the design of iterative schemes for common points of the set of zeros of infinite maximal monotone mappings and the set of fixed points of infinite weakly relatively nonexpansive mappings. From Theorems 3.7, 3.8, 3.9 and 3.10, we see that Theorems 2.1, 2.3, 2.4 and 2.5 are applicable for common zeros of two infinite families of maximal monotone mappings.

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