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KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS AND DUALITY FOR A SEMI-INFINITE PROGRAMMING WITH MULTIPLE INTERVAL-VALUED OBJECTIVE FUNCTIONS

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Abstract. This paper deals with a semi-infinite programming with multiple interval-valued objective functions. We first investigate necessary and sufficient Karush-Kuhn-Tucker optimality conditions for some types of optimal solutions. Then, we formulate types of Mond-Weir and Wolfe dual problems and explore duality relations under convexity assumptions. Some examples are provided to illustrate our results.

Keywords. Multiobjective semi-infinite programming; Interval-valued objective functions; Karush-Kuhn-Tucker optimality conditions; Mond-Weir duality; Wolfe duality.

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1. Introduction

A simultaneous minimization of a finite number of objective functions over an infinite number of constraints is called a multiobjective semi-infinite programming problem. Applied and theoretical aspects of semi-infinite programming problems have been considered by many researchers. For some recent results in this direction, the reader is referred to [1, 2, 3, 4, 5, 6] and the references therein. In practice, the coefficients of objective functions in some optimization problems in engineering, economics and computer models are used to be uncertain or imprecise data due to measurement errors or some unexpected factors. Sometimes one can only determine the range of coefficients, or the interval-valued, of objective functions in these optimization problems. Optimality conditions and duality for optimization problems with one or multiple interval-valued objective functions have been investigated recently in many papers; see, e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein. In [16], the Fritz-John optimality conditions and duality for semi-infinite programming problem with one interval-valued objective function are investigated. It is well-known that Karush-Kuhn-Tucker (KKT) optimality conditions give more information than the Fritz-John optimality conditions since the multipliers corresponding to the objective functions of Fritz-John optimality conditions are able to be equal to zero. Moreover, to the best of our

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knowledge, there is no paper dealing with a semi-infinite programming with multiple interval-valued objective functions.

In this paper, motivated by the above observations, we establish the KKT optimality conditions and investigate duality problems for the semi-infinite programming with multiple interval-valued objective functions. The paper is organized as follows. Section 2 recalls basic concepts and some preliminaries. KKT necessary and sufficient optimality conditions for the semi-infinite programming with multiple interval-valued objective functions are established in the next part by employing some convexity notions. Section 4 is devoted to exploring Mond-Weir and Wolfe dual problems of semi-infinite programming with multiple interval-valued objective functions. Some examples are provided to illustrate our results.

2. Preliminaries

The following notations and definitions will be used throughout the paper. Let $\mathbb{R}^n$ be a finite-dimensional normed space. The notation $\langle \cdot, \cdot \rangle$ is utilized to denote inner product. For a given $\bar{x}$, $\mathscr{U}(\bar{x})$ is the system of the neighborhoods of $\bar{x}$. For a subset $A \subseteq \mathbb{R}^n$, int$A$, cl$A$, aff$A$, and co$A$ stand for its interior, closure, affine hull, convex hull of A, respectively (resp). The cone and the convex cone (containing the origin) generated by A are denoted resp by cone$A$, pos$A$. If $\langle x^*, x\rangle \leq 0$ for all $x^* \in A^*$, where A^* is a subset of the dual space of $\mathbb{R}^n$, we write $\langle A^*, x\rangle \leq 0$. The negative polar cone and strictly negative polar cone of A are defined resp by

$$A^- := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle \leq 0 \, \forall x \in A\},$$

$$A^s := \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle < 0 \, \forall x \in A \setminus \{0\}\}.$$

It is easy to check that $A^s \subset A^-$ and if $A^s \neq \emptyset$ then cl$A^s = A^-$. Moreover, the bipolar theorem, see, e.g., [1] (for $A^{--} = $ cl cone$A$).

Definition 2.1. [17] Let A be a nonempty subset of $\mathbb{R}^n$.

(i) The contingent (or Bouligand) cone of A at $\bar{x} \in$ cl$A$ is

$$T(A,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \exists x_k \to x, \, \forall k \in \mathbb{N}, \bar{x} + \tau_k x_k \in A\}.$$

(ii) The adjacent cone of A at $\bar{x} \in$ cl$A$ is

$$T^b(A,\bar{x}) := \{x \in \mathbb{R}^n \mid \forall \tau_k \downarrow 0, \exists x_k \to x, \, \forall k \in \mathbb{N}, \bar{x} + \tau_k x_k \in A\}.$$

(iii) The cone of the feasible directions of A at $\bar{x}$ is

$$D(A,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \delta > 0, \bar{x} + \tau x \in A, \forall \tau \in (0,\delta)\}.$$

(iv) The cone of the weak feasible directions of A at $\bar{x}$ is

$$F(A,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \, \forall k \in \mathbb{N}, \bar{x} + \tau_k x \in A\}.$$

Remark 2.2. The following properties can be checked directly.

(i) $D(A,\bar{x}) \subset F(A,\bar{x}) \subset T(A,\bar{x})$.
(ii) $D(A,\bar{x}) \subset T^b(A,\bar{x}) \subset T(A,\bar{x})$.
(iii) If A is a convex set then $T^b(A,\bar{x}) = T(A,\bar{x}) = \mathrm{clcone}(A - \bar{x})$.
(iv) If A is a convex set then $D(A,\bar{x}) = F(A,\bar{x}) = \mathrm{cone}(A - \bar{x})$.

Definition 2.3. [18] Let $X \subset \mathbb{R}^n$ be a convex set, $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in X$.

(i) φ is convex at $\bar{x}$ if

$$\varphi(\lambda\bar{x}+(1-\lambda)x) \leq \lambda\varphi(\bar{x})+(1-\lambda)\varphi(x), \forall x \in X, \forall\lambda \in [0,1].$$

(ii) φ is strictly convex at $\bar{x}$ if

$$\varphi(\lambda\bar{x}+(1-\lambda)x) < \lambda\varphi(\bar{x})+(1-\lambda)\varphi(x), \forall x \in X\setminus\{\bar{x}\}, \forall\lambda \in [0,1].$$

(ii) φ is convex/strictly convex on X if φ is convex/strictly convex on each point of X.

Remark 2.4. [18] Let $X \subset \mathbb{R}^n$ be an open convex set, $\varphi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\bar{x} \in X$.

(i) If φ is convex at $\bar{x}$ then

$$\varphi(x)-\varphi(\bar{x}) \geq \langle\nabla\varphi(\bar{x}), x-\bar{x}\rangle, \forall x \in X.$$

(ii) If φ is strictly convex at $\bar{x}$ then

$$\varphi(x)-\varphi(\bar{x}) > \langle\nabla\varphi(\bar{x}), x-\bar{x}\rangle, \forall x \in X\setminus\{\bar{x}\}.$$

Let $\mathscr{K}_C$ denote the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathscr{K}_C = \{[a,b] \mid a,b \in \mathbb{R} \text{ and } a \leq b\}.$$

For $A \in \mathscr{K}_C$, we adopt notation $A = [a^L, a^U]$, where a^L and a^U means the lower and upper bound of A, resp. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be in $\mathscr{K}_C$ and $\lambda \in \mathbb{R}$. Then, one sees from the definition that

$$A+B := \{a+b \mid a \in A, b \in B\} = [a^L+b^L, a^U+b^U],$$

$$\lambda A := \lambda[a^L, a^U] = \begin{cases} [\lambda a^L, \lambda a^U] & \text{if } \lambda \geq 0, \\ [\lambda a^U, \lambda a^L] & \text{if } \lambda < 0. \end{cases}$$

Hence, $-A = [-a^U, -a^L]$ and $A-B = [a^L-b^U, a^U-b^L]$.

Definition 2.5. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two sets in $\mathscr{K}_C$.

(i) $A \leq_{LU} B$ if $a^L \leq b^L$ and $a^U \leq b^U$.

(ii) $A <_{LU} B$ if $A \leq_{LU} B$ and $A \neq B$, or equivalently, $A <_{LU} B$ if

$$\begin{cases} a^L < b^L, \\ a^U \leq b^U, \end{cases} \text{or} \begin{cases} a^L \leq b^L, \\ a^U < b^U, \end{cases} \text{or} \begin{cases} a^L < b^L, \\ a^U < b^U. \end{cases}$$

(iii) ([9]) $A <^s_{LU} B$ if $a^L < b^L$ and $a^U < b^U$.

Let X be an nonempty open subset of $\mathbb{R}^n$. A function $H : X \to \mathscr{K}_C$ is called an interval-valued function if $H(x) = [H^L(x), H^U(x)]$ with $H^L, H^U : X \to \mathbb{R}$ such that $H^L(x) \leq H^U(x)$ for each $x \in X$.

Definition 2.6. Let X be an nonempty open convex subset of $\mathbb{R}^n$, $\bar{x} \in X$ and $H : X \to \mathscr{K}_C$ be an interval-valued function.

(i) It is said that H is LU-convex [14] at $\bar{x}$ if, for all $x \in X$ and $\lambda \in [0,1]$, one has

$$H(\lambda\bar{x}+(1-\lambda)x) \leq_{LU} \lambda H(\bar{x})+(1-\lambda)H(x),$$

and, H is said LU-convex on X if H is LU-convex at each $x \in X$.

(ii) The interval-valued function $H : X \to \mathbb{R}$ is strictly LU-convex [14] at $\bar{x}$ if, for all $x \in X\setminus\{\bar{x}\}$ and $\lambda \in [0,1]$, one has

$$H(\lambda\bar{x}+(1-\lambda)x) <_{LU} \lambda H(\bar{x})+(1-\lambda)H(x),$$

and, H is called strictly LU-convex on X if H is strictly LU-convex at each $x \in X$.
(iii) The interval-valued function $H : X \to \mathbb{R}$ is strongly LU-convex at $\bar{x}$ if, for all $x \in X \setminus \{\bar{x}\}$ and $\lambda \in [0,1]$, one has

$$H(\lambda\bar{x} + (1-\lambda)x) <^s_{LU} \lambda H(\bar{x}) + (1-\lambda)H(x),$$

and, H is called strongly LU-convex on X if H is strongly LU-convex at each $x \in X$.

Remark 2.7. Let X be an nonempty open convex subset of $\mathbb{R}^n$, $\bar{x} \in X$ and $H : X \to \mathscr{K}_C$ be an interval-valued function.
(i) [14] If $H : X \to \mathscr{K}_C$ is LU-convex at $\bar{x}$ then the functions H^L and H^U are convex at $\bar{x}$.
(ii) [14] If $H : X \to \mathscr{K}_C$ is strictly LU-convex at $\bar{x} \in X$ then the functions H^L and H^L are convex at $\bar{x}$ and at least one of them is strictly convex at $\bar{x}$.
(iii) If $H : X \to \mathscr{K}_C$ is strongly LU-convex at $\bar{x}$ then the functions H^L and H^U are strictly convex at $\bar{x}$.

Lemma 2.8. [18] *Let $\{C_t | t \in \Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K = \operatorname{pos}\left(\bigcup_{t\in\Gamma} C_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

Lemma 2.9. [19] *Suppose that S, P are arbitrary (possibly infinite) index sets, $a_s = a(s) = (a_1(s), ..., a_n(s))$ maps S onto $\mathbb{R}^n$, and so does a_p. Suppose that the set $\operatorname{co}\{a_s, s \in S\} + \operatorname{pos}\{a_p, p \in P\}$ is closed. Then the following statements are equivalent:*

$$I: \quad \begin{cases} \langle a_s, x\rangle < 0, s \in S, S \neq \emptyset \\ \langle a_p, x\rangle \leq 0, p \in P \end{cases} \quad \textit{has no solution } x \in \mathbb{R}^n;$$

$$II: \quad 0 \in \operatorname{co}\{a_s, s \in S\} + \operatorname{pos}\{a_p, p \in P\}.$$

Lemma 2.10. [18] *If S is a nonempty compact subset of $\mathbb{R}^n$, then*

(i) $\operatorname{co}S$ *is a compact set;*
(ii) *If $0 \notin \operatorname{co}S$, then $\operatorname{pos}S$ is a closed cone.*

3. KKT OPTIMALITY CONDITIONS

In this section, we consider the following semi-infinite programming with multiple interval-valued objective functions:

$$\text{(P)} \qquad \text{LU}-\min F(x) := (F_1(x), ..., F_m(x)) = ([F_1^L(x), F_1^U(x)], ..., [F_m^L(x), F_m^U(x)])$$
$$\text{s.t.} \quad g_t(x) \leq 0, \quad t \in T,$$

where $F_i : \mathbb{R}^n \to \mathscr{K}_C$ are interval-valued functions for $i \in I := \{1, ..., m\}$, $g_t, t \in T$, are functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $F_i^L, F_i^U, i \in I, g_t, t \in T$ are the continuously differentiable functions. The index set T is an arbitrary nonempty set, not necessary finite. Denote the feasible solution set of (P)

$$\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \ t \in T\}.$$

Denote $\mathbb{R}^{|T|}_+$ the collection of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x} \in \Omega$, denote $T(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^{|T|}_+ | \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $J \subset T(\bar{x})$ such that $\lambda_t > 0$ for all $t \in J$ and $\lambda_t = 0$ for all $t \in T \setminus J$.

Definition 3.1. Let $\bar{x} \in \Omega$ and $\mathscr{U}(\bar{x})$ be the set of neighborhoods of $\bar{x}$.

(i) $\bar{x}$ is a locally type-1 (Pareto) optimal solution [15] of (P), denoted by $\bar{x} \in \mathrm{loc}E(P,1)$, if there exists $U \in \mathscr{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying

$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_k(x) <_{LU} F_k(\bar{x}), & \text{for at least one } k \in I. \end{cases}$$

(ii) $\bar{x}$ is a locally weakly type-1 optimal solution [15] of (P), denoted by $\bar{x} \in \mathrm{loc}WE(P,1)$, if there exists $U \in \mathscr{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying $F_i(x) <_{LU} F_i(\bar{x}), \forall i \in I$.

(iii) $\bar{x}$ is a locally type-2 optimal solution [9] of (P), denoted by $\bar{x} \in \mathrm{loc}E(P,2)$, if there exists $U \in \mathscr{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying

$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_k(x) <^s_{LU} F_k(\bar{x}), & \text{for at least one } k \in I. \end{cases}$$

(iv) $\bar{x}$ is a locally weakly type-2 optimal solution [9] of (P), denoted by $\bar{x} \in \mathrm{loc}WE(P,2)$, if there exists $U \in \mathscr{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying $F_i(x) <^s_{LU} F_i(\bar{x}), \forall i \in I$.

If $U = \mathbb{R}^n$, the word "locally" is omitted. In this case, the type-1 Pareto optimal solution sets is denoted by $E(P,1)$ and the other optimal solution sets are denoted in a similar way.

Remark 3.2. [9] The following relations are immediate:

(i) $E(P,1) \subseteq E(P,2) \subseteq WE(P,2)$,

(ii) $E(P,1) \subseteq WE(P,1) \subseteq WE(P,2)$.

The following examples show that the concepts of optimal solutions in Definition 3.1 are different.

Example 3.3. Consider the following problem

$$\text{(P)} \qquad \mathrm{LU}-\min F(x) = (F_1(x), F_2(x)) = ([-2,-1]x, [0,2])$$
$$\text{s.t.} \quad g_t(x) = -x + t \leq 0, \quad t \in [-1,0].$$

Then, $\Omega = \mathbb{R}_+$ and for $\bar{x} = 0 \in \Omega$, one has

$$F(\bar{x}) = (F_1(\bar{x}), F_2(\bar{x})) = ([0,0],[0,2]).$$

Since, there is no $x \in \Omega$ such that $F_2(x) <_{LU} F_2(\bar{x})$, one gets $\bar{x} \in WE(P,1)$. But $\bar{x} \notin E(P,1)$, since there exists $\hat{x} = 1 \in \Omega$ satisfying

$$F_1(\hat{x}) = [-2,-1] <_{LU} [0,0] = F_1(\bar{x})$$

and

$$F_2(\hat{x}) = [0,2] \leq_{LU} [0,2] = F_2(\bar{x}).$$

Hence,

$$E(P,1) \subsetneqq WE(P,1).$$

From Remark 3.2 (ii), $\bar{x} \in WE(P,1) \subset WE(P,2)$. However, $\bar{x} \notin E(P,2)$, since there exists $\hat{x} = 1 \in \Omega$ satisfying

$$F_1(\hat{x}) = [-2,-1] <^s_{LU} [0,0] = F_1(\bar{x})$$

and

$$F_2(\hat{x}) = [0,2] \leq_{LU} [0,2] = F_2(\bar{x}).$$

Therefore,

$$E(P,2) \subsetneqq WE(P,2).$$

Example 3.4. Consider the following problem

$$\text{(P)} \qquad \text{LU}-\min F(x) = (F_1(x), F_2(x)) = ([-2,0]x, [-3,0]x)$$
$$\text{s.t.} \quad g_t(x) = x - t \leq 0, \quad t \in [0,1].$$

Then, $\Omega = \mathbb{R}_+$ and for $\bar{x} = 0 \in \Omega$, one has

$$F(\bar{x}) = (F_1(\bar{x}), F_2(\bar{x})) = ([0,0],[0,0]).$$

Since there is no $x \in \Omega$ such that $F_2(x) <^s_{LU} F_2(\bar{x})$, we have $\bar{x} \in WE(P,2)$. But $\bar{x} \notin WE(P,1)$ since there exists $\hat{x} = 1 \in \Omega$ satisfying

$$F_1(\hat{x}) = [-2,0] <_{LU} [0,0] = F_1(\bar{x})$$

and

$$F_2(\hat{x}) = [-3,0] <_{LU} [0,0] = F_2(\bar{x}).$$

Hence,

$$WE(P,1) \subsetneqq WE(P,2).$$

Example 3.5. Consider the following problem

$$\text{(P)} \qquad \text{LU}-\min F(x) = (F_1(x), F_2(x)) = ([-2,0]x, [-1,0]x)$$
$$\text{s.t.} \quad g_t(x) = -2x + t \leq 0, \quad t \in [-2,0].$$

Then, $\Omega = \mathbb{R}_+$ and for $\bar{x} = 0 \in \Omega$, one has

$$F(\bar{x}) = (F_1(\bar{x}), F_2(\bar{x})) = ([0,0],[0,0]).$$

Note that, for all $x \in \Omega$,

$$\begin{cases} F_1(x) \not<^s_{LU} F_1(\bar{x}), \\ F_2(x) \not<^s_{LU} F_2(\bar{x}). \end{cases} \tag{3.1}$$

Now, we prove $\bar{x} \in E(P,2)$ by contraposition. Suppose to the contrary that $\bar{x} \notin E(P,2)$. Then, there is $\hat{x} \in \Omega$ satisfying

$$\begin{cases} F_i(\hat{x}) \leq_{LU} F_i(\bar{x}), & \forall i \in I = \{1,2\}, \\ F_k(\hat{x}) <^s_{LU} F_k(\bar{x}), & \text{for at least one } k \in \{1,2\}. \end{cases}$$

which contradicts (3.1). So, $\bar{x} \in E(P,2)$.

But $\bar{x} \notin E(P,1)$, since there exists $\hat{x} = 1 \in \Omega$ satisfying

$$\begin{cases} F_1(\hat{x}) = [-2,0] \leq_{LU} [0,0] = F_1(\bar{x}), F_2(\hat{x}) = [-1,0] \leq_{LU} [0,0] = F_2(\bar{x}) \\ F_1(\hat{x}) = [-2,0] <_{LU} [0,0] = F_1(\bar{x}). \end{cases}$$

Hence,

$$E(P,1) \subsetneqq E(P,2).$$

Lemma 3.6. *A point $\bar{x}$ is a locally weakly type-2 optimal solution of (P) if and only if $\bar{x}$ is a locally weakly efficient solution of the following multiobjective optimization problem:*

$$(MP): \quad \begin{array}{l} \mathbb{R}^{2m}_+ - \min \widetilde{F}(x) = (F^L_1(x), \ldots, F^L_m(x), F^U_1(x), \ldots, F^U_m(x)) \\ \quad s.t.\ g_t(x) \leq 0, t \in T. \end{array}$$

Proof. Let $\bar{x}$ be a locally weakly efficient solution of (MP). Then, $\bar{x} \in \Omega$ and there exists $\bar{U} \in \mathscr{U}(\bar{x})$ such that, for every $x \in \Omega \cap \bar{U}$,

$$\widetilde{F}(x) - \widetilde{F}(\bar{x}) \notin -\mathrm{int}\mathbb{R}^{2m}_{+}. \tag{3.2}$$

Suppose to the contrary that $\bar{x}$ is not a locally weakly type-2 optimal solution of (P). Then, for all $U \in \mathscr{U}(\bar{x})$, there exists $x_0 \in \Omega \cap U$ such that $F_i(x_0) <^s_{LU} F_i(\bar{x}), \forall i \in I$, or equivalently,

$$F_i^L(x_0) < F_i^L(\bar{x}) \text{ and } F_i^U(x_0) < F_i^U(\bar{x}), \forall i \in I.$$

Hence, for $U = \bar{U}$, there exists $\bar{x}_0 \in \Omega \cap \bar{U}$ satisfying $F_i^L(\bar{x}_0) < F_i^L(\bar{x})$ and $F_i^U(\bar{x}_0) < F_i^U(\bar{x}), \forall i \in I$. This implies the existence of $\bar{x}_0 \in \Omega \cap \bar{U}$ such that

$$\widetilde{F}(\bar{x}_0) - \widetilde{F}(\bar{x}) \in -\mathrm{int}\mathbb{R}^{2m}_{+},$$

which contradicts (3.2).

Conversely, let $\bar{x}$ be a locally weakly type-2 optimal solution of (P). Then, there exists $\overline{U} \in \mathscr{U}(\bar{x})$ such that there is no $x \in \Omega \cap \overline{U}$ satisfying $F_i(x) <^s_{LU} F_i(\bar{x}), \forall i \in I$, or equivalently,

$$F_i^L(x) < F_i^L(\bar{x}) \text{ and } F_i^U(x) < F_i^U(\bar{x}), \forall i \in I. \tag{3.3}$$

Suppose to the contrary that $\bar{x}$ is not a locally weakly efficient solution of (MP). Then, for the above $\overline{U}$, there is $\hat{x} \in \Omega \cap \overline{U}$ such that $\widetilde{F}(\hat{x}) - \widetilde{F}(\bar{x}) \in -\mathrm{int}\mathbb{R}^{2m}_{+}$. This leads that

$$F_i^L(\hat{x}) < F_i^L(\bar{x}), F_i^U(\hat{x}) < F_i^U(\bar{x}), \forall i \in I,$$

contradicting with (3.3). $\square$

In the following, we will establish the KKT necessary optimality condition for a locally weakly type-2 optimal solution of (P). Notice that the KKT necessary optimality condition for a locally weakly type-2 optimal solution of (P) is also the KKT necessary optimality condition for the other types of optimal solutions of (P).

Definition 3.7. The (ACQ) holds at $\bar{x} \in \Omega$ if $\left(\bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x}) \right)^{-} \subseteq T(\Omega, \bar{x})$ and $\Delta := \mathrm{pos} \bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x})$ is closed.

Proposition 3.8. *Suppose that $\bar{x} \in \mathrm{loc}WE(P,2)$ and (ACQ) holds at $\bar{x}$. Then, there exist $\alpha^L, \alpha^U \in \mathbb{R}^m_{+}$ with $\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1$ and $\lambda \in \Lambda(\bar{x})$ such that*

$$\sum_{i=1}^{m} \alpha_i^L \nabla F_i^L(\bar{x}) + \sum_{i=1}^{m} \alpha_i^U \nabla F_i^U(\bar{x}) + \sum_{t \in T} \lambda_t \nabla g_t(\bar{x}) = 0.$$

Proof. Since $\bar{x} \in \mathrm{loc}WE(P,2)$, we see that there exists $U \in \mathscr{U}(\bar{x})$ such that there is no $x \in \Omega \cap U$ satisfying

$$F_i(x) <^s_{LU} F_i(\bar{x}), \forall i \in I. \tag{3.4}$$

First, we prove that

$$\left(\bigcup_{i \in I} (\nabla F_i^L(\bar{x}) \cup \nabla F_i^U(\bar{x})) \right)^{s} \cap T(\Omega, \bar{x}) = \emptyset. \tag{3.5}$$

Suppose to the contrary that there exists

$$d \in \left(\bigcup_{i \in I} (\nabla F_i^L(\bar{x}) \cup \nabla F_i^U(\bar{x})) \right)^s \cap T(\Omega, \bar{x}).$$

Then, one has

$$\begin{cases} \langle \nabla F_i^L(\bar{x}), d \rangle < 0, & \forall i \in I, \\ \langle \nabla F_i^U(\bar{x}), d \rangle < 0, & \forall i \in I. \end{cases}$$

By $d \in T(\Omega, \bar{x})$, there exist $\tau_k \downarrow 0$ and $d_k \to d$ such that $\bar{x} + \tau_k d_k \in \Omega$ for all k. We derive from $F_i^L, i \in I$, are continuously differentiable at $\bar{x}$ that

$$F_i^L(\bar{x} + \tau_k d_k) = F_i^L(\bar{x}) + \tau_k \langle \nabla F_i^L(\bar{x}), d_k \rangle + o(\tau_k \|d_k\|), \forall i \in I.$$

Consequently, for all $i \in I$,

$$\frac{F_i^L(\bar{x} + \tau_k d_k) - F_i^L(\bar{x})}{\tau_k} = \langle \nabla F_i^L(\bar{x}), d_k \rangle + \frac{o(\tau_k \|d_k\|)}{\tau_k \|d_k\|} . \|d_k\| \to \langle \nabla F_i^L(\bar{x}), d \rangle < 0, \text{ when } k \to \infty.$$

Thus, for each $i \in I$, there exists $k_i^L > 0$ such that

$$\frac{F_i^L(\bar{x} + \tau_k d_k) - F_i^L(\bar{x})}{\tau_k} < 0, \quad \forall k > k_i^L.$$

Denoting $\bar{k}^L = \max_{i \in I} k_i^L$, we have

$$F_i^L(\bar{x} + \tau_k d_k) < F_i^L(\bar{x}), \forall k > \bar{k}^L, \forall i \in I.$$

Similarly, there exists $\bar{k}^U > 0$ such that

$$F_i^U(\bar{x} + \tau_k d_k) < F_i^U(\bar{x}), \forall k > \bar{k}^U, \forall i \in I.$$

Setting $\bar{k} := \max\{\bar{k}^L, \bar{k}^U\}$, we assure the existence of $k > \bar{k}$ large enough such that $\bar{x} + \tau_k d_k \in \Omega \cap U$ and for all $i \in I$,

$$\begin{cases} F_i^L(\bar{x} + \tau_k d_k) < F_i^L(\bar{x}), \\ F_i^U(\bar{x} + \tau_k d_k) < F_i^U(\bar{x}), \end{cases}$$

i.e.,

$$F_i(\bar{x} + \tau_k d_k) <_{LU}^s F_i(\bar{x}),$$

which contradicts (3.4). Therefore, (3.5) holds. We deduce from (3.5) and (ACQ) that

$$\left(\bigcup_{i \in I} (\nabla F_i^L(\bar{x}) \cup \nabla F_i^U(\bar{x})) \right)^s \cap \left(\bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x}) \right)^- \subset \left(\bigcup_{i \in I} (\nabla F_i^L(\bar{x}) \cup \nabla F_i^U(\bar{x})) \right)^s \cap T(\Omega, \bar{x}) = \emptyset.$$

This leads that there is no $d \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \nabla F_i^L(\bar{x}), d \rangle < 0, & \forall i \in I, \\ \langle \nabla F_i^U(\bar{x}), d \rangle < 0, & \forall i \in I, \\ \langle \nabla g_t(\bar{x}), d \rangle \leq 0, & \forall t \in T(\bar{x}). \end{cases}$$

On the other hand, we derive from Lemma 2.10 that $\mathrm{co}\bigcup_{i\in I}(\nabla F_i^L(\bar{x})\cup\nabla F_i^U(\bar{x}))$ is a compact set, and hence, $\mathrm{co}\bigcup_{i\in I}(\nabla F_i^L(\bar{x})\cup\nabla F_i^U(\bar{x}))+\Delta$ is closed. According to Lemma 2.9, one has

$$0\in \mathrm{co}\bigcup_{i\in I}(\nabla F_i^L(\bar{x})\cup\nabla F_i^U(\bar{x}))+\mathrm{pos}\bigcup_{t\in T(\bar{x})}\nabla g_t(\bar{x}).$$

From Lemma 2.8, there exist $(\alpha^L,\alpha^U):=((\alpha_1^L,...,\alpha_m^L),(\alpha_1^U,...,\alpha_m^U))\in\mathbb{R}_+^m\times\mathbb{R}_+^m$ with $\sum_{i=1}^m(\alpha_i^L+\alpha_i^U)=1$ and $\lambda\in\Lambda(\bar{x})$ such that

$$\sum_{i=1}^m\alpha_i^L\nabla F_i^L(\bar{x})+\sum_{i=1}^m\alpha_i^U\nabla F_i^U(\bar{x})+\sum_{t\in T}\lambda_t\nabla g_t(\bar{x})=0.$$

This completes the proof. □

Now, we investigate a sufficient condition for (ACQ).

Definition 3.9. [19] The (SCQ) holds at $\bar{x}\in\Omega$ if the following conditions are simultaneously satisfied:

(i) g_t, $t\in T$, are convex functions,
(ii) T is a compact set,
(iii) the function $(t,x)\in T\times\mathbb{R}^n\to g_t(x)$ is continuous on $T\times\mathbb{R}^n$,
(iv) there is a $\hat{x}\in\mathbb{R}^n$ such that $g_t(\hat{x})<0$ for all $t\in T$.

The following Lemma can be derived from Theorem 7.9 in [19] with $\partial g_t(\bar{x})=\{\nabla g_t(\bar{x})\}$ for all $t\in T$, where ∂ denotes the subdifferential in the sense of convex analysis.

Lemma 3.10. *If (SCQ) holds at $\bar{x}\in\Omega$, then Δ is closed and $\bigcup_{t\in T(\bar{x})}\nabla g_t(\bar{x})=D(\Omega,\bar{x})^-$.*

Proposition 3.11. *If (SCQ) holds at $\bar{x}\in\Omega$, then (ACQ) holds at $\bar{x}$.*

Proof. It follows from Lemma 3.10 that Δ is closed and $\bigcup_{t\in T(\bar{x})}\nabla g_t(\bar{x})=D(\Omega,\bar{x})^-$. This implies that

$$\left(\bigcup_{t\in T(\bar{x})}\nabla g_t(\bar{x})\right)^-=D(\Omega,\bar{x})^{--}=\mathrm{cl\ cone}D(\Omega,\bar{x}).$$

Since $g_t,t\in T$, are convex, one gets that Ω is a convex set. Hence, by Remark 2.2, $\mathrm{cl\ cone}D(\Omega,\bar{x})=\mathrm{cl\ cone}(\Omega-\bar{x})=T(\Omega,\bar{x})$. So, (ACQ) holds at $\bar{x}$. □

Proposition 3.12. *Let $\bar{x}\in\Omega$. Assume that there exist $\alpha^L,\alpha^U\in\mathbb{R}_+^m$ with $\sum_{i=1}^m(\alpha_i^L+\alpha_i^U)=1$ and $\lambda\in\Lambda(\bar{x})$ such that*

$$\sum_{i=1}^m\alpha_i^L\nabla F_i^L(\bar{x})+\sum_{i=1}^m\alpha_i^U\nabla F_i^U(\bar{x})+\sum_{t\in T}\lambda_t\nabla g_t(\bar{x})=0. \tag{3.6}$$

(i) If $F_i,i\in I$, are LU-convex at $\bar{x}$ and $g_t,t\in T$, are convex at $\bar{x}$, then $\bar{x}\in WE(P,2)$.
(ii) If $F_i,i\in I$, are strongly LU-convex at $\bar{x}$ and $g_t,t\in T$, are convex at $\bar{x}$, then $\bar{x}\in E(P,1)$. So, $\bar{x}\in E(P,2)$ and $\bar{x}\in WE(P,1)$.

Proof. Since, $\bar{x} \in \Omega$ satisfying (3.6), there exists a finite subset J of $T(\bar{x})$ such that

$$\sum_{t \in J} \lambda_t \nabla g_t(\bar{x}) = -\left(\sum_{i=1}^{m} \alpha_i^L \nabla F_i^L(\bar{x}) + \sum_{i=1}^{m} \alpha_i^U \nabla F_i^U(\bar{x}) \right). \tag{3.7}$$

(i) Suppose to the contrary that $\bar{x}$ is not a weakly type-2 optimal solution of (P). Then, there exists a $\hat{x} \in \Omega$ satisfying

$$F_i(\hat{x}) <_{LU}^{s} F_i(\bar{x}), \forall i \in I,$$

or equivalently, for all $i \in I$,

$$F_i^L(\hat{x}) < F_i^L(\bar{x}) \text{ and } F_i^U(\hat{x}) < F_i^U(\bar{x}).$$

The above inequalities together with $\alpha^L, \alpha^U \in \mathbb{R}_+^m$ satisfying $\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1$ implies that

$$\sum_{i=1}^{m} \alpha_i^L (F_i^L(\hat{x}) - F_i^L(\bar{x})) + \sum_{i=1}^{m} \alpha_i^U (F_i^U(\hat{x}) - F_i^U(\bar{x})) < 0. \tag{3.8}$$

It follows from $F_i, i \in I$, are LU-convex at $\bar{x}$. From Remark 2.4 and Remark 2.7, one has

$$F_i^L(\hat{x}) - F_i^L(\bar{x}) \geq \langle \nabla F_i^L(\bar{x}), \hat{x} - \bar{x} \rangle, i \in I,$$

$$F_i^U(\hat{x}) - F_i^U(\bar{x}) \geq \langle \nabla F_i^U(\bar{x}), \hat{x} - \bar{x} \rangle, i \in I.$$

Hence, we derive from the above inequality, (3.7) and (3.8) that

$$\sum_{t \in J} \lambda_t \langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle = -\left(\sum_{i=1}^{m} \alpha_i^L \langle \nabla F_i^L(\bar{x}), \hat{x} - \bar{x} \rangle + \sum_{i=1}^{m} \alpha_i^U \langle \nabla F_i^U(\bar{x}), \hat{x} - \bar{x} \rangle \right) > 0. \tag{3.9}$$

One the other hand, since $\hat{x} \in \Omega$ and $g_t(\bar{x}) = 0$ for all $t \in J$, we get

$$g_t(\hat{x}) \leq g_t(\bar{x}), \forall t \in J.$$

Therefore, by the convexity of $g_t, t \in T$, at $\bar{x}$, one has

$$\sum_{t \in J} \lambda_t \langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle \leq \sum_{t \in J} \lambda_t (g_t(\hat{x}) - g_t(\bar{x})) \leq 0,$$

which contradicts (3.9).

(ii) Suppose to the contrary that $\bar{x}$ is not a type-1 optimal solution of (P). Then, there exists a $\hat{x} \in \Omega$ satisfying

$$F_i(\hat{x}) \leq_{LU} F_i(\bar{x}), \forall i \in I \text{ and } F_k(\hat{x}) <_{LU} F_i(\bar{x}) \text{ for at least one } k \in I,$$

or equivalently, $\begin{cases} F_i^L(\hat{x}) \leq F_i^L(\bar{x}) \\ F_i^U(\hat{x}) \leq F_i^U(\bar{x}) \end{cases}$ for all $i \in I$, and, for at least one $k \in I$,

$$\begin{cases} F_k^L(\hat{x}) \leq F_k^L(\bar{x}) \\ F_k^U(\hat{x}) < F_k^U(\bar{x}), \end{cases} \text{ or } \begin{cases} F_k^L(\hat{x}) < F_k^L(\bar{x}) \\ F_k^U(\hat{x}) \leq F_k^U(\bar{x}), \end{cases} \text{ or } \begin{cases} F_k^L(\hat{x}) < F_k^L(\bar{x}) \\ F_k^U(\hat{x}) < F_k^U(\bar{x}). \end{cases}$$

Hence, $\hat{x} \neq \bar{x}$. The above inequalities together with $\alpha^L, \alpha^U \in \mathbb{R}_+^m$ satisfying $\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1$ imply that

$$\sum_{i=1}^{m} \alpha_i^L (F_i^L(\hat{x}) - F_i^L(\bar{x})) + \sum_{i=1}^{m} \alpha_i^U (F_i^U(\hat{x}) - F_i^U(\bar{x})) \leq 0. \tag{3.10}$$

It follows from $F_i, i \in I$, are strongly LU-convex at $\bar{x}$. From Remark 2.4 and Remark 2.7, one has

$$F_i^L(\hat{x}) - F_i^L(\bar{x}) > \langle \nabla F_i^L(\bar{x}), \hat{x} - \bar{x} \rangle, i \in I,$$

$$F_i^U(\hat{x}) - F_i^U(\bar{x}) > \langle \nabla F_i^U(\bar{x}), \hat{x} - \bar{x} \rangle, i \in I.$$

Hence, we derive from the above inequality, (3.7) and (3.9) that

$$\sum_{t \in J} \lambda_t \langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle = -\left(\sum_{i=1}^{m} \alpha_i^L \langle \nabla F_i^L(\bar{x}), \hat{x} - \bar{x} \rangle + \sum_{i=1}^{m} \alpha_i^U \langle \nabla F_i^U(\bar{x}), \hat{x} - \bar{x} \rangle \right) > 0. \tag{3.11}$$

One the other hand, since $\hat{x} \in \Omega$ and $g_t(\bar{x}) = 0$ for all $t \in J$, we get

$$g_t(\hat{x}) \leq g_t(\bar{x}), \forall t \in J.$$

Therefore, by the convexity of $g_t, t \in T$, at $\bar{x}$, one has

$$\sum_{t \in J} \lambda_t \langle \nabla g_t(\bar{x}), \hat{x} - \bar{x} \rangle \leq \sum_{t \in J} \lambda_t (g_t(\hat{x}) - g_t(\bar{x})) \leq 0,$$

contradicting with (3.11). □

Example 3.13. Consider the following problem

$$\text{(P)} \qquad \begin{array}{ll} \text{LU}-\min F(x) = (F_1(x), F_2(x)) = ([x^2 + x, x^2 + 2], [2x^2, 2x^2 + 2x]) \\ \quad \text{s.t.} \quad g_t(x) = tx - t - 1 \leq 0, \quad t \in T := [-1, 1]. \end{array}$$

Then,

$$\begin{aligned} g_t(x) \leq 0, \forall t \in T \quad &\Leftrightarrow \quad \begin{cases} x \leq 1 + \frac{1}{t}, \ t \in (0,1] \\ x \geq 1 + \frac{1}{t}, \ t \in [-1, 0), \end{cases} \\ &\Leftrightarrow \quad \begin{cases} x \leq \min\limits_{t \in (0,1]} \left\{1 + \frac{1}{t}\right\}, \\ x \geq \max\limits_{t \in [-1,0)} \left\{1 + \frac{1}{t}\right\}, \end{cases} \\ &\Leftrightarrow \quad x \in [0, 2], \text{ i.e., } \Omega = [0, 2]. \end{aligned}$$

Let us take $\bar{x} = 0 \in \Omega$. We can check that $\bar{x} \in \text{loc}WE(P, 2)$. Moreover, by some calculations, one has

$$\nabla F_1^L(\bar{x}) = 1, \nabla F_1^U(\bar{x}) = 0, \nabla F_2^L(\bar{x}) = 0, \nabla F_2^L(\bar{x}) = 2,$$

$$\nabla g_t(x) = t, \forall t \in T, T(\bar{x}) = \{-1\}, \bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x}) = \{-1\}, \text{pos} \bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x}) = -\mathbb{R}_+,$$

$$\left(\bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x}) \right)^- = \mathbb{R}_+, T(\Omega, \bar{x}) = \mathbb{R}_+.$$

Hence, (ACQ) holds at $\bar{x}$ and all assumptions in Proposition 3.8 are satisfied. Now, let $\alpha_1^L = \alpha_1^U = \alpha_2^L = \alpha_2^U = 1/4$ and $\lambda : T \to \mathbb{R}$ be defined by

$$\lambda(t) = \begin{cases} 3/4, & \text{if } t = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\alpha^L, \alpha^U \in \mathbb{R}^2_+$ with $\sum\limits_{i=1}^{2} (\alpha_i^L + \alpha_i^U) = 1$, $\lambda \in \Lambda(\bar{x})$ and

$$\sum_{i=1}^{2} \alpha_i^L \nabla F_i^L(\bar{x}) + \sum_{i=1}^{2} \alpha_i^U \nabla F_i^U(\bar{x}) + \sum_{t \in T} \lambda_t \nabla g_t(\bar{x}) = \frac{1}{4}.1 + \frac{1}{4}.0 + \frac{1}{4}.0 + \frac{1}{4}.2 + \frac{3}{4}(-1) = 0,$$

i.e., the conclusion of Proposition 3.8 is satisfied.

On the other hand, we can check that $F_1^L, F_1^U, F_2^L, F_2^U$ are convex at $\bar{x}=0$ and $g_t, t\in T$, are convex at $\bar{x}$. Hence, all assumptions in Proposition 3.12 (i) hold. Then, it follows that $\bar{x}\in WE(P,2)$. Furthermore, $F_1^L, F_1^U, F_2^L, F_2^U$ are also strictly convex at $\bar{x}=0$. Thus, we deduce from Proposition 3.12 (ii) that $\bar{x}\in E(P,1)$.

4. Duality

In this section, we formulate a Mond-Weir [20] type dual problem and a Wolfe [21] type dual problem of (P) and explore weak and strong duality relations.

Let A_i, B_i be in $\mathscr{K}_C$ for all $i=1,\ldots,m$ and $\mathscr{A}:=(A_1,A_2,\ldots,A_m)$, $\mathscr{B}:=(B_1,B_2,\ldots,B_m)$. In what follows, we use the following notations for convenience:

$$\mathscr{A}\preceq_{LU}\mathscr{B}\Leftrightarrow\begin{cases}A_i\leq_{LU}B_i, & \forall i\in I,\\ A_k<_{LU}B_k, & \text{for at least one } k\in I,\end{cases}\qquad \mathscr{A}\npreceq_{LU}\mathscr{B} \text{ is the negation of } \mathscr{A}\preceq_{LU}\mathscr{B},$$

$\mathscr{A}\prec_{LU}^s\mathscr{B}\Leftrightarrow A_i<_{LU}^s B_i, \forall i\in I$, $\mathscr{A}\nprec_{LU}^s\mathscr{B}$ is the negation of $\mathscr{A}\prec_{LU}^s\mathscr{B}$.

Note that $\bar{x}\in \mathrm{loc}E(P,1)$ ($\bar{x}\in\mathrm{loc}WE(P,2)$) if there exists $U\in\mathscr{U}(\bar{x})$ such that there is no $x\in\Omega\cap U$ satisfying $F(x)\preceq_{LU}F(\bar{x})$ ($F(x)\prec_{LU}^s F(\bar{x})$).

4.1. **Mond-Weir duality.** For $u\in\mathbb{R}^n, (\alpha^L,\alpha^U)\in\mathbb{R}_+^m\times\mathbb{R}_+^m\setminus\{(0,0)\}$ and $\lambda\in\mathbb{R}_+^{|T|}$, denote

$$L(u,\alpha^L,\alpha^U,\lambda):=(F_1(u),\ldots,F_m(u))=([F_1^L(u),F_1^U(u)],\ldots,[F_m^L(u),F_m^U(u)]).$$

We consider the Mond-Weir dual problem of (P) as follows

$$\begin{aligned}(D_{MW})\qquad & \mathrm{LU-max}\, L(u,\alpha^L,\alpha^U,\lambda)=(F_1(u),\ldots,F_m(u))\\ & \text{s.t.}\quad \sum_{i=1}^m\alpha_i^L\nabla F_i^L(u)+\sum_{i=1}^m\alpha_i^U\nabla F_i^U(u)+\sum_{t\in T}\lambda_t\nabla g_t(u)=0,\\ &\qquad\ \ \sum_{t\in T}\lambda_t g_t(u)\geq 0,\\ &\qquad\ \ u\in\mathbb{R}^n,(\alpha^L,\alpha^U)\in\mathbb{R}_+^m\times\mathbb{R}_+^m\setminus\{(0,0)\},\ \lambda\in\mathbb{R}_+^{|T|}.\end{aligned}$$

The feasible set of (D_{MW}) is

$$\begin{aligned}\Omega_{MW}:=\{&(u,\alpha^L,\alpha^U,\lambda)\in\mathbb{R}^n\times\mathbb{R}_+^m\times\mathbb{R}_+^m\times\mathbb{R}_+^{|T|}\mid(\alpha^L,\alpha^U)\neq(0,0),\\ &\sum_{i=1}^m\alpha_i^L\nabla F_i^L(u)+\sum_{i=1}^m\alpha_i^U\nabla F_i^U(u)+\sum_{t\in T}\lambda_t\nabla g_t(u)=0,\ \sum_{t\in T}\lambda_t g_t(u)\geq 0\}.\end{aligned}$$

Definition 4.1. Let $(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\in\Omega_{MW}$.

(i) $(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})$ is a type-1 optimal solution of (D_{MW}), denoted by

$$(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\in E(D_{MW},1)$$

if there is no $(u,\alpha^L,\alpha^U,\lambda)\in\Omega_{MW}$ such that $L(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\preceq_{LU}L(u,\alpha^L,\alpha^U,\lambda)$.

(ii) $(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})$ is a weakly type-2 optimal solution of (D_{MW}), denoted by

$$(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\in WE(D_{MW},2)$$

if there is no $(u,\alpha^L,\alpha^U,\lambda)\in\Omega_{MW}$ such that $L(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\prec_{LU}^s L(u,\alpha^L,\alpha^U,\lambda)$.

Proposition 4.2. *(Weak duality) Let $x\in\Omega$ and $(u,\alpha^L,\alpha^U,\lambda)\in\Omega_{MW}$.*

(i) If $F_i, i\in I$, are LU-convex and $g_t, t\in T$, are convex, then $F(x)\nprec_{LU}^s L(u,\alpha^L,\alpha^U,\lambda)$.

(ii) If $F_i, i\in I$, are strongly LU-convex and $g_t, t\in T$, are convex, then $F(x)\npreceq_{LU}L(u,\alpha^L,\alpha^U,\lambda)$.

Proof. Since $x \in \Omega$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW}$, one has

$$g_t(x) \leq 0, \forall t \in T, \tag{4.1}$$

$$\sum_{t \in T} \lambda_t g_t(u) \geq 0, \tag{4.2}$$

$$\sum_{i=1}^{m} \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^{m} \alpha_i^U \nabla F_i^U(u) + \sum_{t \in T} \lambda_t \nabla g_t(u) = 0. \tag{4.3}$$

(i) Suppose to the contrary that

$$F(x) \prec_{LU}^s L(u, \alpha^L, \alpha^U, \lambda) = F(u),$$

i.e., $F_i(x) <_{LU}^s F_i(u), \forall i \in I$. Then, $F_i^L(x) < F_i^L(u)$ and $F_i^U(x) < F_i^U(u), \forall i \in I$. Hence, for $(\alpha^L, \alpha^U) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \setminus \{(0,0)\}$, we obtain

$$\sum_{i=1}^{m} \alpha_i^L F_i^L(x) + \sum_{i=1}^{m} \alpha_i^U F_i^U(x) < \sum_{i=1}^{m} \alpha_i^L F_i^L(u) + \sum_{i=1}^{m} \alpha_i^U F_i^U(u). \tag{4.4}$$

Since $F_i^L, F_i^U, i \in I$, are convex, we have

$$F_i^L(x) - F_i^L(u) \geq \langle \nabla F_i^L(u), x - u \rangle, i \in I,$$

$$F_i^U(x) - F_i^U(u) \geq \langle \nabla F_i^U(u), x - u \rangle, i \in I.$$

The above inequalities and (4.4) yield

$$\left\langle \sum_{i=1}^{m} \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^{m} \alpha_i^U \nabla F_i^U(u), x - u \right\rangle < 0. \tag{4.5}$$

On the other hand, from (4.1), (4.2) and $\lambda \in \mathbb{R}_+^{|T|}$, one has

$$\sum_{t \in T} \lambda_t g_t(x) \leq 0 \leq \sum_{t \in T} \lambda_t g_t(u).$$

It follows from the convexity of $g_t, t \in T$, and the above inequality that

$$\sum_{t \in T} \lambda_t \langle \nabla g_t(u), x - u \rangle \leq 0. \tag{4.6}$$

By adding (4.5) and (4.6),

$$\left\langle \sum_{i=1}^{m} \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^{m} \alpha_i^U \nabla F_i^U(u) + \sum_{t \in T} \lambda_t \nabla g_t(u), x - u \right\rangle < 0,$$

which contradicts (4.3).

(ii) Suppose to the contrary that $F(x) \preceq_{LU} L(u, \alpha^L, \alpha^U, \lambda) = F(u)$, i.e.,

$$\begin{cases} F_i(x) \leq_{LU} F_i(u), & \forall i \in I, \\ F_k(x) <_{LU} F_k(u), & \text{for at least one } k \in I. \end{cases}$$

This implies that $F_i^L(x) \leq F_i^L(u)$, $F_i^U(x) \leq F_i^U(u)$ for all $i \in I$, and for at least one $k \in I$,

$$\begin{cases} F_k^L(x) < F_k^L(u) \\ F_k^U(x) \leq F_k^U(u), \end{cases} \text{or} \begin{cases} F_k^L(x) \leq F_k^L(u) \\ F_k^U(x) < F_k^U(u), \end{cases} \text{or} \begin{cases} F_k^L(x) < F_k^L(u) \\ F_k^U(x) < F_k^U(u). \end{cases}$$

Hence, $x \neq u$. For $(\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$, we obtain

$$\sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) \leq \sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u). \tag{4.7}$$

Since $F_i^L, F_i^U, i \in I$ are strictly convex and $x \neq u$, we have

$$F_i^L(x) - F_i^L(u) > \langle \nabla F_i^L(u), x-u \rangle, i \in I,$$
$$F_i^U(x) - F_i^U(u) > \langle \nabla F_i^U(u), x-u \rangle, i \in I.$$

The above inequalities and (4.7) yield

$$\left\langle \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u), x-u \right\rangle < 0. \tag{4.8}$$

On the other hand, one has from (4.1), (4.2) and $\lambda \in \mathbb{R}^{|T|}_+$ that

$$\sum_{t\in T} \lambda_t g_t(x) \leq 0 \leq \sum_{t\in T} \lambda_t g_t(u).$$

It follows from the convexity of $g_t, t \in T$, and the above inequality that

$$\sum_{t\in T} \lambda_t \langle \nabla g_t(u), x-u \rangle \leq 0. \tag{4.9}$$

By adding (4.8) and (4.9),

$$\left\langle \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u), x-u \right\rangle < 0,$$

which contradicts (4.3). □

Proposition 4.3. *(Strong duality) Let $\bar{x} \in \mathrm{loc}WE(P,2)$ and (ACQ) holds at $\bar{x}$. Then, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{MW}$ and $F(\bar{x}) = L(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$. Furthermore,*

- *(i) If $F_i, i \in I$, are LU-convex and $g_t, t \in T$, are convex, then $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in WE(D_{MW}, 2)$.*
- *(ii) If $F_i, i \in I$, are strongly LU-convex and $g_t, t \in T$, are convex, then $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{MW}, 1)$.*

Proof. It follows from Proposition 3.8 that there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$\sum_{i=1}^m \bar{\alpha}_i^L \nabla F_i^L(\bar{x}) + \sum_{i=1}^m \bar{\alpha}_i^U \nabla F_i^U(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \nabla g_t(\bar{x}) = 0.$$

Since $\bar{\lambda} \in \Lambda(\bar{x})$, one has $\bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$ and $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) = 0$. Hence, $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{MW}$.

(i) Suppose to the contrary that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \notin WE(D_{MW}, 2)$. By definition, there exists $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW}$ such that

$$F(\bar{x}) = L(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \prec^s_{LU} L(u, \alpha^L, \alpha^U, \lambda),$$

which contradicts Proposition 4.2 (i). So, $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in WE(D_{MW}, 2)$.

(ii) Suppose to the contrary that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \notin E(D_{MW}, 1)$. By definition, there exists $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW}$ such that

$$F(\bar{x}) = L(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq_{LU} L(u, \alpha^L, \alpha^U, \lambda),$$

which contradicts with Proposition 4.2 (ii). So, $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{MW}, 1)$. □

4.2. **Wolfe duality.** For $u \in \mathbb{R}^n, (\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ with $\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1$, $\lambda \in \mathbb{R}^{|T|}_+$ and $e = (1,..,1) \in \mathbb{R}^m$, denote

$$\widetilde{L}(u, \alpha^L, \alpha^U, \lambda) := F(u) + \sum_{t\in T} \lambda_t g_t(u) e = \left([F_1^L(u), F_1^U(u)] + \sum_{t\in T} \lambda_t g_t(u), ..., [F_m^L(u), F_m^U(u)] + \sum_{t\in T} \lambda_t g_t(u) \right).$$

Note that, $\forall b \in \mathbb{R}$, $[F_i^L(u), F_i^U(u)] + b = [F_i^L(u) + b, F_i^U(u) + b], \forall i \in I$. We define the Wolfe type dual problem as follows:

$$\begin{aligned} (D_W) \qquad & \text{LU}-\max \widetilde{L}(u, \alpha^L, \alpha^U, \lambda) = (F_1(u), ..., F_m(u)) + \sum_{t\in T} \lambda_t g_t(u) e \\ \text{s.t.} \quad & \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u) = 0 \\ & u \in \mathbb{R}^n, (\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}, \sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1, \lambda \in \mathbb{R}^{|T|}_+. \end{aligned}$$

The feasible set of (D_W) is

$$\begin{aligned} \Omega_W := \{ (u, \alpha^L, \alpha^U, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \mid & \sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1, \\ & \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u) = 0 \}. \end{aligned}$$

Definition 4.4. Let $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_W$.

(i) $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$ is a type-1 optimal solution of (D_W), denoted by $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_W, 1)$, if there is no $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$ such that

$$\widetilde{L}(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda).$$

(ii) $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$ is a weakly type-2 optimal solution of (D_W), denoted by $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in WE(D_W, 2)$, if there is no $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$ such that

$$\widetilde{L}(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \prec^s_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda).$$

The following proposition describes weak duality relation between the primal problem (P) and the dual problem (D_W).

Proposition 4.5. *(Weak duality) Let $x \in \Omega$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$.*

(i) If $F_i, i \in I$, are LU-convex and $g_t, t \in T$, are convex, then $F(x) \not\prec^s_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda)$.
(ii) If $F_i, i \in I$, are strongly LU-convex and $g_t, t \in T$, are convex, then $F(x) \not\preceq_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda)$.

Proof. Since $x \in \Omega$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$, one has

$$g_t(x) \leq 0, \forall t \in T, \tag{4.10}$$

$$\sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u) = 0. \tag{4.11}$$

(i) Suppose to the contrary that

$$F(x) \prec^s_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda) = F(u) + \sum_{t\in T} \lambda_t g_t(u) e.$$

Then, one has

$$F_i(x) <^s_{LU} F_i(u) + \sum_{t\in T} \lambda_t g_t(u), \forall i \in I,$$

or equivalently,

$$F_i^L(x) < F_i^L(u) + \sum_{t\in T} \lambda_t g_t(u) \text{ and } F_i^U(x) < F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u), \forall i \in I.$$

Since $(\alpha^L, \alpha^U) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \setminus \{(0,0)\}$ with $\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1$, the above inequalities and (4.10) imply

$$\begin{aligned}
& \sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) + \sum_{t\in T} \lambda_t g_t(x) \\
\leq \;& \sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) \\
< \;& \sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u) + \sum_{i=1}^m (\alpha_i^L + \alpha_i^U). \sum_{t\in T} \lambda_t g_t(u) \\
= \;& \sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u). \qquad (4.12)
\end{aligned}$$

Since $F_i^L, F_i^U, i \in I$, are convex and $g_t, t \in T$, are convex, we have

$$\begin{aligned}
F_i^L(x) - F_i^L(u) &\geq \langle \nabla F_i^L(u), x-u \rangle, i \in I, \\
F_i^U(x) - F_i^U(u) &\geq \langle \nabla F_i^U(u), x-u \rangle, i \in I, \\
g_t(x) - g_t(u) &\geq \langle \nabla g_t(u), x-u \rangle, t \in T.
\end{aligned}$$

The above inequalities lead that

$$\begin{aligned}
&\left(\sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) + \sum_{t\in T} \lambda_t g_t(x) \right) - \left(\sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u) \right) \\
&\geq \left\langle \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u), x-u \right\rangle. \qquad (4.13)
\end{aligned}$$

From (4.12) and (4.13), one obtains

$$\left\langle \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u), x-u \right\rangle < 0,$$

contradicting (4.11).

(ii) Suppose to the contrary that

$$F(x) \preceq_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda) = F(u) + \sum_{t\in T} \lambda_t g_t(u) e,$$

i.e.,

$$\begin{cases} F_i(x) \leq_{LU} F_i(u) + \sum_{t\in T} \lambda_t g_t(u), & \forall i \in I, \\ F_k(x) <_{LU} F_k(u) + \sum_{t\in T} \lambda_t g_t(u), & \text{for at least one } k \in I. \end{cases}$$

This implies that $F_i^L(x) \leq F_i^L(u) + \sum_{t\in T} \lambda_t g_t(u)$, $F_i^U(x) \leq_{LU} F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u)$ for all $i \in I$, and for at least one $k \in I$,

$$\begin{cases} F_k^L(x) < F_k^L(u) + \sum_{t\in T} \lambda_t g_t(u) \\ F_k^U(x) \leq F_k^U(u) + \sum_{t\in T} \lambda_t g_t(u), \end{cases} \text{or} \begin{cases} F_k^L(x) \leq F_k^L(u) + \sum_{t\in T} \lambda_t g_t(u) \\ F_k^U(x) < F_k^U(u) + \sum_{t\in T} \lambda_t g_t(u), \end{cases}$$

$$\text{or} \begin{cases} F_k^L(x) < F_k^L(u) + \sum_{t\in T} \lambda_t g_t(u) \\ F_k^U(x) < F_k^U(u) + \sum_{t\in T} \lambda_t g_t(u). \end{cases}$$

Hence, we deduce from the above inequalities and (4.10) that $x \neq u$. For $(\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ with $\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1$, we obtain

$$\begin{aligned} & \sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) + \sum_{t\in T} \lambda_t g_t(x) \\ \leq\ & \sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) \\ \leq\ & \sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u) + \sum_{i=1}^m (\alpha_i^L + \alpha_i^U). \sum_{t\in T} \lambda_t g_t(u) \\ =\ & \sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u). \end{aligned} \tag{4.14}$$

Since $F_i^L, F_i^U, i \in I$, are strictly convex and $x \neq u$, we have

$$F_i^L(x) - F_i^L(u) > \langle \nabla F_i^L(u), x-u \rangle, i \in I,$$

$$F_i^U(x) - F_i^U(u) > \langle \nabla F_i^U(u), x-u \rangle, i \in I.$$

$$g_t(x) - g_t(u) \geq \langle \nabla g_t(u), x-u \rangle, t \in T.$$

The above inequalities and (4.14) yield

$$\begin{aligned} & \left(\sum_{i=1}^m \alpha_i^L F_i^L(x) + \sum_{i=1}^m \alpha_i^U F_i^U(x) + \sum_{t\in T} \lambda_t g_t(x) \right) - \left(\sum_{i=1}^m \alpha_i^L F_i^L(u) + \sum_{i=1}^m \alpha_i^U F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u) \right) \\ & > \left\langle \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u), x-u \right\rangle. \end{aligned} \tag{4.15}$$

From (4.14) and (4.15), one obtains

$$\left\langle \sum_{i=1}^m \alpha_i^L \nabla F_i^L(u) + \sum_{i=1}^m \alpha_i^U \nabla F_i^U(u) + \sum_{t\in T} \lambda_t \nabla g_t(u), x-u \right\rangle < 0,$$

contradicting (4.11). □

Proposition 4.6. *(Strong duality) Let $\bar{x} \in \operatorname{loc}WE(P,2)$ and (ACQ) holds at $\bar{x}$. Then, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ with $\sum_{i=1}^{m} (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_W$ and $F(\bar{x}) = L(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$. Furthermore,*

(i) If $F_i, i \in I$, are LU-convex and $g_t, t \in T$, are convex, then $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in WE(D_W, 2)$.
(ii) If $F_i, i \in I$, are strongly LU-convex and $g_t, t \in T$, are convex, then $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_W, 1)$.

Proof. By Proposition 3.8, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ with $\sum_{i=1}^{m} (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$ and $\bar{\lambda} \in \Lambda(\bar{x})$ such that

$$\sum_{i=1}^m \bar{\alpha}_i^L \nabla F_i^L(\bar{x}) + \sum_{i=1}^m \bar{\alpha}_i^U \nabla F_i^U(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t \nabla g_t(\bar{x}) = 0.$$

Since $\bar{\lambda} \in \Lambda(\bar{x})$, $\bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$, one has

$$F(\bar{x}) = F(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}),$$

i.e., $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_W$ and $F(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$.
(i) Suppose to the contrary that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \notin WE(D_W, 2)$. We find from the definition that there exists $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$ such that

$$F(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \prec^s_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda),$$

which contradicts with Proposition 4.5 (i). So, $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in WE(D_W, 2)$.
(ii) Suppose to the contrary that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \notin E(D_W, 1)$. It follows from the definition that there exists $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$ such that

$$F(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \prec_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda),$$

which contradicts with Proposition 4.5 (ii). So, $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_W, 1)$. □

Example 4.7. Consider the following problem

$$\text{(P)} \qquad \begin{aligned} &\text{LU}-\min F(x) = (F_1(x), F_2(x)) = ([x^2+x, x^2+2], [2x^2, 2x^2+2x]) \\ &\text{s.t.} \quad g_t(x) = tx - t - 1 \le 0, \quad t \in T := [-1, 1]. \end{aligned}$$

Then, the corresponding Wolfe type dual (D_W) is

$$(D_W): \qquad \begin{aligned} &\text{LU}-\max \widetilde{L}(u, \alpha^L, \alpha^U, \lambda) = ([u^2+u, u^2+2], [u^2, 2u^2+2u]) + \sum_{t \in T} \lambda_t (tu - t - 1)(1,1) \\ &\text{s.t.} \quad \alpha_1^L(2u+1) + \alpha_1^U.2u + \alpha_2^L.4u + \alpha_2^U.(4u+2) + \sum_{t \in T} \lambda_t t = 0 \\ &\qquad u \in \mathbb{R}, (\alpha^L, \alpha^U) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \setminus \{(0,0)\}, \sum_{i=1}^2 (\alpha_i^L + \alpha_i^U) = 1, \lambda \in \mathbb{R}^{|T|}_+. \end{aligned}$$

It follows from Example 3.13 that $\Omega = [0, 2]$ and for $\bar{x} = 0 \in \Omega$,

$$\nabla F_1^L(\bar{x}) = 1, \nabla F_1^U(\bar{x}) = 0, \nabla F_2^L(\bar{x}) = 0, \nabla F_2^L(\bar{x}) = 2,$$

$$\nabla g_t(x) = t, \forall t \in T, T(\bar{x}) = \{-1\}, \bigcup_{t \in T(\bar{x})} \nabla g_t(\bar{x}) = \{-1\},$$

i.e., (ACQ) holds at $\bar{x}$ and $\bar{x} \in \text{loc}WE(P, 2)$. Hence, all assumptions in Proposition 4.6 are satisfied. Now, let $\bar{\alpha}_1^L = \bar{\alpha}_1^U = \bar{\alpha}_2^L = \bar{\alpha}_2^U = 1/4$ and $\bar{\lambda} : T \to \mathbb{R}$ be defined by

$$\bar{\lambda}(t) = \begin{cases} 3/4, & \text{if } t = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\bar{\alpha}^L, \bar{\alpha}^U \in \mathbb{R}^2_+$ with $\sum_{i=1}^2 (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$, $\bar{\lambda} \in \Lambda(\bar{x})$ and

$$\sum_{i=1}^2 \bar{\alpha}_i^L \nabla F_i^L(\bar{x}) + \sum_{i=1}^2 \bar{\alpha}_i^U \nabla F_i^U(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \nabla g_t(\bar{x}) = \frac{1}{4}.1 + \frac{1}{4}.0 + \frac{1}{4}.0 + \frac{1}{4}.2 + \frac{3}{4}(-1) = 0,$$

i.e., $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_W$. Morever, since $T(\bar{x}) = \{-1\}$,

$$\bar{\lambda}_t g_t(\bar{x}) = \begin{cases} 3/4.g_{-1}(\bar{x}) = 3/4.0 = 0, & \text{if } t = -1, \\ 0.g_t(\bar{x}) = 0.(-t-1) = 0, & \text{otherwise,} \end{cases}$$

i.e., $\bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$. Therefore,

$$F(\bar{x}) = F(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}) = \widetilde{L}(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) = ([0,2],[0,0]).$$

Moreover, we can check that $F_1^L, F_1^U, F_2^L, F_2^U$ are convex at $\bar{x} = 0$ and $g_t, t \in T$, are convex at $\bar{x}$. Hence, all assumptions in Proposition 4.6 (i) hold. Then, it follows that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in WE(D_W, 2)$. Similarly, one can see that $F_1^L, F_1^U, F_2^L, F_2^U$ are also strictly convex at $\bar{x} = 0$. Thus, we deduce from Proposition 4.6 (ii) that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_W, 1)$.

We can check directly that $(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_W, 1)$ as follows. Since, $\bar{x} \in \Omega$, one has $g_t(\bar{x}) \le 0, t \in T$, and hence, for $\lambda \in \mathbb{R}_+^{|T|}$,

$$\sum_{t\in T} \lambda_t g_t(\bar{x}) \le 0. \tag{4.16}$$

Suppose to the contrary that there is $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_W$ such that

$$F(\bar{x}) = \widetilde{L}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq_{LU} \widetilde{L}(u, \alpha^L, \alpha^U, \lambda),$$

i.e.,

$$\begin{cases} F_i(\bar{x}) \le_{LU} F_i(u) + \sum\limits_{t\in T} \lambda_t g_t(u), & \forall i \in I = \{1,2\}, \\ F_k(\bar{x}) <_{LU} F_k(u) + \sum\limits_{t\in T} \lambda_t g_t(u), & \text{for at least one } k \in I. \end{cases}$$

This implies that $F_i^L(\bar{x}) \le F_i^L(u) + \sum\limits_{t\in T} \lambda_t g_t(u)$, $F_i^U(\bar{x}) \le_{LU} F_i^U(u) + \sum\limits_{t\in T} \lambda_t g_t(u)$ for $i = 1,2$, and for at least one $k \in I = \{1,2\}$,

$$\begin{cases} F_k^L(\bar{x}) < F_k^L(u) + \sum\limits_{t\in T} \lambda_t g_t(u) \\ F_k^U(\bar{x}) \le F_k^U(u) + \sum\limits_{t\in T} \lambda_t g_t(u), \end{cases} \text{or} \begin{cases} F_k^L(\bar{x}) \le F_k^L(u) + \sum\limits_{t\in T} \lambda_t g_t(u) \\ F_k^U(\bar{x}) < F_k^U(u) + \sum\limits_{t\in T} \lambda_t g_t(u), \end{cases}$$

$$\text{or} \begin{cases} F_k^L(\bar{x}) < F_k^L(u) + \sum\limits_{t\in T} \lambda_t g_t(u) \\ F_k^U(\bar{x}) < F_k^U(u) + \sum\limits_{t\in T} \lambda_t g_t(u). \end{cases}$$

Hence, we deduce from the above inequalities and (4.16) that $\bar{x} \ne u$. For $(\alpha^L, \alpha^U) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \setminus \{(0,0)\}$ with $\sum\limits_{i=1}^{2} (\alpha_i^L + \alpha_i^U) = 1$, one has

$$\sum_{i=1}^{2} \alpha_i^L F_i^L(\bar{x}) + \sum_{i=1}^{2} \alpha_i^U F_i^U(\bar{x}) \le \sum_{i=1}^{2} \alpha_i^L F_i^L(u) + \sum_{i=1}^{2} \alpha_i^U F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u). \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\sum_{i=1}^{2} \alpha_i^L F_i^L(\bar{x}) + \sum_{i=1}^{2} \alpha_i^U F_i^U(\bar{x}) + \sum_{t\in T} \lambda_t g_t(\bar{x}) \le \sum_{i=1}^{2} \alpha_i^L F_i^L(u) + \sum_{i=1}^{2} \alpha_i^U F_i^U(u) + \sum_{t\in T} \lambda_t g_t(u). \tag{4.18}$$

Since $F_i^L, F_i^U, i \in \{1,2\}$, are strictly convex, $\bar{x} \ne u$ and $g_t, t \in T$, are convex, one has

$$(\bar{x}^2 + \bar{x}) - (u^2 + u) = F_1^L(\bar{x}) - F_1^L(u) > \langle \nabla F_1^L(u), \bar{x} - u \rangle = (2u+1)(\bar{x} - u),$$

$$(\bar{x}^2 + 1) - (u^2 + 1) = F_2^L(\bar{x}) - F_2^L(u) > \langle \nabla F_2^L(u), \bar{x} - u \rangle = 2u(\bar{x} - u),$$

$$2\bar{x}^2 - 2u^2 = F_1^U(\bar{x}) - F_1^U(u) > \langle \nabla F_1^U(u, \bar{x} - u \rangle = 4u(\bar{x} - u),$$

$$(2\bar{x}^2 + 2\bar{x}) - (2u^2 + 2u) = F_2^U(\bar{x}) - F_2^U(u) > \langle \nabla F_2^U(u), \bar{x} - u \rangle = (4u+2)(\bar{x} - u),$$

$$(t\bar{x} - t - 1) - (tu - t - 1) = g_t(\bar{x}) - g(u) \ge \langle \nabla g_t(u), \bar{x} - u \rangle = t(\bar{x} - u), \forall t \in T.$$

Hence, for $(\alpha^L,\alpha^U)\in\mathbb{R}^2_+\times\mathbb{R}^2_+\setminus\{(0,0)\}$ with $\sum\limits_{i=1}^{2}(\alpha_i^L+\alpha_i^U)=1$, we derive from (4.18) that

$$0\geq\left(\sum_{i=1}^{2}\alpha_i^LF_i^L(\bar{x})+\sum_{i=1}^{2}\alpha_i^UF_i^U(\bar{x})+\sum_{t\in T}\lambda_tg_t(\bar{x})\right)-\left(\sum_{i=1}^{2}\alpha_i^LF_i^L(u)+\sum_{i=1}^{2}\alpha_i^UF_i^U(u)+\sum_{t\in T}\lambda_tg_t(u)\right)$$
$$>\left(\alpha_1^L(2u+1)+\alpha_2^L.2u+\alpha_1^U.4u+\alpha_2^U.(4u+2)+\sum_{t\in T}\lambda_tt\right)(\bar{x}-u). \tag{4.19}$$

Since, $(u,\alpha^L,\alpha^U,\lambda)\in\Omega_W$, we have

$$\alpha_1^L(2u+1)+\alpha_2^L.2u+\alpha_1^U.4u+\alpha_2^U.(4u+2)+\sum_{t\in T}\lambda_tt=0.$$

Hence

$$\left(\alpha_1^L(2u+1)+\alpha_2^L.2u+\alpha_1^U.4u+\alpha_2^U.(4u+2)+\sum_{t\in T}\lambda_tt\right)(\bar{x}-u)=0,$$

which contradicts (4.19).

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