J. Nonlinear Funct. Anal. 2019 (2019), Article ID 2 https://doi.org/10.23952/jnfa.2019.2

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

ON THE k-FRACTIONAL INTEGRAL INEQUALITIES THROUGH THE GENERALIZED s-(α,m)-PREINVEXITY

CHUNYAN LUO[1], TINGSONG DU[1,2,*], MUHAMMAD UZAIR AWAN[3], MARCELA V. MIHAI[4]

[1]Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, China

[2]Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China

[3]Department of Mathematics, Government College University, Faisalabad, Pakistan

[4]Department Scientific-Methodical Sessions, Romanian Mathematical Society-Branch Bucharest, Academy Street No. 14, RO-010014, Bucharest, Romania

Abstract. In this paper, we establish new multi-parameterized integral inequalities for mappings with absolute values of the first derivatives which are generalized s-(α,m)-preinvex, via k-fractional integrations. We also prove Hadamard-type inequalities involving products of two generalized s-(α,m)-preinvex functions in the second sense.

Keywords. Hermite-Hadamard's inequality; Integral inequality; Generalized s-(α,m)-preinvex function; k-fractional integrals.

2010 Mathematics Subject Classification. 26A33, 26D15.

1. INTRODUCTION

In this paper, we use I to denote an interval on the real line $R=(-\infty,\infty)$. Let a and b be real numbers in I such that $a\neq b$. If $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping defined on I, then we have the Hadamard-type inequality grips:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

If $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable mapping on (a,b), then we have the succeeding Simpson's inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\leq\frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^4 \tag{1.2}$$

where $\|f^{(4)}\|_\infty=\sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$.

---

*Corresponding author.

E-mail addresses: luochunyanctgu@gmail.com (C. Luo), tingsongdu@ctgu.edu.cn (T. Du), awan.uzair@gmail.com (M.U. Awan), marcelamihai58@yahoo.com (M.V. Mihai).

Received September 12, 2018; Accepted January 9, 2019.

For recent results with respect to (1.1) and (1.2), we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein.

In 2013, Sarikaya *et al.* [11] established the subsequent Hadamard-type inequality by utilizing Riemann-Liouville fractional integrals.

Theorem 1.1. [11] *Let $f:[a,b]\to\mathbb{R}$ be a positive function along with $0\le a<b$ and let $f\in L^1([a,b])$. If f is a convex function on $[a,b]$, then the forthcoming inequalities for fractional integrals clasp:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[J^{\mu}_{a^+}f(b)+J^{\mu}_{b^-}f(a)\right]\le\frac{f(a)+f(b)}{2},\tag{1.3}$$

where $J^{\mu}_{a^+}f$ and $J^{\mu}_{b^-}f$ denote, respectively, the left-sided and right-sided Riemann-Liouville fractional integrals of order $\mu>0$ defined by

$$J^{\mu}_{a^+}f(x)=\frac{1}{\Gamma(\mu)}\int_a^x(x-t)^{\mu-1}f(t)\mathrm{d}t,\ \ a<x$$

and

$$J^{\mu}_{b^-}f(x)=\frac{1}{\Gamma(\mu)}\int_x^b(t-x)^{\mu-1}f(t)\mathrm{d}t,\ \ x<b,$$

where $\Gamma(\mu)$ is the gamma function defined by $\Gamma(\mu)=\int_0^{\infty}e^{-t}t^{\mu-1}\mathrm{d}t$. $J^{0}_{a^+}f(x)=J^{0}_{b^-}f(x)=f(x)$.

In 2016, Sarikaya and Yildirim [12] obtained another presentation of the Riemann-Liouville fractional Hadamard-type inequality as follows.

Theorem 1.2. [12] *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L^1([a,b])$. If f is a convex mapping on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\left[J^{\mu}_{(\frac{a+b}{2})^+}f(b)+J^{\mu}_{(\frac{a+b}{2})^-}f(a)\right]\le\frac{f(a)+f(b)}{2}\tag{1.4}$$

with $\mu>0$.

In view of the wide applications of the Riemann-Liouville fractional integrals, many authors studied problems which involves this integral operator; see, for example, [13, 14, 15, 16, 17] and the references therein. In [18], Mubeen and Habibullah introduced the following class of fractional integrals.

Definition 1.3. [18] Let $f\in L^1([a,b])$. k-fractional integrals ${}_kJ^{\mu}_{a^+}f(x)$ and ${}_kJ^{\mu}_{b^-}f(x)$ of order $\mu>0$ are defined as

$${}_kJ^{\mu}_{a^+}f(x)=\frac{1}{k\Gamma_k(\mu)}\int_a^x(x-t)^{\frac{\mu}{k}-1}f(t)\mathrm{d}t,\ \ (0\le a<x<b)$$

and

$${}_kJ^{\mu}_{b^-}f(x)=\frac{1}{k\Gamma_k(\mu)}\int_x^b(t-x)^{\frac{\mu}{k}-1}f(t)\mathrm{d}t,\ \ (0\le a<x<b)$$

respectively, where $k>0$ and $\Gamma_k(\mu)$ is the k-gamma function given as $\Gamma_k(\mu)=\int_0^{\infty}t^{\mu-1}e^{-\frac{t^k}{k}}\mathrm{d}t$ with the properties $\Gamma_k(\mu+k)=\mu\Gamma_k(\mu)$ and $\Gamma_k(k)=1$. Note that ${}_kJ^{0}_{a^+}f(x)={}_kJ^{0}_{b^-}f(x)=f(x)$.

If $k=1$, then the k-fractional integrals are reduced to the Riemann-Liouville fractional integrals. For recent results on the k-fractional integral inequalities; see [17, 19, 20] and the references therein.

In 2016, Farid Rehman and Zahra [21] extended Theorem 1.2 to the following form of k-fractional integrals.

Theorem 1.4. [21] *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L^1([a,b])$. If f is a convex mapping on $[a,b]$, then, for $\mu,k>0$, the successive inequalities for k-fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big)\le\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\Big]\le\frac{f(a)+f(b)}{2}. \tag{1.5}$$

The paper aims to establish, based on the k-fractional calculus, some new Hadamard-type and Simpson-type inequalities with multi-parameters via a general k-fractional integral identity. We investigate the functions which have absolute values of the first derivatives which are generalized s-(α,m)-preinvex. We also prove Hadamard-type inequalities for products of two generalized s-(α,m)-preinvex functions in the second sense.

Next, we end this section by recalling some special functions and definitions.

(1) The beta function:

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad x,y>0.$$

(2) The hypergeometric function:

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t,\ \ c>b>0,|z|<1.$$

Definition 1.5. [22] A set $K\subseteq\mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta:K\times K\times(0,1]\to\mathbb{R}^n$ for some fixed $m\in(0,1]$, if $mx+\lambda\eta(y,x,m)\in K$ holds for each $x,y\in K$ and any $\lambda\in[0,1]$.

Definition 1.6. [23] A function $f:[0,\infty)\to[0,\infty)$ is said to be s-(α,m)-convex in the first sense if, for all $x,y\in[0,\infty)$ and $t\in[0,1]$, the following inequality holds for some fixed $s\in(0,1]$:

$$f\big(tx+(1-t)y\big)\le t^{\alpha s}f(x)+m(1-t^{\alpha s})f\Big(\frac{y}{m}\Big), \tag{1.6}$$

where $(\alpha,m)\in(0,1]\times(0,1]$.

Definition 1.7. [23] A function $f:[0,\infty)\to[0,\infty)$ is said to be s-(α,m)-convex in the second sense if, for all $x,y\in[0,\infty)$ and $t\in[0,1]$, the following inequality holds for some fixed $s\in(0,1]$:

$$f\big(tx+(1-t)y\big)\le t^{\alpha s}f(x)+m(1-t^{\alpha})^s f\Big(\frac{y}{m}\Big), \tag{1.7}$$

where $(\alpha,m)\in(0,1]\times(0,1]$.

Definition 1.8. [24] Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$. For every $x,y\in K$, the η-path P_{xv} joining the points x and $v=x+\eta(y,x)$ is defined as

$$P_{xv}=\Big\{z|z=x+t\eta(y,x),t\in[0,1]\Big\}. \tag{1.8}$$

Definition 1.9. [25] Let $K\subseteq\mathbb{R}^n$ be an m-invex set with respect to $\eta:K\times K\times(0,1]\to\mathbb{R}^n$. For every $u,v\in K$ and $m\in(0,1]$, the η_m-path P_{vw} joining the points mv and $w=mv+\eta(u,v,m)$ is defined by

$$P_{vw}=\Big\{z|z=mv+t\eta(u,v,m),t\in[0,1]\Big\}. \tag{1.9}$$

Remark 1.10. If $m=1$ in mapping $\eta(u,v,m)$, then Definition 1.9 is reduced to Definition 1.8.

2. Preliminaries

First, we give the following definitions.

Definition 2.1. Let $A \subseteq \mathbb{R}^n$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \to \mathbb{R}^n$. For some fixed $(\alpha, m) \in (0,1] \times (0,1]$ and $s \in (0,1]$, $f : A \to [0,\infty)$ is said to be generalized s-(α,m)-preinvex in the first sense if

$$f\big(mx + t\eta(y,x,m)\big) \leq m(1-t^{\alpha s})f(x) + t^{\alpha s} f(y)$$

is valid for all $x, y \in A$ and $t \in [0,1]$.

Definition 2.2. Let $A \subseteq \mathbb{R}^n$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \to \mathbb{R}^n$. For some fixed $(\alpha, m) \in (0,1] \times (0,1]$ and $s \in (0,1]$, $f : A \to [0,\infty)$ is said to be generalized s-(α,m)-preinvex in the second sense if

$$f\big(mx + t\eta(y,x,m)\big) \leq m(1-t^{\alpha})^s f(x) + t^{\alpha s} f(y),$$

is valid for all $x, y \in A$ and $t \in [0,1]$.

For one dimension, if we put $s = 1$ in Definition 2.1, then we get the definition of the generalized (α,m)-preinvex function in [21]. Similarly, if we put $\alpha = 1$ in Definition 2.2, then we have the definition of the generalized (s,m)-preinvex function in [22]. Hence, Definitions 2.1 and 2.2 are extensions of preinvex functions.

Throughout this paper, let $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \to \mathbb{R}\backslash\{0\}$ for some fixed $m \in (0,1]$ and let $a, b \in A$ with $a < b$. Assume that $f : A \to \mathbb{R}$ is a differentiable mapping and f' is intergreble on the η_m-path $P_{vw} : w = mv + \eta(u,v,m)$ for arbitrary $u, v \in [a,b]$. Before stating our main results, we give the following notations:

$$\begin{aligned}
I_{f,\eta}(\mu,k;x,\lambda,m,a,b) :=& \frac{1-\lambda}{\eta(b,a,m)}\Big[\eta^{\frac{\mu}{k}}(x,a,m) f\big(ma + \eta(x,a,m)\big) \\
&+ (-1)^{\frac{\mu}{k}}\eta^{\frac{\mu}{k}}(x,b,m) f\big(mb + \eta(x,b,m)\big)\Big] \\
&+ \frac{\lambda}{\eta(b,a,m)}\Big[\eta^{\frac{\mu}{k}}(x,a,m) f(ma) + (-1)^{\frac{\mu}{k}}\eta^{\frac{\mu}{k}}(x,b,m) f(mb)\Big] \\
&- \frac{\Gamma_k(\mu+k)}{\eta(b,a,m)}\Big[{}_kJ^{\mu}_{(ma+\eta(x,a,m))^-} f(ma) + {}_kJ^{\mu}_{(mb+\eta(x,b,m))^+} f(mb)\Big].
\end{aligned}$$

Next, we give some important lemmas for our main results.

Lemma 2.3. *The following identity for k-fractional integrals along with $x \in (a,b)$, $\lambda \in [0,1]$, $\mu > 0$ and $k > 0$ exists*

$$\begin{aligned}
I_{f,\eta}(\mu,k;x,\lambda,m,a,b) =& \frac{(-1)^{\frac{\mu}{k}+1}\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)} \int_0^1 \big(\lambda - t^{\frac{\mu}{k}}\big) f'\big(mb + t\eta(x,b,m)\big)\mathrm{d}t \\
&- \frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{\eta(b,a,m)} \int_0^1 \big(\lambda - t^{\frac{\mu}{k}}\big) f'\big(ma + t\eta(x,a,m)\big)\mathrm{d}t.
\end{aligned} \tag{2.1}$$

Proof. By integration by parts and changing the variable, we can state

$$\begin{aligned}
&\int_0^1 (\lambda - t^{\frac{\mu}{k}}) f'\big(mb + t\eta(x,b,m)\big)\mathrm{d}t \\
&= \Big[(\lambda - t^{\frac{\mu}{k}}) \frac{1}{\eta(x,b,m)} f\big(mb + t\eta(x,b,m)\big)\Big|_0^1 \\
&\quad + \int_0^1 \Big(\frac{\mu}{k} t^{\frac{\mu}{k}-1}\Big) \frac{1}{\eta(x,b,m)} f\big(mb + t\eta(x,b,m)\big)\mathrm{d}t\Big] \\
&= \frac{1}{\eta(x,b,m)} \Big[(\lambda - 1) f\big(mb + \eta(x,b,m)\big) - \lambda f(mb) \\
&\quad + \int_0^1 \frac{\mu}{k} t^{\frac{\mu}{k}-1} f\big(mb + t\eta(x,b,m)\big)\mathrm{d}t\Big] \\
&= \frac{1}{\eta(x,b,m)} \Big[(\lambda - 1) f\big(mb + \eta(x,b,m)\big) - \lambda f(mb)\Big] \\
&\quad + \frac{\frac{\mu}{k}}{\eta(b,a,m)} \int_{mb}^{mb+\eta(x,b,m)} \left(\frac{u - mb}{\eta(x,b,m)}\right)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u \\
&= \frac{1}{\eta(x,b,m)} \Big[(\lambda - 1) f\big(mb + \eta(x,b,m)\big) - \lambda f(mb)\Big] \\
&\quad + (-1)^{\frac{\mu}{k}} \frac{\Gamma_k(\mu + k)}{\eta(b,a,m)\eta^{\frac{\mu}{k}}(x,b,m)} {}_kJ^{\mu}_{(mb+\eta(x,b,m))^+} f(mb).
\end{aligned} \tag{2.2}$$

Similarly, one has

$$\begin{aligned}
&\int_0^1 (\lambda - t^{\frac{\mu}{k}}) f'\big(ma + t\eta(x,a,m)\big)\mathrm{d}t \\
&= \frac{1}{\eta(x,a,m)} \Big[(\lambda - 1) f\big(ma + \eta(x,a,m)\big) - \lambda f(ma)\Big] \\
&\quad + \frac{\Gamma_k(\mu + k)}{\eta(b,a,m)\eta^{\frac{\mu}{k}}(x,a,m)} {}_kJ^{\mu}_{(ma+\eta(x,a,m))^-} f(ma).
\end{aligned} \tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $\frac{\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)}$ and $\frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{\eta(b,a,m)}$, respectively, and adding the resulting identities, we get the desired result immediately. □

Remark 2.4. (a) In Lemma 2.3, if $k = \mu = 1$ and $\eta(b,a,m) = b - ma$ with $m = 1$, then we get [26, Lemma 2.1].
(b) In Lemma 2.3, if $\lambda = 0$, $k = \mu = 1$ and $\eta(b,a,m) = b - ma$ with $m = 1$, then we have [27, Lemma 1].
(c) In Lemma 2.3, if $\lambda = 1 = k$, then we get [28, Lemma 4.1].
In addition, we also have the following:
(ci) for $\mu = 1$, we have [28, Lemma 3.1],
(cii) for $\eta(b,a,m) = b - ma$ with $m = 1$, we obtain [29, Lemma 2],
(ciii) for $\eta(b,a,m) = b - ma$ with $m = 1 = \mu$, we get [30, Lemma 1].

Lemma 2.5. *For $t \in [0,1]$, we have $(1 - t^{\alpha})^s \leq 2^{1-s} - t^{s\alpha}$ for some fixed $\alpha \in (0,1]$ and $s \in (0,1]$.*

Proof. Let $f(t) = t^{\alpha s} + (1 - t^{\alpha})^s$ for $t \in [0,1]$ with $\alpha, s \in (0,1]$. Obviously, $f(\cdot)$ is increasing on $\big[0, 2^{-\frac{1}{\alpha}}\big]$ and decreasing on $\big[2^{-\frac{1}{\alpha}}, 1\big]$. So, $f(t) \leq f\big(2^{-\frac{1}{\alpha}}\big) = 2^{1-s}$ for $\forall\, t \in [0,1]$. This ends the proof. □

3. MAIN RESULTS

Using Lemma 2.3, we now state the forthcoming theorem.

Theorem 3.1. *If $|f'|^q$ for $q \geq 1$ is generalized s-(α,m)-preinvex in the second sense with $(\alpha,s) \in (0,1] \times (0,1]$, then for $x \in (a,b)$, $\lambda \in [0,1]$, $\mu > 0$ and $k > 0$, the following inequality for k-fractional integral holds:*

$$\begin{aligned} \left|I_{f,\eta}(\mu,k;x,\lambda,m,a,b)\right| &\leq \left(\frac{2\mu}{\mu+k}\lambda^{\frac{\mu}{k}+1} - \lambda + \frac{k}{\mu+k}\right)^{1-\frac{1}{q}} \\ &\times \left\{ \left|\frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{\eta(b,a,m)}\right| \left[C_1(\mu,\alpha,k,s,\lambda)|f'(x)|^q + mC_2(\mu,\alpha,k,s,\lambda)|f'(a)|^q\right]^{\frac{1}{q}} \right. \\ &\left. + \left|\frac{\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)}\right| \left[C_1(\mu,\alpha,k,s,\lambda)|f'(x)|^q + mC_2(\mu,\alpha,k,s,\lambda)|f'(b)|^q\right]^{\frac{1}{q}} \right\}, \end{aligned} \tag{3.1}$$

where

$$C_1(\mu,\alpha,k,s,\lambda) = \frac{k}{\mu+ks\alpha+k}\left(1-2\lambda^{\frac{k}{\mu}(s\alpha+1)+1}\right) + \frac{\lambda}{s\alpha+1}\left(2\lambda^{\frac{k}{\mu}(s\alpha+1)}-1\right)$$

and

$$C_2(\mu,\alpha,k,s,\lambda) = \begin{cases} \frac{1}{\alpha}\beta(\frac{k+\mu}{k\alpha},s+1), & \lambda = 0, \\ 2^{1-s}\left(\frac{2\mu}{\mu+k}\lambda^{\frac{k}{\mu}+1} - \lambda + \frac{k}{\mu+k}\right) \\ \quad -\frac{k}{\mu+ks\alpha+k}\left(1-2\lambda^{\frac{k}{\mu}(\alpha s+1)+1}\right) \\ \quad -\frac{\lambda}{s\alpha+1}\left(2\lambda^{\frac{k}{\mu}(\alpha s+1)}-1\right), & 0<\lambda<1, \\ \frac{1}{\alpha}\left[\beta(\frac{1}{\alpha},s+1) - \beta(\frac{k+\mu}{k\alpha},s+1)\right], & \lambda = 1. \end{cases}$$

Here, we denote

$$I_1 = \int_0^1 \left|f'\left(mb+t\eta(x,b,m)\right)\right|^q \left|\lambda - t^{\frac{\mu}{k}}\right| \mathrm{d}t$$

and

$$I_2 = \int_0^1 \left|f'\left(ma+t\eta(x,a,m)\right)\right|^q \left|\lambda - t^{\frac{\mu}{k}}\right| \mathrm{d}t.$$

Proof. Using Lemma 2.3, the s-(α,m)-preinvexity of $|f'|^q$ and the power mean inequality, we have

$$\begin{aligned} &\left|I_{f,\eta}(\mu,k;x,\lambda,m,a,b)\right| \\ &\leq \left(\int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| \mathrm{d}t\right)^{1-\frac{1}{q}} \left[\left|\frac{\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)}\right| (I_1)^{\frac{1}{q}} + \left|\frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{\eta(b,a,m)}\right| (I_2)^{\frac{1}{q}}\right], \end{aligned}$$

where

$$\int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| \mathrm{d}t = \frac{2\mu}{\mu+k}\lambda^{\frac{k}{\mu}+1} - \lambda + \frac{k}{\mu+k}$$

and

$$I_1 \le |f'(x)|^q \int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| (t^\alpha)^s \mathrm{d}t + m|f'(b)|^q \int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| (1-t^\alpha)^s \mathrm{d}t.$$

Note that

$$\int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| (t^\alpha)^s \mathrm{d}t = \frac{k}{\mu + ks\alpha + k}\Big(1 - 2\lambda^{\frac{k}{\mu}(s\alpha+1)+1}\Big) + \frac{\lambda}{s\alpha+1}\Big(2\lambda^{\frac{k}{\mu}(s\alpha+1)} - 1\Big).$$

If $\lambda = 0$, then

$$\int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| (1-t^\alpha)^s \mathrm{d}t = \frac{1}{\alpha}\beta\Big(\frac{k+\mu}{k\alpha}, s+1\Big).$$

If $\lambda = 1$, then

$$\int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| (1-t^\alpha)^s \mathrm{d}t = \frac{1}{\alpha}\Big[\beta\Big(\frac{1}{\alpha}, s+1\Big) - \beta\Big(\frac{k+\mu}{k\alpha}, s+1\Big)\Big].$$

If $0 < \lambda < 1$, we find from Lemma 2.5 that

$$\begin{aligned}
&\int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| (1-t^\alpha)^s \mathrm{d}t \le \int_0^1 \left|\lambda - t^{\frac{\mu}{k}}\right| \big(2^{1-s} - t^{s\alpha}\big)\mathrm{d}t \\
&= 2^{1-s}\Big(\frac{2\mu}{\mu+k}\lambda^{\frac{k}{\mu}+1} - \lambda + \frac{k}{\mu+k}\Big) \\
&\quad - \frac{k}{\mu + ks\alpha + k}\Big(1 - 2\lambda^{\frac{k}{\mu}(s\alpha+1)+1}\Big) - \frac{\lambda}{s\alpha+1}\Big(2\lambda^{\frac{k}{\mu}(s\alpha+1)} - 1\Big).
\end{aligned}$$

Similarly, one has

$$I_2 \le C_1(\mu,\alpha,k,s,\lambda)|f'(x)|^q + mC_2(\mu,\alpha,k,s,\lambda)\left|f'(a)\right|^q.$$

This completes the proof of Theorem 3.1. □

As a special case Theorem 3.1, we give the following result.

Corollary 3.2. *In Theorem 3.1,*
(a) if $s = 1$, then

$$\begin{aligned}
&\left|I_{f,\eta}(\mu,k;x,\lambda,m,a,b)\right| \\
&\le \Big(\frac{2\mu}{\mu+k}\lambda^{\frac{\mu}{k}+1} - \lambda + \frac{k}{\mu+k}\Big)^{1-\frac{1}{q}} \\
&\quad \times \Bigg\{\left|\frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{\eta(b,a,m)}\right|\Big[C_1(\mu,\alpha,k,1,\lambda)\left|f'(x)\right|^q + mC_2(\mu,\alpha,k,1,\lambda)\left|f'(a)\right|^q\Big]^{\frac{1}{q}} \\
&\quad + \left|\frac{\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)}\right|\Big[C_1(\mu,\alpha,k,1,\lambda)\left|f'(x)\right|^q + mC_2(\mu,\alpha,k,1,\lambda)\left|f'(b)\right|^q\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

(b) if $x=\frac{a+b}{2}$, $\lambda=\frac{1}{3}$, $s=1$ *and* $\eta(b,a,m)=b-ma$ *with* $m=1$, *then*

$$\begin{aligned}
&\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1} I_{f,\eta}\left(\mu,k;\frac{a+b}{2},\frac{1}{3},1,a,b\right)\right| \\
&=\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right. \\
&\quad\left.-\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+1)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)+{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)\right]\right| \\
&\le\left(\frac{2\mu}{(\mu+k)3^{\frac{\mu}{k}+1}}+\frac{2k-\mu}{3(\mu+k)}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{4}\right) \\
&\quad\times\left\{\left[C_1\left(\mu,\alpha,k,1,\frac{1}{3}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+C_2\left(\mu,\alpha,k,1,\frac{1}{3}\right)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[C_1\left(\mu,\alpha,k,1,\frac{1}{3}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+C_2\left(\mu,\alpha,k,1,\frac{1}{3}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

In particular, if $\mu=\alpha=k=1$, *then we get the following Simpson type inequality*

$$\begin{aligned}
&\left|I_{f,\eta}\left(1,1;\frac{a+b}{2},\frac{1}{3},1,a,b\right)\right| \\
&=\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{4}\right)\left\{\left[\frac{8}{81}\left|f'(a)\right|^q+\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{8}{81}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

(c) if $x=\frac{a+b}{2}$, $\lambda=\frac{1}{2}$, $s=1$ *and* $\eta(b,a,m)=b-ma$ *with* $m=1$, *then*

$$\begin{aligned}
&\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1} I_{f,\eta}\left(\mu,k;\frac{a+b}{2},\frac{1}{2},1,a,b\right)\right| \\
&=\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right. \\
&\quad\left.-\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)+{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)\right]\right| \\
&\le\left(\frac{2\mu}{\mu+k}\left(\frac{1}{2}\right)^{\frac{\mu}{k}+1}+\frac{k-\mu}{2(\mu+k)}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{4}\right) \\
&\quad\times\left\{\left[C_1\left(\mu,\alpha,k,1,\frac{1}{2}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+C_2\left(\mu,\alpha,k,1,\frac{1}{2}\right)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[C_1\left(\mu,\alpha,k,1,\frac{1}{2}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+C_2\left(\mu,\alpha,k,1,\frac{1}{2}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

In particular, if $\mu=\alpha=1=k$, then we have the following averaged midpoint-trapezoid type inequality

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\leq\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right)\left\{\left[\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 3.3. Put $\mu=\alpha=1=k$ and $\eta(b,a,m)=b-ma$ with $m=1$ in (a) of Corollary 3.2.
(i) If $\lambda=1$, then we have Theorem 7 in [30].
(ii) if $\lambda=0$ or $\lambda=1$ with $x=\frac{a+b}{2}$, then we can get the same inequality as (2.3) and (2.4) of Corollary 2.1 in [26], respectively.

If $|f'|^q$ is generalized s-(α,m)-preinvex, then we obtain the following result.

Theorem 3.4. *If $|f'|^q$ for $q>1$ is generalized s-(α,m)-preinvex in the second sense with $(\alpha,s)\in(0,1]\times(0,1]$ and $p^{-1}+q^{-1}=1$, then, for $x\in[a,b]$, $\lambda\in[0,1]$, $0<\mu$ and $k>0$, the following inequality for k-fractional integral holds:*

$$\begin{aligned}\left|I_{f,\eta}(\mu,k;x,\lambda,m,a,b)\right|&\leq\frac{(\phi(\mu,k;\lambda,p))^{\frac{1}{p}}}{|\eta(b,a,m)|}\\&\times\left\{\left|\eta^{\frac{\mu}{k}+1}(x,b,m)\right|\left[\frac{1}{s\alpha+1}|f'(x)|^q+m|f'(b)|^q\left(2^{1-s}-\frac{1}{s\alpha+1}\right)\right]^{\frac{1}{q}}\right.\\&\left.+\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|\left[\frac{1}{s\alpha+1}|f'(x)|^q+m|f'(a)|^q\left(2^{1-s}-\frac{1}{s\alpha+1}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}\tag{3.2}$$

where

$$\phi(\mu,k;\lambda,p)=\begin{cases}\frac{k}{\mu p+k}, & \lambda=0,\\ \lambda^p\left(2\lambda^{\frac{k}{\mu}}-1\right)+\frac{k}{\mu p+k}\left(1-2\lambda^{p+\frac{k}{\mu}}\right), & 0<\lambda<1,\\ \frac{k}{\mu}\beta\left(\frac{k}{\mu},p+1\right), & \lambda=1.\end{cases}$$

Proof. Using Lemma 2.3 and Hölder's inequality, we have

$$\begin{aligned}\left|I_{f,\eta}(\mu,k;x,\lambda,m,a,b)\right|&\leq\left|\frac{\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)}\right|\left(\int_0^1|\lambda-t^{\frac{\mu}{k}}|^p\mathrm{d}t\right)^{\frac{1}{p}}[J_1]^{\frac{1}{q}}\\&+\left|\frac{\eta^{\frac{\mu}{k}+1}(x,b,m)}{\eta(b,a,m)}\right|\left(\int_0^1|\lambda-t^{\frac{\mu}{k}}|^p\mathrm{d}t\right)^{\frac{1}{p}}[J_2]^{\frac{1}{q}}.\end{aligned}$$

If $\lambda=0$, then

$$\int_0^1\left|\lambda-t^{\frac{\mu}{k}}\right|^p\mathrm{d}t=\frac{k}{\mu p+k}.$$

If $\lambda=1$, then

$$\int_0^1\left|\lambda-t^{\frac{\mu}{k}}\right|^p\mathrm{d}t=\frac{k}{\mu}\beta\left(\frac{k}{\mu},p+1\right).$$

If $0<\lambda<1$, then

$$\begin{aligned}\int_0^1\left|\lambda-t^{\frac{\mu}{k}}\right|^p\mathrm{d}t&=\int_0^{\lambda^{\frac{k}{\mu}}}\left(\lambda-t^{\frac{\mu}{k}}\right)^p\mathrm{d}t+\int_{\lambda^{\frac{k}{\mu}}}^1\left(t^{\frac{\mu}{k}}-\lambda\right)^p\mathrm{d}t\\&\leq\int_0^{\lambda^{\frac{k}{\mu}}}\left(\lambda^p-t^{\frac{\mu p}{k}}\right)\mathrm{d}t+\int_{\lambda^{\frac{k}{\mu}}}^1\left(t^{\frac{\mu p}{k}}-\lambda^p\right)\mathrm{d}t\\&=\lambda^p\left(2\lambda^{\frac{k}{\mu}}-1\right)+\frac{k}{\mu p+k}\left(1-2\lambda^{p+\frac{k}{\mu}}\right).\end{aligned}$$

Here we use

$$(\lambda-t^{\frac{\mu}{k}})^p\leq\lambda^p-t^{\frac{\mu p}{k}}$$

for $t\in[0,\lambda^{\frac{k}{\mu}}]$, and

$$(t^{\frac{\mu}{k}}-\lambda)^p\leq t^{\frac{\mu p}{k}}-\lambda^p$$

for $t\in[\lambda^{\frac{k}{\mu}},1]$ which follow from $(A-B)^p\leq A^p-B^p$ for any $A\geq B\geq0$ and $p\geq1$. Using the generalized s-(α,m)-preinvexity of $|f'|^q$ and Lemma 2.5, one arrives at

$$\begin{aligned}J_1&=\int_0^1\left|f'\left(mb+t\eta(x,b,m)\right)\right|^q\mathrm{d}t\\&\leq\int_0^1\left[(t^\alpha)^s|f'(x)|^q+m|f'(b)|^q(1-t^\alpha)^s\right]\mathrm{d}t\\&\leq\frac{1}{s\alpha+1}|f'(x)|^q+m|f'(b)|^q\left(2^{1-s}-\frac{1}{s\alpha+1}\right)\end{aligned}$$

and

$$\begin{aligned}J_2&=\int_0^1\left|f'\left(ma+t\eta(x,a,m)\right)\right|^q\mathrm{d}t\\&\leq\frac{1}{s\alpha+1}|f'(x)|^q+m|f'(a)|^q\left(2^{1-s}-\frac{1}{s\alpha+1}\right).\end{aligned}$$

This completes the proof. □

Next, we give a special case of Theorem 3.4.

Corollary 3.5. *In Theorem 3.4,*

(a) if $x=\frac{a+b}{2}$, $\lambda=0$, $\alpha=1$ and $\eta(b,a,m)=b-ma$ with $m=1$, then

$$\begin{aligned}&\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1}I_{f,\eta}\left(\mu,k;\frac{a+b}{2},0,1,a,b\right)\right|\\&\leq\left(\frac{k}{\mu p+k}\right)^{\frac{1}{p}}\left(\frac{b-a}{4}\right)\left\{\left[\frac{1}{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(2^{1-s}-\frac{1}{s+1}\right)|f'(b)|^q\right]^{\frac{1}{q}}\right.\\&\quad\left.+\left[\frac{1}{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(2^{1-s}-\frac{1}{s+1}\right)|f'(a)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

In particular, if $\mu = k = 1 = s$, then we get the following Hadamard-type inequality

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|$$
$$\le \frac{b-a}{4}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

(b) if $x = \frac{a+b}{2}$, $\lambda = 1$, $\alpha = 1$ and $\eta(b,a,m) = b - ma$ with $m = 1$, then

$$\left|\Big(\frac{2}{b-a}\Big)^{\frac{\mu}{k}-1} I_{f,\eta}\Big(\mu,k;\frac{a+b}{2},1,1,a,b\Big)\right|$$
$$\le \left(\frac{k}{\mu}\beta\Big(\frac{k}{\mu},p+1\Big)\right)^{\frac{1}{p}}\Big(\frac{b-a}{4}\Big)\left\{\left[\frac{1}{s+1}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\Big(2^{1-s}-\frac{1}{s+1}\Big)|f'(b)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{s+1}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\Big(2^{1-s}-\frac{1}{s+1}\Big)|f'(a)|^q\right]^{\frac{1}{q}}\right\}.$$

In particular, $\mu = k = 1 = s$, then we get the following Hadamard-type inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\le \Big(\frac{b-a}{4}\Big)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

(c) if $x = \frac{a+b}{2}$, $\lambda = \frac{1}{3}$, $\alpha = 1$ and $\eta(b,a,m) = b - ma$ with $m = 1$, then

$$\left|\Big(\frac{2}{b-a}\Big)^{\frac{\mu}{k}-1} I_{f,\eta}\Big(\mu,k;\frac{a+b}{2},\frac{1}{3},1,a,b\Big)\right|$$
$$\le \left(\phi\Big(\mu,k;\frac{1}{3},p\Big)\right)^{\frac{1}{p}}\Big(\frac{b-a}{4}\Big)\left\{\left[\Big(\frac{1}{s+1}\Big)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\Big(2^{1-s}-\frac{1}{s+1}\Big)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\Big(\frac{1}{s+1}\Big)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\Big(2^{1-s}-\frac{1}{s+1}\Big)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

In particular, if $\mu = k = 1 = s$, then we get the following Simpson type inequality

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \left(\frac{3^{p+1}-p-3}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\frac{b-a}{4}\left\{\left[\frac{1}{2}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\frac{1}{2}\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{2}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\frac{1}{2}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

(d) if $x=\frac{a+b}{2}$, $\lambda=\frac{1}{2}$, $\alpha=1$ *and* $\eta(b,a,m)=b-ma$ *with* $m=1$, *then*

$$\begin{aligned}&\Big|\Big(\frac{2}{b-a}\Big)^{\frac{\mu}{k}-1}I_{f,\eta}\Big(\mu,k;\frac{a+b}{2},\frac{1}{2},1,a,b\Big)\Big|\\&\le\Big(\phi\Big(\mu,k;\frac{1}{2},p\Big)\Big)^{\frac{1}{p}}\Big(\frac{b-a}{4}\Big)\Bigg\{\Bigg[\Big(\frac{1}{s+1}\Big)\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q+\Big(2^{1-s}-\frac{1}{s+1}\Big)\big|f'(a)\big|^q\Bigg]^{\frac{1}{q}}\\&\quad+\Bigg[\Big(\frac{1}{s+1}\Big)\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q+\Big(2^{1-s}-\frac{1}{s+1}\Big)\big|f'(b)\big|^q\Bigg]^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

In particular, if $\mu=k=1=s$, *then we have the following averaged midpoint-trapezoid type inequality*

$$\begin{aligned}&\Big|\frac{1}{4}\Big[f(a)+2f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big|\\&\le\Big(\frac{2^p-1}{2^p(p+1)}\Big)^{\frac{1}{p}}\frac{b-a}{4}\Bigg\{\Bigg[\frac{1}{2}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{1}{2}\big|f'(a)\big|^q\Bigg]^{\frac{1}{q}}+\Bigg[\frac{1}{2}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{1}{2}\big|f'(b)\big|^q\Bigg]^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

We next establish k-fractional integral inequality related to products of two generalized s-(α,m)-preinvex functions.

Theorem 3.6. *If* f *is* s-(α_1,m)*-preinvex in the second sense,* g *is* s-(α_2,m)*-preinvex in the second sense for* $(\alpha_i,s)\in(0,1]\times(0,1]$ *and* $i=1,2$, *then the following inequality holds:*

$$\begin{aligned}&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}\Big[{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^+}(fg)\big(ma+\eta(b,a,m)\big)+{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^-}(fg)(ma)\Big]\\&\le\Phi_1m^2f(a)g(a)+\Phi_2mf(a)g(b)+\Phi_3mf(b)g(a)+\Phi_4f(b)g(b),\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}\Phi_1&=2^{2-2s}-2^{-s}{}_2F_1\Big[-\alpha_1s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]-2^{-s}{}_2F_1\Big[-\alpha_2s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]\\&\quad+2^{-(\alpha_1+\alpha_2)s-1}\frac{\mu}{\mu+(\alpha_1+\alpha_2)ks}+2^{-1}{}_2F_1\Big[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]\\&\quad-2^{-s-\alpha_1s}\frac{\mu}{\mu+\alpha_1ks}-2^{-s-\alpha_2s}\frac{\mu}{\mu+\alpha_2ks},\end{aligned}$$

$$\begin{aligned}\Phi_2&=2^{-s}{}_2F_1\Big[-\alpha_2s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]-2^{-1}{}_2F_1\Big[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]\\&\quad+2^{-s-\alpha_2s}\frac{\mu}{\mu+\alpha_2ks}-2^{-(\alpha_1+\alpha_2)s-1}\frac{\mu}{\mu+(\alpha_1+\alpha_2)ks},\end{aligned}$$

$$\begin{aligned}\Phi_3&=2^{-s}{}_2F_1\Big[-\alpha_1s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]-2^{-1}{}_2F_1\Big[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]\\&\quad+2^{-s-\alpha_1s}\frac{\mu}{\mu+\alpha_1ks}-2^{-(\alpha_1+\alpha_2)s-1}\frac{\mu}{\mu+(\alpha_1+\alpha_2)ks}\end{aligned}$$

and

$$\Phi_4=2^{-1}{}_2F_1\Big[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\Big]+\frac{\mu2^{-(\alpha_1+\alpha_2)s-1}}{\mu+(\alpha_1+\alpha_2)ks}.$$

Proof. Using the generalized s-(α_1,m)-preinvexity of f, the generalized s-(α_2,m)-preinvexity of g and Lemma 2.5, we get

$$\begin{aligned}
&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}\,{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^+}(fg)\big(ma+\eta(b,a,m)\big)\\
&=\frac{\mu 2^{\frac{\mu}{k}-1}}{k\eta^{\frac{\mu}{k}}(b,a,m)}\int_{ma+\frac{1}{2}\eta(b,a,m)}^{ma+\eta(b,a,m)}\big(ma+\eta(b,a,m)-u\big)^{\frac{\mu}{k}-1}f(u)g(u)\mathrm{d}u\\
&=\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}f\left(ma+\frac{1+t}{2}\eta(b,a,m)\right)f\left(ma+\frac{1+t}{2}\eta(b,a,m)\right)\mathrm{d}t\\
&\leq\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}\left[m\left(1-\left(\frac{1+t}{2}\right)^{\alpha_1}\right)^s f(a)+\left(\frac{1+t}{2}\right)^{s\alpha_1}f(b)\right]\\
&\quad\times\left[m\left(1-\left(\frac{1+t}{2}\right)^{\alpha_2}\right)^s g(a)+\left(\frac{1+t}{2}\right)^{s\alpha_2}g(b)\right]\mathrm{d}t\\
&\leq\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}\left[m\left(2^{1-s}-\left(\frac{1+t}{2}\right)^{s\alpha_1}\right)f(a)+\left(\frac{1+t}{2}\right)^{s\alpha_1}f(b)\right]\\
&\quad\times\left[m\left(2^{1-s}-\left(\frac{1+t}{2}\right)^{s\alpha_2}\right)g(a)+\left(\frac{1+t}{2}\right)^{s\alpha_2}g(b)\right]\mathrm{d}t\\
&=\Bigg(2^{1-2s}-2^{-s}{}_2F_1\left[-\alpha_1 s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]-2^{-s}{}_2F_1\left[-\alpha_2 s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\\
&\quad+2^{-1}{}_2F_1\left[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\Bigg)m^2f(a)g(a)\\
&\quad+\Bigg(2^{-s}{}_2F_1\left[-\alpha_2 s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\\
&\quad-2^{-1}{}_2F_1\left[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\Bigg)mf(a)g(b)\\
&\quad+\Bigg(2^{-s}{}_2F_1\left[-\alpha_1 s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\\
&\quad-2^{-1}{}_2F_1\left[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\Bigg)mf(b)g(a)\\
&\quad+\Bigg(2^{-1}{}_2F_1\left[-(\alpha_1+\alpha_2)s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\Bigg)f(b)g(b)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}\,{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^-}(fg)(ma)\\
&\leq\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}\left[m\left(1-\left(\frac{1-t}{2}\right)^{\alpha_1}\right)^s f(a)+\left(\frac{1-t}{2}\right)^{s\alpha_1}f(b)\right]\\
&\quad\times\left[m\left(1-\left(\frac{1-t}{2}\right)^{\alpha_2}\right)^s g(a)+\left(\frac{1-t}{2}\right)^{s\alpha_2}g(b)\right]\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\le \frac{\mu}{2k}\int_0^1 (1-t)^{\frac{\mu}{k}-1}\left[m\left(2^{1-s}-\left(\frac{1-t}{2}\right)^{s\alpha_1}\right)f(a)+\left(\frac{1-t}{2}\right)^{s\alpha_1}f(b)\right]\\
&\quad\times\left[m\left(2^{1-s}-\left(\frac{1-t}{2}\right)^{s\alpha_2}\right)g(a)+\left(\frac{1-t}{2}\right)^{s\alpha_2}g(b)\right]\mathrm{d}t\\
&=\Bigg(2^{1-2s}-2^{-s-\alpha_1 s}\frac{\mu}{\mu+\alpha_1 ks}-2^{-s-\alpha_2 s}\frac{\mu}{\mu+\alpha_2 ks}\\
&\quad+2^{-(\alpha_1+\alpha_2)s-1}\frac{\mu}{\mu+(\alpha_1+\alpha_2)ks}\Bigg)m^2 f(a)g(a)\\
&\quad+\left(2^{-s-\alpha_2 s}\frac{\mu}{\mu+\alpha_2 ks}-2^{-(\alpha_1+\alpha_2)s-1}\frac{\mu}{\mu+(\alpha_1+\alpha_2)ks}\right)mf(a)g(b)\\
&\quad+\left(2^{-s-\alpha_1 s}\frac{\mu}{\mu+\alpha_1 ks}-2^{-(\alpha_1+\alpha_2)s-1}\frac{\mu}{\mu+(\alpha_1+\alpha_2)ks}\right)mf(b)g(a)\\
&\quad+\frac{\mu 2^{-(\alpha_1+\alpha_2)s-1}}{\mu+(\alpha_1+\alpha_2)ks}f(b)g(b).
\end{aligned}$$

This completes the proof. □

Corollary 3.7. *In Theorem 3.6, if $\eta(b,a,m)$ with $m=1$ is reduced to $\eta(b,a)$ and $\alpha_1=\alpha_2=\alpha$, then*

$$\begin{aligned}
&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a)}\left[{}_kJ^{\mu}_{(a+\frac{1}{2}\eta(b,a))^+}(fg)\big(a+\eta(b,a)\big)+{}_kJ^{\mu}_{(a+\frac{1}{2}\eta(b,a))^-}(fg)(a)\right]\\
&\le\Bigg(2^{2-2s}-2^{1-s}{}_2F_1\left[-\alpha s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]-2^{1-s-\alpha s}\frac{\mu}{\mu+\alpha ks}\\
&\quad+2^{-2\alpha s-1}\frac{\mu}{\mu+2\alpha ks}+2^{-1}{}_2F_1\left[-2\alpha s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\Bigg)f(a)g(a)\\
&\quad+\Bigg(2^{-s}{}_2F_1\left[-\alpha s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]-2^{-1}{}_2F_1\left[-2\alpha s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]\\
&\quad+2^{-s-\alpha s}\frac{\mu}{\mu+\alpha ks}-2^{-2\alpha s-1}\frac{\mu}{\mu+2\alpha ks}\Bigg)\big[f(a)g(b)+f(b)g(a)\big]\\
&\quad+\left(2^{-1}{}_2F_1\left[-2\alpha s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]+\frac{\mu 2^{-2\alpha s-1}}{\mu+2\alpha ks}\right)f(b)g(b).
\end{aligned}$$

In particular, if $\eta(b,a)=b-a$ and $s=1=\alpha$, then

$$\begin{aligned}
&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}(fg)(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}(fg)(a)\right]\\
&\le\frac{4k^2+3\mu k+\mu^2}{4(\mu+k)(\mu+2k)}\big[f(a)g(a)+f(b)g(b)\big]\\
&\quad+\frac{3\mu k+\mu^2}{4(\mu+k)(\mu+2k)}\big[f(a)g(b)+f(b)g(a)\big].
\end{aligned}$$

Corollary 3.8. *In Theorem 3.6, if $\eta(b,a,m)=b-ma$ with $m=1=s$, $\alpha_1=\alpha_2=\alpha$ and $g(x)=1$, then*

$$\begin{aligned}&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\right]\\&\le\left(1-2^{-1}{}_2F_1\left[-\alpha,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]-2^{-1-\alpha}\frac{\mu}{\mu+\alpha k}\right)f(a)\\&\quad+\left(2^{-1}{}_2F_1\left[-\alpha,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}\right]+2^{-1-\alpha}\frac{\mu}{\mu+\alpha k}\right)f(b).\end{aligned}$$

In particular, if $\alpha=1$, then

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\right]\le\frac{f(a)+f(b)}{2},$$

which is the right hand side of inequality (1.4) proved by Farid, Rehman and Zahra in [21].

Another k-fractional integral inequality with respect to the products of two generalized s-(α,m)-preinvex functions as follows.

Theorem 3.9. *If f is s_1-(α,m)-preinvex in the second sense, g is s_2-(α,m)-preinvex in the second sense for $(\alpha,s_i)\in(0,1]\times(0,1]$ and $i=1,2$, then the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}\left[{}_kJ^{\mu}_{(ma)^+}(fg)\big(ma+\eta(b,a,m)\big)+{}_kJ^{\mu}_{(ma+\eta(b,a,m))^-}(fg)(ma)\right]\\&\le\frac{\mu}{2k}\Big[m^2\Upsilon_1f(a)g(a)+m\Upsilon_2f(a)g(b)+m\Upsilon_3f(b)g(a)+\Upsilon_4f(b)g(b)\Big],\end{aligned}\tag{3.4}$$

where

$$\begin{aligned}\Upsilon_1=&\,2^{3-s_1-s_2}\frac{k}{\mu}-2^{1-s_1}\beta\left(\alpha s_2+1,\frac{\mu}{k}\right)-2^{1-s_2}\beta\left(\alpha s_1+1,\frac{\mu}{k}\right)\\&+\beta\left(\alpha(s_1+s_2)+1,\frac{\mu}{k}\right)-\frac{2^{1-s_1}k}{\mu+\alpha ks_2}-\frac{2^{1-s_2}k}{\mu+\alpha ks_1}+\frac{k}{\mu+\alpha k(s_1+s_2)},\end{aligned}$$

$$\Upsilon_2=2^{1-s_2}\beta\left(\alpha s_1+1,\frac{\mu}{k}\right)-\beta\left(\alpha(s_1+s_2)+1,\frac{\mu}{k}\right)+\frac{k2^{1-s_1}}{\mu+\alpha ks_2}-\frac{k}{\mu+\alpha k(s_1+s_2)},$$

$$\Upsilon_3=2^{1-s_1}\beta\left(\alpha s_2+1,\frac{\mu}{k}\right)-\beta\left(\alpha(s_1+s_2)+1,\frac{\mu}{k}\right)+\frac{k2^{1-s_2}}{\mu+\alpha ks_1}-\frac{k}{\mu+\alpha k(s_1+s_2)}$$

and

$$\Upsilon_4=\beta\left(\alpha(s_1+s_2)+1,\frac{\mu}{k}\right)+\frac{k}{\mu+\alpha k(s_1+s_2)}.$$

Proof. The proof of Theorem 3.9 is analogous to that of Theorem 3.6 and so is omitted here. ☐

Corollary 3.10. *In Theorem 3.9, if $\eta(b,a,m)$ with $m=1$ is reduced to $\eta(b,a)$ and $s_1=s_2=s$, then*

$$\begin{aligned}&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a)}\left[{}_kJ^{\mu}_{a^+}(fg)\big(a+\eta(b,a)\big)+{}_kJ^{\mu}_{(a+\eta(b,a))^-}(fg)(a)\right]\\&\le\frac{\mu}{2k}\left\{\left[2^{3-2s}\frac{k}{\mu}-2^{2-s}\beta\left(\alpha s+1,\frac{\mu}{k}\right)+\beta\left(2\alpha s+1,\frac{\mu}{k}\right)\right.\right.\\&\quad\left.-\frac{2^{2-s}k}{\mu+\alpha ks}+\frac{k}{\mu+2\alpha ks}\right]f(a)g(a)+\left[2^{1-s}\beta\left(\alpha s+1,\frac{\mu}{k}\right)-\beta\left(2\alpha s+1,\frac{\mu}{k}\right)\right.\\&\quad\left.+\frac{k2^{1-s}}{\mu+\alpha ks}-\frac{k}{\mu+2\alpha ks}\right]\left[f(a)g(b)+f(b)g(a)\right]\\&\quad\left.+\left[\beta\left(2\alpha s+1,\frac{\mu}{k}\right)+\frac{k}{\mu+2\alpha ks}\right]f(b)g(b)\right\}.\end{aligned}$$

In particular, if $\eta(b,a)=b-a$ and $k=1=s=\alpha$, then

$$\begin{aligned}&\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[J^{\mu}_{a^+}f(b)g(b)+J^{\mu}_{b^-}f(a)g(a)\right]\\&\le\frac{\mu^2+\mu+2}{2(\mu+1)(\mu+2)}\Big[f(a)g(a)+f(b)g(b)\Big]+\frac{\mu}{(\mu+1)(\mu+2)}\Big[f(a)g(b)+f(b)g(a)\Big],\end{aligned}$$

which is Theorem 2.1 established by Chen in [31].

Corollary 3.11. *In Theorem 3.9, if $\eta(b,a,m)=b-ma$ with $m=1=\alpha$, $s_1=1=s_2$ and $g(x)=1$, then*

$$\frac{\Gamma_k(\mu+k)}{2(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{a^+}f(b)+{}_kJ^{\mu}_{b^-}f(a)\right]\le\frac{f(a)+f(b)}{2}.$$

In particular, if $k=1$, then

$$\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[J^{\mu}_{a^+}f(b)+J^{\mu}_{b^-}f(a)\right]\le\frac{f(a)+f(b)}{2},$$

which is the right side of inequality 1.3 proved by Sarikaya et al. in [11].

4. Conclusion

In this paper, we introduced the class of generalized s-(α,m)-preinvex mappings defined on m-invex, and we proved two multi-parameterized inequalities and two product-type inequalities of k-fractional Hadamard-type for such mappings. Some subresults can be derived from our main results by choosing different mappings η and the special parameter values for λ, k and μ.

Acknowledgements
This paper was supported by the National Natural Science Foundation of China (No. 61374028) and the Research Fund for Excellent Dissertation of China Three Gorges University (No. 2018SSPY132 and No. 2018SSPY134).

References

[1] T. Du, Y. Li, Z. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions, Appl. Math. Comput. 293 (2017), 358-369.

[2] S. Erden, H. Budak, M.Z. Sarikaya, An Ostrowski type inequality for twice differentiable mappings and applications, Math. Model. Anal. 21 (2016), 522-532.

[3] D.Y. Hwang, S.S. Dragomir, Extensions of the Hermite-Hadamard inequality for r-preinvex functions on an invex set, Bull. Aust. Math. Soc. 95 (2017), 412-423.

[4] M.A. Khan, Y. Chu, T.U. Khan, J. Khan, Some new inequalities of Hermite–Hadamard type for s-convex functions with applications, Open Math. 15 (2017), 1414-1430.

[5] M.A. Khan, Y. Chu, A. Kashuri, R. Liko, G. Ali, Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations, J. Funct. Spaces, 2018 (2018), Article ID 6928130.

[6] M.A. Latif, S.S. Dragomir, Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions through preinvexity, Georgian Math. J. 23 (2016), 97-104.

[7] M.V. Mihai, M. A. Noor, K.I. Noor, M.U. Awan, Some integral inequalities for harmonic h-convex functions involving hypergeometric functions, Appl. Math. Comput. 252 (2015), 257-262.

[8] M.V. Mihai, M.A. Noor, M.U. Awan, Trapezoidal like inequalities via harmonic h-convex functions on the co-ordinates in a rectangle from the plane, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. 111 (2017), 765-779.

[9] M.A. Noor, K.I. Noor, F. Safdar, Integral inequalities via generalized (α,m)-convex functions, J. Nonlinear Funct. Anal. 2017 (2017), Article ID 32.

[10] E. Set, S.S. Karataş, M.A. Khan, Hermite-Hadamard type inequalities obtained via fractional integral for differentiable m-convex and (α,m)-convex function, Int. J. Anal. 2016 (2016), Article ID 4765691.

[11] M.Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model. 57 (2013), 2403-2407.

[12] M.Z. Sarikaya, H. Yildirim, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, Miskolc Math. Notes 17 (2016), 1049-1059.

[13] S.S. Dragomir, M.I. Bhatti, M. Iqbal, M. Muddassar, Some new Hermite-Hadamard's type fractional integral inequalities, J. Comput. Anal. Appl. 18 (2015), 655-661.

[14] S.R. Hwang, K.L. Tseng, K.C. Hsu, New inequalities for fractional integrals and their application, Turkish J. Math. 40 (2016), 471-486.

[15] M.A. Noor, K.I. Noor, M.V. Mihai, M.U. Awan, Fractional Hermite-Hadamard inequalities for some classes of differentiable preinvex functions, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 78 (2016), 163-174.

[16] S. Qaisar, M. Iqbal, M. Muddassar, New Hermite-Hadamard's inequalities for preinvex functions via fractional integrals, J. Comput. Anal. Appl. 20 (2016), 1318-1328.

[17] M.U. Awan, M.A. Noor, M.V. Mihai, K.I. Noor, On bounds involving k-Appell's hypergeometric functions, J. Inequal. Appl. 2017 (2017), Article ID 118.

[18] S. Mubeen, G. M. Habibullah, k-fractional integrals and application, Int. J. Contemp. Math. Sci. 7 (2012), 89-94.

[19] E. Set, M. Tomar, M.Z. Sarikaya, On generalized Grüss type inequalities for k-fractional integrals, Appl. Math. Comput. 269 (2015), 29-34.

[20] H. Wang, T. Du, Y. Zhang, k-fractional integral trapezium-like inequalities through (h,m)-convex and (α,m)-convex mappings, J. Inequal. Appl. 2017 (2017), Article ID 311.

[21] G. Farid, A.U. Rehman, M. Zahra, On Hadamard-type inequalities for k-fractional integrals, Konuralp J. Math. 4 (2016), 79-86.

[22] T. Du, J. Liao, L. Chen, M.U. Awan, Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α,m)-preinvex functions, J. Inequal. Appl. 2016 (2016), Article ID 306.

[23] M. Muddassar, M. I Bhatti, W. Irshad, Generalisations of integral inequalities of Hermite-Hadamard type through convexity, Bull. Aust. Math. Soc. 88 (2013), 320-330.

[24] T. Antczak, Mean value in invexity analysis, Nonlinear Anal. 60 (2005), 1473-1484.

[25] Y. Zhang, T. Du, H. Wang, Some new k-fractional integral inequalities containing multiple parameters via generalized (s;m)-preinvexity, Ital. J. Pure Appl. Math. in press.

[26] Y. Li, T. Du, Some improvements on Hermite-Hadamard's inequalities for s-convex functions, J. Math. Study 49 (2016), 82-92.

[27] M. Alomari, M. Darus, S.S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. 23 (2010), 1071-1076.

[28] A. Kashuri, R. Liko, Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions, Proyecciones 36 (2017), 45-80.

[29] M.E. Özdemir, M. Avci, H. Kavurmaci, Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals, arXiv:1202.0380 [math.FA], (2012).

[30] H. Kavurmaci, M. Avci, M. E. Özdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, J. Inequal. Appl. 2011 (2011), Article ID 86.

[31] F. X. Chen, A note on Heimite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals, Ital. J. Pure Appl. Math. 33 (2014), 299-306.