J. Nonlinear Funct. Anal. 2019 (2019), Article ID 19 https://doi.org/10.23952/jnfa.2019.19

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

MULTIPLE SOLUTIONS FOR HARDY NONLOCAL FRACTIONAL ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper, we use Ekeland's variational principle to study the existence of at least three nontrivial solutions for the following critical nonlocal fractional Hardy elliptic equation $(-\Delta)^s u - \gamma\frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha} + \lambda f(x,u)$ in $\mathbb{R}^N$, where $N > 2s$, $0 < s < 1$, γ, λ are real parameters, $2_s^*(\alpha) = \frac{2(N-\alpha)}{N-2s}$ is critical Hardy-Sobolev exponent with $\alpha \in [0,2s)$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a suitable function and $(-\Delta)^s$ is the fractional Laplace operator.

Keywords. Fractional Laplacian; Solution; Hardy-Sobolev exponent; Ekeland's variational principle.

2010 Mathematics Subject Classification. 35R11, 35A15, 35J60, 35J20.

1. INTRODUCTION

In this paper, we investigate the existence and multiplicity of solutions for the following critical nonlocal fractional Hardy elliptic equation

$$(-\Delta)^s u - \gamma\frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha} + \lambda f(x,u), \ in\ \mathbb{R}^N, \tag{1.1}$$

where $N > 2s$, $0 < s < 1$, γ, λ are real parameters, $2_s^*(\alpha) = \frac{2(N-\alpha)}{N-2s}$ is critical Hardy-Sobolev exponent with $\alpha \in [0,2s)$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a suitable function and $(-\Delta)^s$ is the fractional Laplace operator, which, up to normalization factors, may be defined as

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{N+2s}}dy$$

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Received December 5, 2018; Accepted May 3, 2019.

for $x \in R^N$. Moreover, Let $\pounds$ be the Schwartz space of rapidly decaying C^∞ functions in R^N. Then, for any $u \in \pounds$ and $s \in (0,1)$, $(-\Delta)^s$ is defined as

$$\begin{aligned}(-\Delta)^s u(x) &= C(N,s)P.V.\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy \\ &= C(N,s)\lim_{\varepsilon\to 0}\int_{\pounds B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,\end{aligned}$$

where $\pounds B_\varepsilon(x) = R^N \setminus B_\varepsilon(x)$ and the symbol $P.V.$ stands for the Cauchy principal value and $C(N,s)$ is a dimensional constant that depends on N,s, precisely given by

$$C(N,s) := \left(\int_{R^N} \frac{1-cos(\varsigma_1)}{|\varsigma|^{N+2s}}d\varsigma\right)^{-1}.$$

Fractional and nonlocal operators have recently been studied; see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references therein. These types of operators arise in a quite natural way in many different physical situations such as continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcomes of the stochastic stabilisation of the Levy processes (for more study, see [22, 23, 24] and the references therein). For the basic properties of fractional Sobolev spaces, we refer the readers to [14, 25]. Also, we know that the topological methods for fractional problems have been first set in [18, 19]. Recently, by using the mountain pass lemma, Ghoussoub and Shakerian [26] established the existence of a nontrivial weak solution to the problem

$$(-\Delta)^s u - \gamma\frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha}, \quad u>0, \quad \text{in } \mathbb{R}^N.$$

Moreover, Yang and Wu [27] showed the existence of nontrivial solutions for doubly critical nonlocal elliptic problems in $\mathbb{R}^N$. For other recent results in fractional and nonlocal operators, the reader is referred to, for example, [28, 29, 30, 31] and the references therein.

In [32], Chen considered the following doubly critical problem involving the fractional Laplacian

$$(-\Delta)^s u - \gamma\frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha} + \frac{|u|^{2_s^*(\beta)-2}u}{|x|^\beta}, \quad u>0, \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $s \in (0,1)$, $0 < \alpha, \beta < 2s < n$ with $\alpha \neq \beta$, $\gamma < \gamma_H$. Applying the mountain pass lemma and a concentration compactness principle, Chen proved the existence of positive solutions to (1.2). For other recent results for related problems, one can see [8, 33, 34].

In this paper, motivated by [35, 36], we study the multiplicity of nontrivial solutions for problem (1.1).

Now, we state our main result.

Theorem 1.1. *Assume that $f(x,u)$ is measurable in x and continuously differentiable in u, $f(x,0)=0$ for every $x \in \mathbb{R}^N$, there exist $l \in (2,2_s^*)$ and constants $\mu_1 \in (\frac{1}{2_s^*(\alpha)-1},1)$, $\mu_2 \in (2,2_s^*(\alpha))$, $0<\mu_3<\mu_4$ such that, for any $u \in H_0^s(\mathbb{R}^N)$,*

$$\begin{aligned}\mu_3\int_{\mathbb{R}^N} g(x)|u|^l dx &\leq \mu_2\int_{R^N} F(x,u)dx \leq \int_{R^N} f(x,u)udx \\ &\leq \mu_1\int_{R^N} f_u(x,u)u^2dx \leq \mu_4\int_{\mathbb{R}^N} g(x)|u|^l dx,\end{aligned} \tag{1.3}$$

where $F(x,u)=\int_0^u f(x,s)ds$ and $g(x)>0$, $g \in L^\infty(\mathbb{R}^N)$ and there exists $\rho_0>0$ such that $suppg \subset B_{\rho_0}(0)$, where $B_{\rho_0}(x) = \{y \in \mathbb{R}^N : |y-x| < \rho_0\}$. Then, there exists $\lambda^* = \lambda^*(N,s,\gamma,\alpha,l)$, such that for every

$\lambda > \lambda^*$, there exist three different nontrivial weak solutions of problem (1.1). *Moreover, these solutions are one positive, one negative and the other one sign-changing.*

2. Preliminaries and proof of the main result

In this paper, by employing Ekeland's variational principle and a similar argument in [37], we investigate the existence and multiplicity of solutions for critical nonlocal fractional Hardy elliptic equation 1.1. In order to prove our main results, we need need to present some preliminaries of the variational framework, definitions and lemmas which will play an important role to solve problem (1.1).

We first give some useful notations and basic results of fractional Sobolev space that will be used in proof of the main results. Let $0 < s < 1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

We know that, if $p = 2$, then $W^{s,2}(\mathbb{R}^N) := H^s(\mathbb{R}^N)$. Also, $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space of $g \in L^2(\mathbb{R}^N)$ such that the map $(x,y) \mapsto \frac{g(x)-g(y)}{|x-y|^{\frac{N+2s}{2}}}$ is in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$.
Let us consider $H^s(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

We introduce the space $H^s_0(\mathbb{R}^N)$ as the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

We know that $(H^s_0(\mathbb{R}^N), [\cdot]_s)$ is a uniformly convex Banach space. By [38, Theorem 1], one has

$$\|u\|^2_{2^*_s} \le C_N \frac{s(1-s)}{N-2s} [u]^2_s, \ \forall\, u \in H^s_0(\mathbb{R}^N), \tag{2.1}$$

and by [39, Theorem 1.1], one has

$$\gamma_1 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx \le [u]^2_s, \ \forall\, u \in H^s_0(\mathbb{R}^N), \tag{2.2}$$

where

$$\gamma_1 := 2\pi^{\frac{N}{2}} \frac{\Gamma\left(\frac{N+2s}{4}\right)^2 |\Gamma(-s)|}{\Gamma\left(\frac{N-2s}{4}\right)^2 \Gamma\left(\frac{N+2s}{2}\right)}. \tag{2.3}$$

Moreover, the constant γ_1 is optimal. If $\gamma < \gamma_1$, it follows from the Hardy inequality (2.2) that

$$\|u\| := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx \right)^{\frac{1}{2}}$$

is well defined on $H_0^s(\mathbb{R}^N)$. Since

$$\left(1-\frac{\lambda_+}{\gamma_1}\right)[u]_s^2 \le \|u\|^2 \le \left(1+\frac{\lambda_-}{\gamma_1}\right)[u]_s^2, \ \forall\, u \in H_0^s(\mathbb{R}^N), \tag{2.4}$$

where $\lambda_+ = \max\{\gamma, 0\}$ and $\lambda_- = \max\{-\gamma, 0\}$, then $\|u\|$ is comparable to $[u]_s$. Thus, fractional Sobolev embedding $H_0^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ and fractional Hardy embedding $H_0^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, |x|^{-2s})$ are continuous, but they are not compact. Combining the Hardy inequality and the Sobolev inequality, we obtain the Hardy-Sobolev inequality. Indeed, let $\alpha \in [0, 2s)$ be a real number. Then $H_0^s(\mathbb{R}^N)$ is continuously embedded in the weighted space $L^{2_s^*(\alpha)}(\mathbb{R}^N, |x|^{-\alpha})$. Here, taking the smallest constant associated to this embedding, we let

$$S(N,s,\gamma,\alpha) \quad = \quad \inf_{u \in H_0^s(\mathbb{R}^N)\setminus\{0\}} \frac{\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}dx}{\left(\int_{\mathbb{R}^N} |u(x)|^{2_s^*}|x|^{-\alpha}dx\right)^{\frac{2}{2_s^*(\alpha)}}}. \tag{2.5}$$

By [25, Proposition 3.6], we have $[u]_s = \|(-\triangle)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}$ for any $u \in H^s(\mathbb{R}^N)$, i.e.

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy = \int_{\mathbb{R}^N} |(-\triangle)^{\frac{s}{2}} u(x)|^2 dx. \tag{2.6}$$

So,

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy = \int_{\mathbb{R}^N} (-\triangle)^{\frac{s}{2}} u(x) \cdot -\triangle)^{\frac{s}{2}} v(x)dx, \tag{2.7}$$

for any $u, v \in H^s(\mathbb{R}^N)$.

Lemma 2.1. *Assume that* $0 < s < 1$.

(i) (The fractional Hardy inequality [40]*) For all $u \in H^s(\mathbb{R}^N)$, we have*

$$\gamma_H \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}\, dx \le \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\, dx, \tag{2.8}$$

where $\gamma_H = 4^s \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$ *is the best constant in the above inequality on* $\mathbb{R}^N$.

(ii) (The fractional Hardy-Sobolev inequality [26]*) Assume* $0 \le \alpha \le 2s < N$. *Then, there exist positive constants c and C, such that for all* $u \in H^s(\mathbb{R}^N)$,

$$\Big(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\, dx\Big)^{\frac{2}{2_s^*(\alpha)}} \le c \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\, dx := c[u]_s^2. \tag{2.9}$$

Moreover, if $\gamma < \gamma_H$, *then*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}\, dx\right)^{\frac{2}{2_s^*(\alpha)}} \le C_\alpha \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\, dx - \gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}\, dx\right), \tag{2.10}$$

for all $u \in H^s(\mathbb{R}^N)$.

We say that $u \in H_0^s(\mathbb{R}^N)$ is a weak solution of problem (1.1) if

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \left(\varphi(x)-\varphi(y)\right) dxdy - \gamma \int_{\mathbb{R}^N} \frac{u(x)}{|x|^{2s}}\varphi(x)dx \\ = \int_{R^N} \frac{|u|^{2_s^*(\alpha)-2}}{|x|^\alpha} u(x)\varphi(x)dx - \lambda \int_{\mathbb{R}^N} f(x,u)\varphi(x)dx\end{aligned}$$

for any $\varphi \in H_0^s(\mathbb{R}^N)$. The energy functional $J : H_0^s(\mathbb{R}^N) \to \mathbb{R}$, which is defined by the formula

$$\begin{aligned} J(u) &= \frac{1}{2}\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}dx\right) \\ &\quad -\frac{1}{2_s^*(\alpha)}\int_{\mathbb{R}^N} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} F(x,u)dx. \end{aligned} \tag{2.11}$$

So a solution of (1.1) is a nontrivial critical point of J. From supp$g \subset B_{\rho_0}(0)$, the Hölder inequality and (2.10), we can get

$$\begin{aligned} \int_{\mathbb{R}^N} g(x)|u(x)|^l dx &\leq \|g\|_\infty \left(\int_{B_{\rho_0}(0)} 1dx\right)^{\frac{2_s^*-l}{2_s^*}} \left(\int_{B_{\rho_0}(0)} |u(x)|^{2_s^*}dx\right)^{\frac{l}{2_s^*}} \\ &= \|f\|_\infty \left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\rho_0^N\right)^{\frac{2_s^*-l}{2_s^*}} \left(\int_{B_{\rho_0}(0)} |u(x)|^{2_s^*}dx\right)^{\frac{l}{2_s^*}} \\ &\leq \|g\|_\infty \left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\rho_0^N\right)^{\frac{2_s^*-l}{2_s^*}} C_0^{\frac{2_s^*}{2}}\|u\|^l. \end{aligned} \tag{2.12}$$

Recall that a sequence $\{u_n\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_c$ sequence for functional J if $J(u_n) \to c$ and $J'(u_n) \to 0$. If any $(\mathrm{PS})_c$ sequence $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, we say that J satisfies the $(\mathrm{PS})_c$ condition. Set

$$\begin{aligned} M_1 : &= \Bigg\{u \in H_0^s(\mathbb{R}^N) : \int_{R^N} u_+ > 0, \iint_{\mathbb{R}^{2N}} \frac{|u_+(x)-u_+(y)|^2}{|x-y|^{N+2s}}dxdy \\ &\quad -\gamma\int_{\mathbb{R}^N} \frac{|u_+(x)|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N} \frac{|u_+|^{2_s^*(\alpha)}}{|x|^\alpha}dx = \lambda\int_{\mathbb{R}^N} f(x,u)u_+dx\Bigg\}, \end{aligned}$$

$$\begin{aligned} M_2 : &= \Bigg\{u \in H_0^s(\mathbb{R}^N) : \int_{R^N} u_- > 0, \iint_{\mathbb{R}^{2N}} \frac{|u_-(x)-u_-(y)|^2}{|x-y|^{N+2s}}dxdy \\ &\quad -\gamma\int_{\mathbb{R}^N} \frac{|u_-(x)|^2}{|x|^{2s}}dx - \int_{R^N} \frac{|u_-|^{2_s^*(\alpha)}}{|x|^\alpha}dx = -\lambda\int_{\mathbb{R}^N} f(x,u)u_-dx\Bigg\}, \end{aligned}$$

$M_3 = M_1 \bigcap M_2$,

$$K_1 := \{u \in M_1; u \geq 0\}, \quad K_2 := \{u \in M_2; u \leq 0\}, \quad K_3 := M_3,$$

where $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$.

Lemma 2.2. *Under the assumptions of Theorem 1.1, for every $u_0 \in H_0^s(\mathbb{R}^N)$, $u_0 > 0 (u_0 < 0)$, there exists $t_\lambda > 0$ such that $t_\lambda u_0 \in M_1 (\in M_2)$. Furthermore, $\lim_{\lambda\to\infty} t_\lambda = 0$.*

Proof. For $u \in H_0^s(\mathbb{R}^N)$, we define the functional

$$\chi_1(u) := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} f(x,u)udx.$$

Given $u_0 > 0$, we claim that $\chi_1(t_\lambda u_0) = 0$ for some $t_\lambda > 0$. To this end, we find from (1.3) and (2.12) that

$$\begin{aligned}
\chi_1(tu_0) &= t^2\left[\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_{\mathbb{R}^N}\frac{|u_0(x)|^2}{|x|^{2s}}dx\right] \\
&\quad -t^{2_s^*(\alpha)}\int_{\mathbb{R}^N}\frac{|u_0|^{2_s^*(\alpha)}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} f(x,tu_0)tu_0dx \\
&\geq t^2\left[\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_{\mathbb{R}^N}\frac{|u_0(x)|^2}{|x|^{2s}}dx\right] \\
&\quad -t^{2_s^*(\alpha)}\int_{\mathbb{R}^N}\frac{|u_0|^{2_s^*(\alpha)}}{|x|^\alpha}dx - \lambda t^l\mu_4\int_{\mathbb{R}^N} g(x)|u_0|^l dx \\
&\geq At^2 - Bt^{2_s^*(\alpha)} - \lambda\mu_4 t^l C,
\end{aligned}$$

where $A = \iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_{\mathbb{R}^N}\frac{|u_0(x)|^2}{|x|^{2s}}dx$, $B = \int_{\mathbb{R}^N}\frac{|u_0|^{2_s^*(\alpha)}}{|x|^\alpha}dx$ and

$$C = \|g\|_\infty\left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\rho_0^N\right)^{\frac{2_s^*-l}{2_s^*}} C_0^{\frac{2_s^*}{2}}\|u_0\|^l.$$

Similarly we can get

$$\chi_1(tu_0) \leq At^2 - Bt^{2_s^*(\alpha)} - \lambda\mu_3 t^l C.$$

Since $2 < l < 2_s^*(\alpha)$, Bolzano's theorem yields that there exists $t = t_\lambda$ such that $\varphi_1(t_\lambda u) = 0$. Also, from $\chi_1(tu_0) \leq At^2 - \lambda\mu_3 Ct^l$, we choose t_1 such that $At_1^2 - \lambda\mu_3 Ct_1^l = 0$. So, $t_1 = \left(\frac{A}{\lambda\mu_3 C}\right)^{\frac{1}{l-2}}$. Choosing $t_\lambda \in [0, t_1]$, we obtain the desired conclusion immediately. □

Corollary 2.3. *By Lemma 2.2, for any $u_0, u_1 \in H_0^s(\mathbb{R}^N)$, $u_0 > 0$ and $u_1 < 0$ with disjoint supports, there exists $t_{1\lambda}, t_{2\lambda} > 0$ such that $t_\lambda^* u_0 + t_\lambda^{**} u_1 \in M_3$. Moreover $t_\lambda^*, t_\lambda^{**} \to 0$ as $\lambda \to \infty$.*

Lemma 2.4. *There exist η_1 and $\eta_2 > 0$ such that, for every $u \in K_i$, $i = 1, 2, 3$,*

$$\|u\|^2 = \int_{\mathbb{R}^N}\frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx + \lambda\int_{\mathbb{R}^N} f(x,u)udx \leq \eta_1 J(u) \leq \eta_2\|u\|^2.$$

Proof. Since $u \in K_i$, one sees that the equality is clear. In view of (1.3), one can get

$$\begin{aligned}
J(u) &= \frac{1}{2}\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy - \gamma\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}dx\right) \\
&\quad -\frac{1}{2_s^*(\alpha)}\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} F(x,u)dx \\
&= \frac{1}{2}\left(\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx + \lambda\int_{\mathbb{R}^N} f(x,u)udx\right) - \frac{1}{2_s^*(\alpha)}\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} F(x,u)dx \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*(\alpha)}\right)\int_{R^N}\frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha} + \frac{\lambda}{2}\int_{\mathbb{R}^N} f(x,u)udx - \lambda\int_{\mathbb{R}^N} F(x,u)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2_s^*(\alpha)}\right)\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha} + \left(\frac{1}{2} - \frac{1}{\mu_2}\right)\lambda\int_{\mathbb{R}^N} f(x,u)udx.
\end{aligned}$$

Since $2 < \mu_2 < 2_s^*(\alpha)$, one finds that the first inequality hold. By $F(x,u) \geq 0$, the last inequality is clear. □

Lemma 2.5. *There exists $C>0$ such that*

$$\|u_+\| \geq C, \quad \forall\, u \in K_1,$$
$$\|u_-\| \geq C, \quad \forall\, u \in K_2,$$
$$\|u_+\|, \|u_-\| \geq C, \quad \forall\, u \in K_3.$$

Proof. By the definition of $K_i (i=1,2,3)$, $2<l<2_s^*(\alpha)$, (1.3), (2.10) and (2.12), one can get

$$\begin{aligned}\|u_\pm\|^2 &= \lambda \int_{R^N} f(x,u)u_\pm dx + \int_{\mathbb{R}^N} \frac{|u_\pm|^{2_s^*(\alpha)}}{|x|^\alpha} dx \\ &\leq c_1 \int_{\mathbb{R}^N} g(x)|u_\pm|^l dx + \int_{\mathbb{R}^N} \frac{|u_\pm|^{2_s^*(\alpha)}}{|x|^\alpha} dx \\ &\leq c_1 \|g\|_\infty \left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\rho_0^N\right)^{\frac{2_s^*-l}{2_s^*}} C_0^{\frac{2_s^*}{2}} \|u_\pm\|^l + C_\alpha^{\frac{2_s^*(\alpha)}{2}} \|u_\pm\|^{2_s^*(\alpha)},\end{aligned}$$

where c_1 is a positive constant. Therefore, we have the conclusion. □

Lemma 2.6. *There exists $\Theta>0$ such that $J(u) \geq \Theta\|u\|^2$, $\forall u \in H_0^s(\mathbb{R}^N)$ if $\|u\|$ is small enough.*

Proof. In view of (1.3), (2.10) and (2.12), we get

$$\begin{aligned} J(u) &= \frac{1}{2}\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx\right) \\ &\quad - \frac{1}{2_s^*(\alpha)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha} dx - \lambda \int_{\mathbb{R}^N} F(x,u) dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2_s^*(\alpha)} \int_{\mathbb{R}^N} \frac{|u|^{2_s^*(\alpha)}}{|x|^\alpha} dx - c_2 \int_{\mathbb{R}^N} g(x)|u|^l dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2_s^*(\alpha)} C_\alpha^{\frac{2_s^*(\alpha)}{2}} \|u\|^{2_s^*(\alpha)} - c_2\|g\|_\infty \left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\rho_0^N\right)^{\frac{2_s^*-l}{2_s^*}} C_0^{\frac{2_s^*}{2}} \|u\|^l \\ &= \|u\|^2 \left(\frac{1}{2} - \frac{1}{2_s^*(\alpha)} C_\alpha^{\frac{2_s^*(\alpha)}{2}} \|u\|^{2_s^*(\alpha)-2} - c_2\|g\|_\infty \left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}\rho_0^N\right)^{\frac{2_s^*-l}{2_s^*}} C_0^{\frac{2_s^*}{2}} \|u\|^{l-2}\right). \end{aligned}$$

Since $2<l<2_s^*(\alpha)$, we find that $J(u) \geq C\|u\|^2$ if $\|u\|$ is small enough. □

Lemma 2.7. *M_i is a C^1 sub-manifold with co-dimension $1(i=1,2), 2(i=3)$. The sets $K_i(i=1,2,3)$ are complete. Moreover, for every $u \in M_i$,*

$$T_u H_0^s(\mathbb{R}^{\mathbb{N}}) = T_u M_1 \oplus span\{u_+\},$$

$$T_u H_0^s(\mathbb{R}^{\mathbb{N}}) = T_u M_2 \oplus span\{u_-\},$$

$$T_u H_0^s(\mathbb{R}^{\mathbb{N}}) = T_u M_3 \oplus span\{u_+, u_-\},$$

where $T_u M$ is the tangent space at u of the Banach manifold M. Moreover, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof. Similar to the method in [35], we set

$$\bar{M}_1 := \left\{ u \in H_0^s(\mathbb{R}^{\mathbb{N}}) : \int_{\mathbb{R}^N} u_+ dx > 0 \right\};$$
$$\bar{M}_2 := \left\{ u \in H_0^s(\mathbb{R}^{\mathbb{N}}) : \int_{\mathbb{R}^N} u_- dx > 0 \right\};$$
$$\bar{M}_3 := \bar{M}_1 \bigcap \bar{M}_2.$$

Notice that $M_i \subset \bar{M}_i$ and $\bar{M}_i$ is open in $H_0^s(\mathbb{R}^{\mathbb{N}})$. We claim that M_i is a C^1 sub-manifold of $\bar{M}_i$. To this end, we will construct a C^1 function $\varphi_i : \bar{M}_i \to \mathbb{R}^d$ with $d = 1 (i = 1,2), d = 2 (i = 3)$ respectively and M_i will be the inverse image of a regular value of φ_i. So, we define

$$\begin{aligned}
\varphi_1(u) &= \iint_{\mathbb{R}^{2N}} \frac{|u_+(x) - |u_+(y)|^2}{|x-y|^{N+2s}} dxdy - \gamma \int_{\mathbb{R}^N} \frac{|u_+(x)|^2}{|x|^{2s}} dx \\
&\quad - \int_{\mathbb{R}^N} \frac{|u_+|^{2_s^*(\alpha)}}{|x|^{\alpha}} dx - \lambda \int_{\mathbb{R}^N} f(x,u) u_+ dx, \quad \forall\, u \in \bar{M}_1, \\
\varphi_2(u) &= \iint_{\mathbb{R}^{2N}} \frac{|u_-(x) - |u_-(y)|^2}{|x-y|^{N+2s}} dxdy - \gamma \int_{\mathbb{R}^N} \frac{|u_-(x)|^2}{|x|^{2s}} dx \\
&\quad - \int_{\mathbb{R}^N} \frac{|u_-|^{2_s^*(\alpha)}}{|x|^{\alpha}} dx - \lambda \int_{\mathbb{R}^N} f(x,u) u_- dx, \quad \forall\, u \in \bar{M}_2, \\
\varphi_3(u) &= (\varphi_1(u), \varphi_2(u)), \quad \forall\, u \in \bar{M}_3.
\end{aligned}$$

Clearly, $M_i = \varphi_i^{-1}(0)$. We now claim that φ_i is of class C^1 (we will prove that $\varphi_1 \in C^1$, and by similar method, $\varphi_2 \in C^1$). To this end, set

$$\psi(u) = \int_{\mathbb{R}^N} f(x,u) u_+ dx.$$

It suffices to show that $\psi \in C^1$,

$$< \psi'(u), u_+ >= \int_{\mathbb{R}^N} [f_u(x,u) u_+^2 + f(x,u) u_+] dx, \ \forall u \in H_0^s(\mathbb{R}^{\mathbb{N}}).$$

First we prove the existence of the Gateaux derivative of ψ. Since $f(x,u)$ is continuously differentiable in u, one finds from the mean value theorem that there exists $0 < \theta < 1$ and $0 < |\mu| < 1$ such that

$$\frac{f(x,u(x) + \mu u_+(x)) - f(x,u(x))}{\mu u_+(x)} = f_u(x, u(x) + \theta \mu u_+(x)). \tag{2.13}$$

Using (1.3), (2.13) and the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
< \psi'(u), u_+ > &= \lim_{\mu \to 0^+} \frac{\psi(u + \mu u_+) - \psi(u)}{\mu} \\
&= \lim_{\mu \to 0^+} \int_{\mathbb{R}^N} \frac{f(x,u(x) + \mu u_+(x))(u_+(x) + \mu u_+(x)) - f(x,u(x)) u_+(x)}{\mu} dx \\
&= \lim_{\mu \to 0^+} \int_{\mathbb{R}^N} \Bigg[\frac{f(x,u(x) + \mu u_+(x)) - f(x,u(x))}{\mu} u_+(x) \\
&\qquad + f(x,u(x) + \mu u_+(x)) u_+(x) \Bigg] dx \\
&= \int_{\mathbb{R}^N} [f_u(x,u(x)) u_+^2(x) + f(x,u(x)) u_+(x)] dx.
\end{aligned}$$

Hence, we can easily get

$$
\begin{aligned}
<\varphi_1'(u),u_+> &= 2\iint_{\mathbb{R}^{2N}} \frac{|u_+(x)-u_+(y)|^2}{|x-y|^{N+2s}} - 2\gamma\int_{R^N}\frac{|u_+|^2}{|x|^{2s}} \\
&\quad -2_s^*(\alpha)\int_{\mathbb{R}^N}\frac{|u_+|^{2_s^*(\alpha)}}{|x|^\alpha} - \lambda\int_{\mathbb{R}^N}[f_u(x,u)u_+^2+f(x,u)u_+]dx.
\end{aligned}
$$

Next, we show that $\psi'(\cdot): H_0^s(\mathbb{R}^{\mathbb{N}}) \to \left(H_0^s(\mathbb{R}^{\mathbb{N}})\right)^*$ is continuous. Assume that $u_n \to u$ in $H_0^s(\mathbb{R}^{\mathbb{N}})$. From (1.3) and continuity f and f_u, one has

$$
\begin{aligned}
\|\psi'(u_n)-\psi'(u)\|_{H_0^s(\mathbb{R}^{\mathbb{N}})} &= \sup_{\|u_+\|\le 1}\int_{\mathbb{R}^N}\Big|(f_u(x,u_n(x))-f_u(x,u(x)))u_+^2(x) \\
&\quad +(f(x,u_n(x))-f(x,u(x)))u_+(x)dx\Big| \\
&= \sup_{\|u_+\|\le 1}\Big[\int_{\mathbb{R}^N}|f_u(x,u_n(x))-f_u(x,u(x))||u_+^2(x)| \\
&\quad +|f(x,u_n(x))-f(x,u(x))||u_+(x)|dx\Big] \to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Hence $\|\psi'(u_n)-\psi'(u)\|_{H_0^s(\mathbb{R}^{\mathbb{N}})} \to 0$. This shows that ψ' is continuous and so $\varphi_1 \in C^1$. Thus, we only need to prove that 0 is a regular value for φ_i. Note that $2 < 2_s^*(\alpha)$ and $\frac{1}{2_s^*(\alpha)-1} < \mu_1 < 1$. Then for $u \in M_1$, we conclude from (1.3) that

$$
\begin{aligned}
<\varphi_1'(u),u_+> &= 2\iint_{\mathbb{R}^{2N}}\frac{|u_+(x)-|u_+(y)|}{|x-y|^{N+2s}}(u_+(x)-u_+(y))\,dxdy - 2\gamma\int_{\mathbb{R}^N}\frac{|u_+(x)|}{|x|^{2s}}u_+(x)dx \\
&\quad -2_s^*(\alpha)\int_{\mathbb{R}^N}\frac{|u_+|^{2_s^*(\alpha)-2}}{|x|^\alpha}u_+(x)dx - \lambda\left(\int_{\mathbb{R}^N}f(x,u)u_+ + f_u(x,u)u_+^2\right)dx. \\
&\le 2\|u\|^2 - 2_s^*(\alpha)\int_{\mathbb{R}^N}\frac{|u_+|^{2_s^*(\alpha)-2}}{|x|^\alpha}u_+(x)dx - \lambda\left(\int_{\mathbb{R}^N}f(x,u)u_+ + f_u(x,u)u_+^2\right)dx \\
&\le 2\left(\|u\|^2 - \int_{\mathbb{R}^N}\frac{|u_+|^{2_s^*(\alpha)-2}}{|x|^\alpha}u_+(x)dx\right) - \lambda\left(\int_{\mathbb{R}^N}f(x,u)u_+ + f_u(x,u)u_+^2\right)dx \\
&\le 2\left(\lambda\int_{\mathbb{R}^N}f(x,u)u_+dx\right) - \lambda\int_{\mathbb{R}^N}f(x,u)u_+dx - \lambda\int_{\mathbb{R}^N}f_u(x,u)u_+^2dx \\
&\le \lambda\int_{\mathbb{R}^N}f(x,u)u_+dx - \lambda\int_{\mathbb{R}^N}f_u(x,u)u_+^2dx \\
&\le \lambda\mu_1\int_{\mathbb{R}^N}f_u(x,u)u_+^2dx - \lambda\int_{\mathbb{R}^N}f_u(x,u)u_+^2dx \\
&\le \lambda\mu_1\left(1-\frac{1}{\mu_1}\right)\int_{\mathbb{R}^N}f_u(x,u)u_+^2dx \\
&\le \lambda\mu_4\left(1-\frac{1}{\mu_1}\right)\int_{\mathbb{R}^N}g(x)|u_+|^l dx < 0.
\end{aligned}
$$

Therefore M_1 is a C^1 sub-manifold of $H_0^s(\mathbb{R}^N)$. By similar method, we have that M_2 is a C^1 sub-manifold of $H_0^s(\mathbb{R}^N)$. Also, for $u \in M_3$, we have $< \varphi_1'(u), u_- >=< \varphi_2'(u), u_+ >= 0$. Then M_3 is also a C^1 sub-manifold of $H_0^s(\mathbb{R}^N)$. The remainder of the proof can be proved by using Lemma 2.5 and Lemma 5 in [35]. So, we omit it here. □

Lemma 2.8. *The restricted functional $J|_{K_i}$ satisfies the $(PS)_c$ condition for every*

$$0 < c < \frac{2s-\alpha}{2(N-\alpha)} \left(S(N,s,\gamma,\alpha)\right)^{\frac{N-\alpha}{2s-\alpha}}.$$

Proof. From [34], we have that functional J satisfies the $(\mathrm{PS})_c$ condition for every

$$0 < c < \frac{2s-\alpha}{2(N-\alpha)} \left(S(N,s,\gamma,\alpha)\right)^{\frac{N-\alpha}{2s-\alpha}}.$$

Therefore, by similar method in the proof of Lemma 7 in [35], we have the desired conclusion. □

We note that if $u \in K_i$ be a critical point of the restricted functional $J|_{K_i}$, then we know from Lemma 2.8 that u is also a critical point of the unrestricted functional J, and so a weak solution to (1.1).

We now prove Theorem 1.1 by using Ekeland's variational principle [37].

Proof of Theorem 1.1. To prove Theorem 1.1, we need to check that functional $J|_{K_i}$ satisfies the hypotheses of Ekeland's variational principle [37]. The fact that J is bounded below over K_i is a direct consequence of the construction of the manifold K_i. Then, by Ekeland's variational principle, there exists $v_k \in K_i$ such that

$$J(v_k) \to c_i := \inf_{K_i} J(u), \quad J'|_{K_i}(v_k) \to 0.$$

We have to check that if we choose λ large, we have that $0 < c < \frac{2s-\alpha}{2(N-\alpha)} \left(S(N,s,\gamma,\alpha)\right)^{\frac{N-\alpha}{2s-\alpha}}$. This easily follows from Lemma 2.2. For instance, for μ_1, we choose $w_0 \geq 0$ such that

$$\mu_1 \leq J(t_\lambda w_0) \leq \frac{1}{2} t_\lambda^2 \|w_0\|^2.$$

Hence $\mu_1 \to 0$ as $\lambda \to 0$. Moreover, it follows from the estimate of t_λ in Lemma 2.2 that $0 < c < \frac{2s-\alpha}{2(N-\alpha)} \left(S(N,s,\gamma,\alpha)\right)^{\frac{N-\alpha}{2s-\alpha}}$ for $\lambda > \lambda*$. The other cases are similar. From Lemma 2.8, it follows that v_k has a convergent subsequence that we still call v_k. Therefore J has a critical point in K_i, $(i = 1,2,3)$. By construction, one finds that one of them is positive, other is negative and the last one changes sign. □

Acknowledgements

The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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