J. Nonlinear Funct. Anal. 2019 (2019), Article ID 26 https://doi.org/10.23952/jnfa.2019.26

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

EXTRAGRADIENT ALGORITHMS FOR SPLIT PSEUDOMONOTONE EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

SHENGHUA WANG[1,*], YIFAN ZHANG[1], WENXIN WANG[1], HAICHAO GUO[2]

[1]Department of Mathematics and Physics, North China Electric Power University, Baoding 071003, China

[2]Department of Electrical Engineering, North China Electric Power University, Baoding 071003, China

Abstract. In this paper, we propose some extragradient iterative algorithms to find a common solution for split pseudomonotone equilibrium problems and the fixed point problems of nonself nonexpansive mappings and prove the weak and strong convergence of the proposed algorithms in Hilbert spaces. A numerical example in an infinite dimensional space is given to illustrate our main convergence theorems.

Keywords. Split equilibrium problem; Pseudomonotone equilibrium problem; Equilibrium problem; Split feasible problem.

2010 Mathematics Subject Classification. 47H05, 65K15.

1. Introduction

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T: C \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \ \forall x, y \in C.$$

Denote the fixed point set of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : x = Tx\}$. Let $f: C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem is to find $z \in C$ such that

$$f(z, y) \ge 0, \ \forall y \in C.$$

The set of all solutions of the equilibrium problem is denoted by $EP(f)$, that is,

$$EP(f) = \{z \in C : f(z, y) \ge 0, \ \forall y \in C\}.$$

The equilibrium problem is said to be monotone if bifunction f is monotone, that is,

$$f(x, y) + f(y, x) \le 0, \ \forall x, y \in C.$$

*Corresponding author.

E-mail addresses: sheng-huawang@163.com (S. Wang).

Received January 23, 2019; Accepted June 16, 2019.

In 2007, Takahashi and Takahashi [1] introduced the following viscosity approximation method to find a common solution of a monotone equilibrium problem and a fixed point problem of a nonexpansive mapping in a Hilbert space:

$$\begin{cases} x_0 \in H, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)Tu_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $h : H \to H$ is a contraction, $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$. They proved that $\{x_n\}$ generated by (1.1) strongly converges to the element $z = P_{EP(f)\cap Fix(T)}h(z)$ under some certain conditions on $\{\alpha_n\}$ and $\{r_n\}$.

Another strong convergent algorithm called the hybrid method was introduced by Tada and Takahashi [2]:

$$\begin{cases} x_0 \in C_0 = D_0 = C, \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n T u_n, \\ C_n = \{v \in H : \|z_n - v\| \leq \|x_n - v\|\}, \\ D_n = \{v \in H : \langle x_0 - x_n, v - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$. They proved that $\{x_n\}$ generated by (1.2) strongly converges to the element $P_{EP(f)\cap Fix(T)}x_0$ under some certain conditions on $\{\alpha_n\}$ and $\{r_n\}$.

On more iterative algorithms for the monotone equilibrium problem and the fixed point problem, we refer the readers to [3, 4, 5, 6, 7, 8] and the references therein.

The split inverse problem (SIP) was first introduced by Censor and Elving [9] for investigating a class of inverse problems which arise in signal processing and image restoration problems. Let X and Y be two spaces and A be a bounded linear operator from X to Y. There are two inverse problems involved in X and Y denoted by IP_1 and IP_2, respectively. The SIP is stated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \text{IP}_1$$
$$\text{and such that}$$
$$\text{the point } y^* = Ax^* \in Y \text{ solves } \text{IP}_2.$$

Note that SIPs can be split variational inequality problems and split equilibrium problems if IP_1 and IP_2 are variational inequality problems and equilibrium problems, respectively. Censor, Gibali and Reich [10] first investigated a split variational inequality problem in Hilbert spaces. Let H_1 and H_2 be Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be the given operators. Let $C \subset H_1$ and $Q \subset H_2$ be the nonempty closed and convex subsets. The split variational inequality problem is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^*\rangle \geq 0 \ \forall x \in C$$
$$\text{and such that}$$

$$\text{the point } y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \ \forall y \in Q.$$

In 2012, He [11] introduced a split equilibrium problem for the monotone bifunction which was also mentioned in [12]. Let H_1 and H_2 be two real Hilbert spaces. Let C and K be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be two monotone bifunctions. The split equilibrium problem is to find $x^* \in C$ such that

$$f(x^*, x) \geq 0, \ \forall x \in C,$$

and

$$y^* = Ax^* \in K \text{ such that } g(y^*, y) \geq 0, \ \forall y \in K.$$

In [11], He constructed the following iterative algorithm to solve the split monotone equilibrium problem:

$$\begin{cases} x_1 \in C_1 = C, \\ f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ g(w_n, z) + \dfrac{1}{r_n}\langle z - w_n, w_n - Au_n \rangle \geq 0, \ \forall z \in K, \\ y_n = P_C\big(u_n + \mu A^*(w_n - Au_n)\big), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \ \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $A^* : H_2 \to H_1$ is the adjoint operator of A, $\mu \in (0, 1/\|A^*\|^2)$ and $\{r_n\} \subset (0, \infty)$. He proved that $\{x_n\}$ generated by (1.13) strongly converges to an element $x^* \in \Omega$, where $\Omega = \{x \in C : x \in EP(f) \text{ and } Ax \in EP(g)\}$.

Subsequently, many authors investigated iterative algorithms for approximating solutions of the split monotone equilibrium problem; see [13, 14, 15, 16, 17, 18] and the references therein. However, for finding solutions of the monotone equilibrium problem or the split monotone equilibrium problem, we have to face the difficulty of computing u_n at each step by solving the following problem: find $u_n \in C$ such that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C. \tag{1.4}$$

In fact, it is very difficult to obtain u_n by the above inequality. An equilibrium problem is said to be pseudomonotone if the bifunction f is pseudomonotone, that is,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \ \forall x, y \in C.$$

Obviously, the class of pesudomonotone equilibrium problems includes that of monotone equilibrium problems.

The pseudomonotone equilibrium problem was recently investigated by many authors; see [19, 20, 21, 22, 23] and the reference therein.

In 2008, Tran, Muu and Nguyen [24] introduced the following extragradient method to solve the pseudomonotone equilibrium problem: given $x_0 \in C$, find successively y_n and x_{n+1} by

$$\begin{cases} y_n = \operatorname{argmin}_{y \in C}\{\lambda_n f(x_n, y) + \dfrac{1}{2}\|y - x_n\|^2\}, \\ x_{n+1} = \operatorname{argmin}_{y \in C}\{\lambda_n f(y_n, y) + \dfrac{1}{2}\|y - x_n\|^2\}, \ \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\lambda_n\}$ is a sequence in $(0,1]$ with some conditions and f satisfies the Lipschitz-type property. They proved that the sequence $\{x_n\}$ generated by (1.5) strongly converges to some $x^* \in EP(f)$ under some certain conditions. Note that their convergence theorems were established in the framework of $\mathbb{R}^m$.

In 2012, Vuong, Strodiot and Nguyen [25] considered a hybrid projection algorithm for finding the common element of the fixed point set of a pseudocontraction S and the solution set of the equilibrium problem on the pseudomonotone bifunction f by the following manner: $x_0 \in C$ and

$$\begin{cases} y_n = \mathrm{argmin}_{y\in C}\{\lambda_n f(x_n,y) + \frac{1}{2}\|y-x_n\|^2\}, \\ z_n = \mathrm{argmin}_{y\in C}\{\lambda_n f(y_n,y) + \frac{1}{2}\|y-x_n\|^2\}, \\ t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)Sz_n], \\ C_n = \{z \in C : \|t_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap D_n} x_0, \ \forall n \in \mathbb{N}, \end{cases}$$

where $S : C \to C$ is a strict pseudo-contraction, f satisfies the Lipschitz-type property and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ are the sequences in $(0,1)$. They obtained strong convergence theorems under some certain conditions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$.

In 2013, Anh [26] introduced the following algorithm to find an element in C which is the fixed point of a nonexpansive mapping T on C and is also a solution of the pseudomonotone equilibrium problem in a Hilbert space H: $x_0 \in C$ and

$$\begin{cases} y_n = \mathrm{argmin}_{y\in C}\{\lambda_n f(x_n,y) + \frac{1}{2}\|y-x_n\|^2\}, \\ t_n = \mathrm{argmin}_{y\in C}\{\lambda_n f(y_n,y) + \frac{1}{2}\|y-x_n\|^2\}, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tt_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}, \{\lambda_n\} \subset (0,1)$ and f satisfies the Lipschitz-type property. Anh proved the strong convergence of the sequence $\{x_n\}$ generated by (1.6) provided $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and some conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$. Here, y_n and t_n are obtained by solving strongly convex problems, which are easier to compute than u_n in (1.4). So, the algorithms of solving the pseudomonotone equilibriums problems avoid the difficulties of computing u_n that exists in the algorithms of solving the monotone equilibrium problems and split monotone equilibrium problems.

Motivated by results mentioned above, we introduce some extragradient methods to solve the split pseudomonotone equilibrium problem and the fixed point problem for nonself nonexpansive mappings in Hilbert spaces and prove the weak and strong convergence of our algorithms. We also give a numerical example in an infinite dimensional space to illustrate the main results of this paper.

2. PRELIMINARIES

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. For all $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x-z, z-y\rangle \geq 0,\ \forall y \in C.$$

Proposition 2.1. *For all $x \in H$ and $z \in C$, it holds*

$$\|P_C x - z\|^2 \leq \|x-z\|^2 - \|P_C x - x\|^2.$$

Proposition 2.2. *Let $t \in [0,1]$. The following equality holds:*

$$\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ \forall x,y \in H.$$

Let $f : C \times C \to \mathbb{R}$ be a bifunction. f is said to be

(a1) γ-strong monotone ($\gamma > 0$) on C if

$$f(x,y) + f(y,x) \leq -\gamma\|x-y\|^2,\ \forall x,y \in C;$$

(a2) monotone on C if

$$f(x,y) + f(y,x) \leq 0,\ \forall x,y \in C;$$

(a3) pseudomonotone on C if

$$f(x,y) \geq 0 \quad \text{implies} \quad f(y,x) \leq 0,\ \forall x,y \in C.$$

The following examples show the relations among (a1), (a2) and (a3):

Example 2.3. Let $C = [0,\infty)$ and define a bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = \arctan(x-y)e^y$$

for all $x,y \in C$. Since $f(x,y) + f(y,x) = \arctan(x-y)(e^y - e^x) \leq 0$ for all $x,y \in C$, we find that f is monotone. However, f is not strong monotone.

Example 2.4. Let $C = [0,\infty)$. Set $a > 1$ and define a bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = (x-y)a^x, \quad \forall x,y \in C.$$

Then f is pseudomonotone. Since $f(x,y) + f(y,x) = (x-y)(a^x - a^y) \geq 0$ for all $x,y \in C$, we find that f is not monotone.

In fact, we have $(a1) \Longrightarrow (a2) \Longrightarrow (a3)$.

We say that bifunction f is Lipschitz-type continuous if there exist constants $c_1, c_2 > 0$ such that

$$f(x,z) \leq f(x,y) + f(y,z) + c_1\|x-y\|^2 + c_2\|y-z\|^2,\ \forall x,y,z \in C.$$

An example of a Lipschitz-type continuous bifunction f, which can be found in [26, 27], is defined by

$$f(x,y) = \langle Fx, y-x\rangle,\ \forall x,y \in C,$$

where $F : C \to H$ is Lipschitz continuous on C with Lipschitz constant $L > 0$. The Lipschitz constants for bifunction f are $c_1 = c_2 = L/2$.

From now, we give the following conditions on bifunction f:

(b1) $f(x,x) = 0$ for all $x \in C$ and f is pseudomonotone on C;

(b2) f is Lipschitz-type continuous;

(b3) $f(x,y)$ is convex and subdifferentiable in the second argument;

(b4) $f(x,y)$ is weakly continuous on $C\times C$, that is, if $\{x_n\},\{y_n\}\subset C$ weakly converges to $x,y\in C$, respectively, then $f(x_n,y_n)\to f(x,y)$.

It is easy to prove that $EP(f)$ is weakly closed and convex if f satisfies the conditions (b1), (b3) and (b4).

The bifunction f in the following example satisfies conditions (b1)–(b4), which are used in Section 3.

Example 2.5. [28] Let $H=\mathbb{R}^n$ $(n\geq 2)$ and $C=\{(x_1,\cdots,x_n):x_1,\cdots,x_n\geq 0\}$. Define a bifunction $f:C\times C\to\mathbb{R}$ by

$$f(x,y)=2(y_n-x_n)\|x\|$$

for all $x=(x_1,\cdots,x_n),y=(y_1,\cdots,y_n)\in C$. Then f is pseudomonotone and Lipschitz-type continuous with the constants $c_1=c_2=1$.

Lemma 2.6. [26] *Assume that $EP(f)\neq\emptyset$ and let $x\in C$. Suppose that the following strongly convex problems hold:*

$$y=\operatorname{argmin}\{\frac{1}{2}\|z-x\|^2+\lambda f(x,z):z\in C\},$$

$$t=\operatorname{argmin}\{\frac{1}{2}\|a-x\|^2+\lambda f(y,a):a\in C\},$$

where $\lambda>0$. Then

$$\lambda\big(f(x,z)-f(x,y)\big)\geq\langle y-x,y-z\rangle,\ \forall z\in C, \tag{2.1}$$

and

$$\|t-w\|^2\leq\|x-w\|^2-(1-2\lambda c_1)\|x-y\|^2-(1-2\lambda c_2)\|t-y\|^2,\ \forall w\in EP(f). \tag{2.2}$$

Lemma 2.7. [29] *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T:C\to H$ be a nonexpansive mapping. If $Fix(T)\neq\emptyset$, then $Fix(P_CT)=Fix(T)$.*

3. Main results

In this section, let H_1 and H_2 be two real Hilbert spaces and let $C_1\subset H_1$ and $C_2\subset H_2$ be nonempty closed convex subsets. Let $\langle\cdot,\cdot\rangle_1$ (resp., $\|\cdot\|_1$) and $\langle\cdot,\cdot\rangle_2$ (resp., $\|\cdot\|_2$) denote the inner products (resp., norms) in H_1 and H_2, respectively. However, for convenience, we use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to denote the inner product and the norm in H_1 and H_2 without the confusion.

Let $T:C_1\to H_1$ be a nonexpansive nonself-mapping. Let $A:H_1\to H_2$ be a linear bounded operator and let A^* be the adjoint operator of A. Let $f_1:C_1\times C_1\to\mathbb{R}$ and $f_2:C_2\times C_2\to\mathbb{R}$ be the bifunctions satisfying conditions (b1)–(b4) with the common Lipschitz constants $c_1,c_2>0$ (this assumption is feasible), that is,

$$f_i(x,z)\leq f_i(x,y)+f_i(y,z)+c_1\|x-y\|+c_2\|y-z\|,\ \forall x,y,z\in C_i,\ i=1,2. \tag{3.1}$$

Assume that $Fix(T)\cap\Omega\neq\emptyset$, where $\Omega=\{p\in C_1:p\in EP(f_1),Ap\in EP(f_2)\}$.

Lemma 3.1. *Let $x\in C_1$ and $\lambda\in(0,\min\{\frac{1}{2c_1},\frac{1}{2c_2}\})$. Suppose that the following strongly convex problems hold:*

$$y=\operatorname{argmin}\{\frac{1}{2}\|z-P_{C_2}Ax\|^2+\lambda f_2(P_{C_2}Ax,z):z\in C_2\},$$

$$t=\operatorname{argmin}\{\frac{1}{2}\|a-P_{C_2}Ax\|^2+\lambda f_2(y,a):a\in C_2\}.$$

Then

$$\|Ax-t\|^2\leq 2\langle Ax-Ap,Ax-t\rangle,\ \forall p\in\Omega.$$

Proof. For any fixed $p \in \Omega$, it follows that $Ap \in EP(f_2)$. Since $2\lambda c_1 < 1$ and $2\lambda c_2 < 1$, we get from 2.2 that

$$\begin{aligned}
\|Ax-t\|^2 &= \|Ax-Ap-(t-Ap)\|^2 \\
&= \|Ax-Ap\|^2 - 2\langle Ax-Ap, t-Ap\rangle + \|t-Ap\|^2 \\
&\le \|Ax-Ap\|^2 - 2\langle Ax-Ap, t-Ap\rangle + \|P_{C_2}Ax-Ap\|^2 \\
&\quad -(1-2\lambda c_1)\|y-P_{C_2}Ax\|^2 - (1-2\lambda c_2)\|y-t\|^2 \\
&\le \|Ax-Ap\|^2 - 2\langle Ax-Ap, t-Ap\rangle + \|P_{C_2}Ax-Ap\|^2 \\
&= \|Ax-Ap\|^2 - 2\langle Ax-Ap, t-Ap\rangle + \|P_{C_2}Ax-P_{C_2}Ap\|^2 \\
&\le \|Ax-Ap\|^2 - 2\langle Ax-Ap, t-Ap\rangle + \|Ax-Ap\|^2 \\
&= 2\langle Ax-Ap, Ax-t\rangle.
\end{aligned} \tag{3.2}$$

This completes the proof. □

Lemma 3.2. *Let $x \in C_1$, $\gamma > 0$ and $\lambda \in (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$. Let $t \in C_2$ be defined as in Lemma 3.1. Then*

$$\|x-\gamma A^*(Ax-t)-p\|^2 \le \|x-p\|^2 - \gamma(1-\gamma\|A\|^2)\|Ax-t\|^2, \ \forall p \in \Omega.$$

Proof. For any fixed $p \in \Omega$, we find from Lemma 3.1, that

$$\begin{aligned}
\|x-\gamma A^*(Ax-t)-p\|^2 &= \|x-p\|^2 + \gamma^2\|A^*(Ax-t)\|^2 + 2\gamma\langle x-p, A^*(t-Ax)\rangle \\
&= \|x-p\|^2 + \gamma^2\|A^*(Ax-t)\|^2 + 2\gamma\langle Ax-Ap, t-Ax\rangle \\
&\le \|x-p\|^2 + \gamma^2\|A\|^2\|Ax-t\|^2 - \gamma\|Ax-t\|^2 \\
&= \|x-p\|^2 - \gamma(1-\gamma\|A\|^2)\|Ax-t\|^2.
\end{aligned} \tag{3.3}$$

This completes the proof. □

Now, we give the first algorithm to solve the split pseudomonotone equilibrium problem and the fixed point problem as follows.

Algorithm 3.3. Initialization: Choose $\gamma \in (0, 1/M^2)$, where M is a boundedness above of $\|A\|$, $\{\lambda_n\} \in [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $x_0 \in C_1$. Set $n = 0$.

Step 1. Solve the strongly convex problems:

$$\begin{cases} y_n = \operatorname{argmin}\{\frac{1}{2}\|z-P_{C_2}Ax_n\|^2 + \lambda_n f_2(P_{C_2}Ax_n, z) : z \in C_2\}, \\ t_n = \operatorname{argmin}\{\frac{1}{2}\|t-P_{C_2}Ax_n\|^2 + \lambda_n f_2(y_n, t) : t \in C_2\}. \end{cases}$$

Step 2. Set $v_n = P_{C_1}[x_n - \gamma A^*(Ax_n - t_n)]$ and solve the strongly convex problems:

$$\begin{cases} z_n = \operatorname{argmin}\{\frac{1}{2}\|y-v_n\|^2 + \lambda_n f_1(v_n, y) : y \in C_1\}, \\ w_n = \operatorname{argmin}\{\frac{1}{2}\|a-v_n\|^2 + \lambda_n f_1(z_n, a) : a \in C_1\}. \end{cases}$$

Step 3. If $x_n = Tx_n$, $t_n = Ax_n$ and $w_n = x_n$, then $x_n \in Fix(T) \cap \Omega$, stop. Otherwise, go to Step 4.

Step 4. Set $x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_{C_1}T[\beta_n x_n + (1-\beta_n)w_n]$. Set $n = n+1$ and go to Step 1.

Theorem 3.4. *If* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$ *and* $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$*, then the sequence* $\{x_n\}$ *generated in Algorithm 3.3 weakly converges to the element* $\bar{x}=\lim_{n\to\infty}P_{Fix(T)\cap\Omega}x_n$*.*

Proof. If $t_n=Ax_n$ for some $n\in\mathbb{N}$, then $Ax_n\in C_2$ and $v_n=x_n$. By 2.2, we have

$$\begin{aligned}\|t_n-w\|^2 &\le \|P_{C_2}Ax_n-w\|^2-(1-2\lambda_n c_1)\|P_{C_2}Ax_n-y_n\|^2\\ &\quad -(1-2\lambda_n c_2)\|t_n-y_n\|^2,\ \forall w\in EP(f_2).\end{aligned}\tag{3.4}$$

Since $t_n=Ax_n$, we have $Ax_n=y_n$. Using 2.1 and (b1), we have

$$\begin{aligned}\lambda_n f_2(Ax_n,z)&=\lambda_n\big(f_2(P_{C_2}Ax_n,z)-f_2(P_{C_2}Ax_n,y_n)\big)\\ &\ge\langle y_n-P_{C_2}Ax_n,y_n-z\rangle\\ &=0,\ \forall z\in C_2.\end{aligned}\tag{3.5}$$

Thus, it follows from $\lambda_n>0$ that $Ax_n\in EP(f_2)$. Similarly, we can prove that $x_n\in EP(f_1)$ if $x_n=w_n$. Hence $x_n\in\Omega$. To show the convergence of Algorithm 3.3, we assume that the stop criterion at Step 3 can not be satisfied for all $n\in\mathbb{N}$. Set $l_n=\beta_n x_n+(1-\beta_n)w_n$ for each $n\in\mathbb{N}$. Using Proposition 2.2, Lemma 3.2 and (2.2), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ \le\ &\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|Tl_n-p\|^2\\ \le\ &\alpha_n\|x_n-p\|^2+(1-\alpha_n)\big[\beta_n\|x_n-p\|^2+(1-\beta_n)\|w_n-p\|^2-\beta_n(1-\beta_n)\|x_n-w_n\|^2\big]\\ \le\ &\alpha_n\|x_n-p\|^2+(1-\alpha_n)\big[\beta_n\|x_n-p\|^2+(1-\beta_n)(\|v_n-p\|^2-(1-2\lambda_n c_1)\|z_n-v_n\|^2\\ &-(1-2\lambda_n c_2)\|w_n-z_n\|^2)-\beta_n(1-\beta_n)\|x_n-w_n\|^2\big]\\ \le\ &\alpha_n\|x_n-p\|^2+(1-\alpha_n)\big[\beta_n\|x_n-p\|^2+(1-\beta_n)(\|x_n-\gamma A^*(Ax_n-t_n)-p\|^2\\ &-(1-2\lambda_n c_1)\|z_n-v_n\|^2-(1-2\lambda_n c_2)\|w_n-z_n\|^2)-\beta_n(1-\beta_n)\|x_n-w_n\|^2\big]\\ \le\ &\alpha_n\|x_n-p\|^2+(1-\alpha_n)\big[\beta_n\|x_n-p\|^2+(1-\beta_n)\big(\|x_n-p\|^2-\gamma(1-\gamma\|A\|^2)\|Ax_n-t_n\|^2\\ &-(1-2\lambda_n c_1)\|z_n-v_n\|^2-(1-2\lambda_n c_2)\|w_n-z_n\|^2\big)-\beta_n(1-\beta_n)\|x_n-w_n\|^2\big]\\ =\ &\|x_n-p\|^2-(1-\alpha_n)(1-\beta_n)\big[\gamma(1-\gamma\|A\|^2)\|Ax_n-t_n\|^2+(1-2\lambda_n c_1)\|z_n-v_n\|^2\\ &+(1-2\lambda_n c_2)\|w_n-z_n\|^2+\beta_n\|x_n-w_n\|^2\big]\\ \le\ &\|x_n-p\|^2-(1-\alpha_n)(1-\beta_n)\big[\gamma(1-\gamma M^2)\|Ax_n-t_n\|^2+(1-2\lambda_n c_1)\|z_n-v_n\|^2\\ &+(1-2\lambda_n c_2)\|w_n-z_n\|^2+\beta_n\|x_n-w_n\|^2\big]\\ \le\ &\|x_n-p\|^2,\ \forall p\in Fix(T)\cap\Omega.\end{aligned}\tag{3.6}$$

It follows that $\lim_{n\to\infty}\|x_n-p\|$ exists. Define $h_n=P_{Fix(T)\cap\Omega}x_n\in Fix(T)\cap\Omega$ for each $n\in\mathbb{N}$. By (3.6), we have

$$\|x_{n+1}-h_n\|\le\|x_n-h_n\|,\ \forall n\in\mathbb{N}.\tag{3.7}$$

From $h_{n+1}=P_{Fix(T)\cap\Omega}x_{n+1}$, $h_n\in Fix(T)\cap\Omega$ and (3.7), we have

$$\|x_{n+1}-h_{n+1}\|\le\|x_{n+1}-h_n\|\le\|x_n-h_n\|,\ \forall n\in\mathbb{N}.\tag{3.8}$$

Hence $\{\|x_n-h_n\|\}$ is a convergent sequence. Since $h_n\in Fix(T)\cap\Omega$ for each $n\in\mathbb{N}$, we obtain from (3.7) that

$$\|x_{n+m}-h_n\|\le\|x_{n+m-1}-h_n\|\le\cdots\le\|x_n-h_n\|,\ \forall n,m\in\mathbb{N}.\tag{3.9}$$

From the definition of h_{n+m} and Proposition 2.1, it follows that

$$\|h_n - h_{n+m}\|^2 + \|h_{n+m} - x_{n+m}\|^2 \le \|h_n - x_{n+m}\|^2 \le \|h_n - x_n\|^2. \tag{3.10}$$

Hence

$$\|h_n - h_{n+m}\|^2 \le \|h_n - x_n\|^2 - \|h_{n+m} - x_{n+m}\|^2, \ \forall n, m \in \mathbb{N}. \tag{3.11}$$

Since $\{\|h_n - x_n\|\}$ is convergent, we have

$$\lim_{n\to\infty} \|h_n - h_{n+m}\| = 0, \ \forall m \in \mathbb{N}. \tag{3.12}$$

It follows that $\{h_n\}$ is a Cauchy sequence. Since $Fix(T) \cap \Omega$ is nonempty closed, $\{h_n\}$ converges strongly to some $\bar{x} \in Fix(T) \cap \Omega$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\big[\gamma(1-\gamma M^2)\|Ax_n - t_n\|^2 + (1-2\lambda_n c_1)\|z_n - v_n\|^2 \\
&\quad + (1-2\lambda_n c_2)\|w_n - z_n\|^2 + \beta_n\|x_n - w_n\|^2\big] \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

By the hypothesis on γ, $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$, we have

$$\lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|Ax_n - t_n\| = \lim_{n\to\infty} \|w_n - z_n\| = \lim_{n\to\infty} \|z_n - v_n\| = 0. \tag{3.13}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $\hat{x} \in C_1$. From (3.13), it follows that $\{z_{n_k}\}$ and $\{v_{n_k}\}$ also converge weakly to $\hat{x}$.

Next, we show that $\hat{x} \in \Omega$. From (2.1), we have

$$\lambda_{n_k}\big(f_1(v_{n_k}, y) - f_1(v_{n_k}, z_{n_k})\big) \ge \langle z_{n_k} - v_{n_k}, z_{n_k} - y\rangle, \ \forall y \in C_1. \tag{3.14}$$

Letting $k \to \infty$ in (3.14), we have

$$f_1(\hat{x}, y) \ge 0, \ \forall y \in C_1.$$

It follows that $\hat{x} \in EP(f_1)$. Using (3.13), we have

$$\|P_{C_2}Ax_n - t_n\| = \|P_{C_2}Ax_n - P_{C_2}t_n\| \le \|Ax_n - t_n\| \to 0 \text{ as } n \to \infty. \tag{3.15}$$

By (2.2), we have

$$\begin{aligned}
\|t_n - w\|^2 &\le \|P_{C_2}Ax_n - w\|^2 - (1-2\lambda_n c_1)\|P_{C_2}Ax_n - y_n\|^2 \\
&\quad - (1-2\lambda_n c_2)\|t_n - y_n\|^2, \ \forall w \in EP(f_2).
\end{aligned}$$

Using this inequality and (3.15), we obtain

$$\begin{aligned}
&(1-2\lambda_n c_1)\|P_{C_2}Ax_n - y_n\|^2 + (1-2\lambda_n c_2)\|t_n - y_n\|^2 \\
&\le \|P_{C_2}Ax_n - w\|^2 - \|t_n - w\|^2 \\
&\le \|P_{C_2}Ax_n - t_n\|\big(\|P_{C_2}Ax_n - w\| + \|t_n - w\|\big) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.16}$$

From (3.16) and the hypothesis on $\{\lambda_n\}$, it follows that

$$\lim_{n\to\infty} \|y_n - P_{C_2}Ax_n\| = \lim_{n\to\infty} \|t_n - y_n\| = 0. \tag{3.17}$$

By (3.13) and (3.17), we have $\lim_{n\to\infty} \|y_n - Ax_n\| = 0$. Since A is linear bounded and $\{x_{n_k}\}$ weakly converges to $\hat{x} \in C_1$, we obtain that $\{Ax_{n_k}\}$ weakly converges to $A\hat{x} \in H_2$. Since $\{y_n\} \subset C_2$ and $\|y_n -$

$Ax_n\| \to 0$, it follows that $\{y_{n_k}\}$ weakly converges to $A\hat{x} \in C_2$. Further, from (3.17), we also have that $\{P_{C_2}Ax_{n_k}\}$ weakly converges to $A\hat{x}$. By (2.1), we have

$$\lambda_{n_k}\big(f_2(P_{C_2}Ax_{n_k},z) - f_2(P_{C_2}Ax_{n_k},y_{n_k})\big) \geq \langle y_{n_k} - P_{C_2}Ax_{n_k}, y_{n_k} - z\rangle,\ \forall z \in C_2. \tag{3.18}$$

Letting $k \to \infty$ in (3.18), we can obtain

$$f_2(A\hat{x},z) \geq 0,\ \forall z \in C_2.$$

It follows that $A\hat{x} \in EP(f_2)$. Therefore, $\hat{x} \in \Omega$.

Now, we show that $\hat{x} \in Fix(T)$. By (3.3) and the definition of l_n, we have

$$\|l_n - x_n\| = (1-\beta_n)\|x_n - w_n\| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

By Proposition 2.2, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|P_{C_1}Tl_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - P_{C_1}Tl_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - P_{C_1}Tl_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - P_{C_1}Tl_n\|^2.\end{aligned}$$

Since the limit of $\{\|x_n - p\|^2\}$ exists, we have

$$\alpha_n(1-\alpha_n)\|P_{C_1}Tl_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0 \text{ as } n \to \infty.$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} \|P_{C_1}Tl_n - x_n\| = 0. \tag{3.20}$$

Combining (3.19) with (3.20), we obtain that

$$\|P_{C_1}Tl_n - l_n\| \leq \|P_{C_1}Tl_n - x_n\| + \|x_n - l_n\| \to 0 \text{ as } n \to \infty \tag{3.21}$$

and that $\{P_{C_1}Tl_{n_k}\}$ and $\{l_{n_k}\}$ weakly converge to $\hat{x} \in C_1$. If $\hat{x} \neq P_{C_1}T\hat{x}$, by (3.21) and Opial's inequality [30], we have

$$\begin{aligned}\liminf_{k\to\infty}\|l_{n_k} - \hat{x}\| &< \liminf_{k\to\infty}\|l_{n_k} - P_{C_1}T\hat{x}\| \\ &\leq \liminf_{k\to\infty}\big(\|l_{n_k} - P_{C_1}Tl_{n_k}\| + \|P_{C_1}Tl_{n_k} - P_{C_1}T\hat{x}\|\big) \\ &\leq \liminf_{k\to\infty}\|l_{n_k} - \hat{x}\|,\end{aligned}$$

which is a contradiction. Hence $\hat{x} \in Fix(P_{C_1}T)$. From Lemma 2.7, it follows that $\hat{x} \in Fix(T)$.

Finally, we prove that $\hat{x} = \bar{x}$. From $h_n = P_{Fix(T)\cap\Omega}x_n$ and $\hat{x} \in Fix(T)\cap\Omega$, we have

$$\langle h_{n_k} - \hat{x}, x_{n_k} - h_{n_k}\rangle \geq 0. \tag{3.22}$$

Let $k \to \infty$ in (3.22). Since $\{h_n\}$ strongly converges to $\bar{x} \in Fix(T)\cap\Omega$, we have

$$-\|\bar{x} - \hat{x}\|^2 = \langle \bar{x} - \hat{x}, \hat{x} - \bar{x}\rangle \geq 0.$$

Hence $\bar{x} = \hat{x}$. Therefore, $\{x_n\}$ converges weakly to a point $\hat{x} \in Fix(T)\cap\Omega$, where

$$\hat{x} = \lim_{n\to\infty} P_{Fix(T)\cap\Omega}x_n.$$

This completes the proof. □

Next, we give a hybrid projection algorithm to solve the split pseudomonotone equilibrium problem and the fixed point problem as follows.

Algorithm 3.5. Initialization: Choose $\gamma \in (0, 1/M^2)$, where M is a boundedness above of $\|A\|$, $\{\lambda_n\} \in [\delta_1, \delta_2]$ with $0 < \delta_1 < \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and take $x_0 \in C_1$. Set $n = 0$.

Step 1. Solve the strongly convex problems:

$$\begin{cases} y_n = \operatorname{argmin}\{\frac{1}{2}\|z - P_{C_2}Ax_n\|^2 + \lambda_n f_2(P_{C_2}Ax_n, z) : z \in C_2\}, \\ t_n = \operatorname{argmin}\{\frac{1}{2}\|t - P_{C_2}Ax_n\|^2 + \lambda_n f_2(y_n, t) : t \in C_2\}. \end{cases}$$

Step 2. Set $v_n = P_{C_1}[x_n - \gamma A^*(Ax_n - t_n)]$ and solve the strongly convex problems:

$$\begin{cases} z_n = \operatorname{argmin}\{\frac{1}{2}\|y - v_n\|^2 + \lambda_n f_1(v_n, y) : y \in C_1\}, \\ w_n = \operatorname{argmin}\{\frac{1}{2}\|a - v_n\|^2 + \lambda_n f_1(z_n, a) : a \in C_1\}. \end{cases}$$

Step 3. If $x_n = Tx_n$, $t_n = Ax_n$ and $w_n = x_n$, then $x_n \in Fix(T) \cap \Omega$, stop. Otherwise, go to Step 4.

Step 4. Put $h_n = \alpha_n x_n + (1-\alpha_n)T[\beta_n x_n + (1-\beta_n)w_n]$ and

$$\begin{cases} D_n = \{v \in C_1 : \|h_n - v\| \le \|x_n - v\|\}, \\ U_n = \cap_{j=0}^n D_j, \\ x_{n+1} = P_{U_n} x_0. \end{cases}$$

Set $n = n+1$ and go to Step 1.

Lemma 3.6. *The sequence $\{x_n\}$ generated in Algorithm 3.5 is well defined and strongly converges to some $\hat{x} \in C_1$. If $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then*

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \|t_n - Ax_n\| = \lim_{n\to\infty} \|z_n - v_n\| = \lim_{n\to\infty} \|w_n - z_n\| = 0.$$

Proof. It is obvious that U_n is closed and convex since D_n is closed and convex for all $n \in \mathbb{N}$. Now, we show that

$$Fix(T) \cap \Omega \subset D_n, \ \forall n \in \mathbb{N}.$$

For any fixed $w \in Fix(T) \cap \Omega$, by (2.2) and Lemma 3.2, we have

$$\begin{aligned} \|w_n - w\|^2 &\le \|v_n - w\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2 - (1 - 2\lambda_n c_1)\|z_n - v_n\|^2 \\ &\le \|x_n - w\|^2 - \gamma(1 - \gamma\|A\|^2)\|Ax_n - t_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2 \\ &\quad - (1 - 2\lambda_n c_1)\|z_n - v_n\|^2 \\ &\le \|x_n - w\|^2 - \gamma(1 - \gamma M^2)\|Ax_n - t_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2 \\ &\quad - (1 - 2\lambda_n c_1)\|z_n - v_n\|^2 \\ &\le \|x_n - w\|^2, \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned} \|h_n - w\| &\le \alpha_n\|x_n - w\| + (1-\alpha_n)\big\|T[\beta_n x_n + (1-\beta_n)w_n] - w\big\| \\ &\le \alpha_n\|x_n - w\| + (1-\alpha_n)\big[\beta_n\|x_n - w\| + (1-\beta_n)\|w_n - w\|\big] \\ &\le \alpha_n\|x_n - w\| + (1-\alpha_n)\|x_n - w\| \\ &= \|x_n - w\|, \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.24}$$

It follows that $w \in D_n$. Hence $Fix(T) \cap \Omega \subset D_n$, which implies $Fix(T) \cap \Omega \subset U_n$ for each $n \in \mathbb{N}$. Therefore, the sequence $\{x_n\}$ is well defined. Since $z = P_{Fix(T)\cap\Omega}x_0 \in Fix(T) \cap \Omega \subset U_n$, we have $\|x_{n+1} - x_0\| \leq \|z - x_0\|$. So, $\{\|x_n - x_0\|\}$ is bounded.

On the other hand, since $U_{n+1} \subset U_n$, we have $x_{n+2} = P_{U_{n+1}}x_0 \in U_{n+1} \subset U_n$ It follows that $\|x_{n+1} - x_0\| \leq \|x_{n+2} - x_0\|$. Hence, the limit of $\{\|x_n - x_0\|\}$ exists. Since $U_m \subset U_n$ and $x_{m+1} = P_{U_m}x_0 \in U_m \subset U_n$ for all $m \geq n$, we have $\langle x_{n+1} - x, x_{m+1} - x_{n+1}\rangle \geq 0$, which implies

$$\begin{aligned}
& \|x_{m+1} - x_{n+1}\|^2 \\
= & \|x_{m+1} - x_0 - (x_{n+1} - x_0)\|^2 \\
= & \|x_{m+1} - x_0\|^2 + \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, x_{m+1} - x_0\rangle \\
= & \|x_{m+1} - x_0\|^2 + \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, x_{m+1} - x_{n+1} + x_{n+1} - x_0\rangle \\
= & \|x_{m+1} - x_0\|^2 - \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_0, x_{m+1} - x_{n+1}\rangle \\
\leq & \|x_{m+1} - x_0\|^2 - \|x_{n+1} - x_0\|^2
\end{aligned}$$

for all $m, n \in \mathbb{N}$ with $m \geq n$. Since the limit of $\{\|x_{n+1} - x_0\|\}$ exists, we obtain

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0, \tag{3.25}$$

which shows that $\{x_n\}$ is a Cauchy sequence. Hence one can assume that $x_n \to \hat{x} \in C_1$ as $n \to \infty$. Since $x_{n+1} \in D_n$, by (3.25) with $m = n+1$, we have

$$\|h_n - x_{n+1}\| \leq \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.26}$$

Set $l_n = \beta_n x_n + (1-\beta_n)w_n$ for each $n \in \mathbb{N}$. Since $\limsup_{n\to\infty} \alpha_n < 1$, by the definition of h_n, (3.25) with $m = n+1$ and (3.26), we have

$$\|Tl_n - x_n\| \leq \frac{1}{1-\alpha_n}\left[\|h_n - x_{n+1}\| + \|x_{n+1} - x_n\|\right] \to 0 \text{ as } n \to \infty. \tag{3.27}$$

From (3.23), we have

$$\begin{aligned}
\|Tl_n - w\|^2 &\leq \beta_n\|x_n - w\|^2 + (1-\beta_n)\|w_n - w\|^2 \\
&\leq \|x_n - w\|^2 - (1-\beta_n)\big[\gamma(1-\gamma M^2)\|Ax_n - t_n\|^2 \\
&\quad + (1-2\lambda_n c_2)\|w_n - z_n\|^2 + (1-2\lambda_n c_1)\|z_n - v_n\|^2\big].
\end{aligned} \tag{3.28}$$

Combining (3.27) with (3.28), we have

$$\begin{aligned}
&(1-\beta_n)\left[\gamma(1-\gamma M^2)\|Ax_n - t_n\|^2 + (1-2\lambda_n c_2)\|w_n - z_n\|^2 + (1-2\lambda_n c_1)\|z_n - v_n\|^2\right] \\
\leq\ & \|x_n - Tl_n\|(\|x_n - w\| + \|Tl_n - w\|) \\
\to\ & 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, $\lambda_n \leq \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and $\gamma(1-\gamma M^2) > 0$, we have

$$\lim_{n\to\infty} \|Ax_n - t_n\| = \lim_{n\to\infty} \|w_n - z_n\| = \lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.29}$$

From the definition of h_n, Proposition 2.2 and (3.23), we have

$$\begin{aligned}
&\|h_n - w\|^2 \\
&\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|\beta_n(x_n - w) + (1-\beta_n)(w_n - w)\|^2 \\
&= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\left[\beta_n\|x_n - w\|^2 + (1-\beta_n)\|w_n - w\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2\right] \\
&\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\left[\beta_n\|x_n - w\|^2 + (1-\beta_n)\|x_n - w\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2\right] \\
&= \|x_n - w\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|x_n - w_n\|^2, \ \forall n \in \mathbb{N}.
\end{aligned}$$

By (3.24)-(3.26), we have

$$\begin{aligned}
(1-\alpha_n)\beta_n(1-\beta_n)\|x_n - w_n\|^2 &\le \|x_n - h_n\|(\|x_n - w\| + \|h_n - w\|) \\
&\le 2\|x_n - h_n\|\|x_n - w\| \\
&\le 2\big(\|x_n - x_{n+1}\| + \|x_{n+1} - h_n\|\big)\|x_n - w\| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $\limsup_{n\to\infty}\alpha_n < 1$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.30}$$

This completes the proof. □

Theorem 3.7. *If* $\limsup_{n\to\infty}\alpha_n < 1$ *and* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$*, then the sequence* $\{x_n\}$ *given in Algorithm 3.5 strongly converges to the element* $P_{Fix(T)\cap\Omega}x_0$.

Proof. Assume that the stop criterion can not be satisfied for all $n \in \mathbb{N}$. By the definition of l_n and (3.30), we have

$$\|l_n - x_n\| = (1-\beta_n)\|x_n - w_n\| \to 0 \text{ as } n \to \infty, \tag{3.31}$$

which implies that $\{l_n\}$ converges to $\hat{x}$. Using (3.27) and (3.31), we have

$$\|Tl_n - l_n\| \le \|Tl_n - x_n\| + \|x_n - l_n\| \to 0 \text{ as } n \to \infty.$$

Hence $\hat{x} \in Fix(T)$.

Now, we show that $\hat{x} \in \Omega$. From (2.1), it follows that

$$\lambda_n\big(f_1(v_n, y) - f_1(v_n, z_n)\big) \ge \langle z_n - v_n, z_n - y\rangle, \ \forall y \in C_1. \tag{3.32}$$

Letting $n \to \infty$ in (3.32), it follows from (b2), (b3) and (3.29) that

$$f_1(\hat{x}, y) \ge 0, \ \forall y \in C_1.$$

This implies that $\hat{x} \in EP(f_1)$. By (3.29), we have

$$\|P_{C_2}Ax_n - t_n\| = \|P_{C_2}Ax_n - P_{C_2}t_n\| \le \|Ax_n - t_n\| \to 0 \text{ as } n \to \infty. \tag{3.33}$$

For any $w \in EP(f_2)$, it follows from (2.2) that

$$\|t_n - w\|^2 \le \|P_{C_2}Ax_n - w\|^2 - (1-2\lambda_n c_1)\|P_{C_2}Ax_n - y_n\|^2 - (1-2\lambda_n c_2)\|t_n - y_n\|^2,$$

which implies from (3.33) that

$$\begin{aligned}&(1-2\lambda_n c_1)\|P_{C_2}Ax_n-y_n\|^2+(1-2\lambda_n c_2)\|t_n-y_n\|^2\le\|P_{C_2}Ax_n-w\|^2-\|t_n-w\|^2\\&\le\|P_{C_2}Ax_n-t_n\|\big(\|P_{C_2}Ax_n-w\|+\|t_n-w\|\big)\\&\to 0 \text{ as } n\to\infty.\end{aligned}\tag{3.34}$$

Since $\lambda_n\le\delta_2<\min\{\frac{1}{2c_1},\frac{1}{2c_2}\}$, it follows from 3.34 that

$$\lim_{n\to\infty}\|y_n-P_{C_2}Ax_n\|=\lim_{n\to\infty}\|t_n-y_n\|=0.\tag{3.35}$$

Combining (3.29) with (3.35), we have

$$\lim_{n\to\infty}\|y_n-Ax_n\|=0.\tag{3.36}$$

Since A is linear bounded and $\{x_n\}$ strongly converges to $\hat{x}\in C_1$, $\{Ax_n\}$ strongly converges to $A\hat{x}\in H_2$. Since $\{y_n\}\subset C_2$ and $y_n-Ax_n\to 0$, it follows that $\{y_n\}$ strongly converges to $A\hat{x}\in C_2$. Further, it follows from (3.35) that $\{P_{C_2}Ax_n\}$ also strongly converges to $A\hat{x}$. By (2.1), we have

$$\lambda_n\big(f_2(P_{C_2}Ax_n,z)-f_2(P_{C_2}Ax_n,y_n)\big)\ge\langle y_n-P_{C_2}Ax_n,y_n-z\rangle,\ \forall z\in C_2.\tag{3.37}$$

Let $n\to\infty$ in (3.37). Since $\lambda_n\ge\delta_1>0$, it follows from (b1), (b3) and (3.35) that

$$f_2(A\hat{x},z)\ge 0,\ \forall z\in C_2.$$

It follows that $A\hat{x}\in EP(f_2)$. Therefore, $\hat{x}\in\Omega$ and hence $\hat{x}\in Fix(T)\cap\Omega$.

Finally, we prove that $\hat{x}=P_{Fix(T)\cap\Omega}x_0$. From $x_{n+1}=P_{U_n}x_0$ and $Fix(T)\cap\Omega\subset U_n$ for all $n\in\mathbb{N}$, we have

$$\langle x_0-x_{n+1},x_{n+1}-v\rangle\ge 0,\ \forall v\in Fix(T)\cap\Omega,\ n\in\mathbb{N}.\tag{3.38}$$

Taking the limit in (3.38) and noting that $x_n\to\hat{x}$ as $n\to\infty$, we obtain

$$\langle x_0-\hat{x},\hat{x}-v\rangle\ge 0,\ \forall v\in Fix(T)\cap\Omega.$$

Therefore, it follows that $\hat{x}=P_{Fix(T)\cap\Omega}x_0$. This completes the proof. □

If $T=I$ (the identity mapping), then we get the following results.

Corollary 3.8. *Let H_i be a Hilbert space. Let C_i be a nonempty closed and convex subset of H_i and let $f_i:C_i\times C_i\to\mathbb{R}$ be the bifunction satisfying the conditions* (b1)-(b4) *for each $i=1,2$. Assume that f_1 and f_2 enjoy the common Lipschitz constants c_1 and c_2 and $\Omega\neq\emptyset$. Let $A:H_1\to H_2$ be a linear bounded operator with the adjoint operator A^*. Take $x_0\in C_1$ and define the sequence $\{x_n\}$ by the following manner:*

$$\begin{cases}y_n=\operatorname{argmin}\{\frac{1}{2}\|z-P_{C_2}Ax_n\|^2+\lambda_n f_2(P_{C_2}Ax_n,z):z\in C_2\},\\ t_n=\operatorname{argmin}\{\frac{1}{2}\|t-P_{C_2}Ax_n\|^2+\lambda_n f_2(y_n,t):t\in C_2\},\\ v_n=P_{C_1}[x_n-\gamma A^*(Ax_n-t_n)],\\ z_n=\operatorname{argmin}\{\frac{1}{2}\|y-v_n\|^2+\lambda_n f_1(v_n,y):y\in C_1\},\\ w_n=\operatorname{argmin}\{\frac{1}{2}\|a-v_n\|^2+\lambda_n f_1(z_n,a):a\in C_1\},\\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)(\beta_n x_n+(1-\beta_n)w_n),\ n\in\mathbb{N},\end{cases}\tag{3.39}$$

where $\gamma \in (0, 1/M^2)$*,* M *is a boundedness above of* $\|A\|$*,* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$*,* $\{\lambda_n\} \in [\delta_1, \delta_2]$ *with* $0 < \delta_1 < \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$*. If the sequences* $\{\alpha_n\}, \{\beta_n\}$ *satisfy the following conditions:*

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0 \quad and \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,$$

then $\{x_n\}$ *generated by (3.39) weakly converges to the element* $\hat{x} = \lim_{n\to\infty} P_\Omega x_n$*.*

Corollary 3.9. *Let* H_i *be a Hilbert space. Let* C_i *be a nonempty closed and convex subset of* H_i *and let* $f_i : C_i \times C_i \to \mathbb{R}$ *be the bifunction satisfying the conditions* (b1)-(b4) *for each* $i = 1, 2$*. Assume that* f_1 *and* f_2 *enjoy the common Lipschitz constants* c_1 *and* c_2 *and* $\Omega \neq \emptyset$*. Let* $A : H_1 \to H_2$ *be a linear bounded operator with the adjoint operator* A^**. Take* $x_0 \in C_1$ *and define the sequence* $\{x_n\}$ *by the following manner:*

$$\begin{cases} y_n = \operatorname{argmin}\{\frac{1}{2}\|z - P_{C_2}Ax_n\|^2 + \lambda_n f_2(P_{C_2}Ax_n, z) : z \in C_2\}, \\ t_n = \operatorname{argmin}\{\frac{1}{2}\|t - P_{C_2}Ax_n\|^2 + \lambda_n f_2(y_n, t) : t \in C_2\}, \\ v_n = P_{C_1}[x_n - \gamma A^*(Ax_n - t_n)], \\ z_n = \operatorname{argmin}\{\frac{1}{2}\|y - v_n\|^2 + \lambda_n f_1(v_n, y) : y \in C_1\}, \\ w_n = \operatorname{argmin}\{\frac{1}{2}\|a - v_n\|^2 + \lambda_n f_1(z_n, a) : a \in C_1\}, \\ h_n = \alpha_n x_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)w_n), \\ D_n = \{z \in C_1 : \|h_n - z\| \leq \|x_n - z\|\}, \\ U_n = \cap_{j=0}^n D_j, \\ x_{n+1} = P_{U_n} x_0, \ n \in \mathbb{N}, \end{cases} \tag{3.40}$$

where $\gamma \in (0, 1/M^2)$*,* M *is a boundedness above of* $\|A\|$*,* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$*,* $\{\lambda_n\} \in [\delta_1, \delta_2]$ *with* $0 < \delta_1 < \delta_2 < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$*. If the sequences* $\{\alpha_n\}, \{\beta_n\}$ *satisfy the following conditions:*

$$\limsup_{n\to\infty} \alpha_n < 1 \quad and \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,$$

then $\{x_n\}$ *generated by (3.40) strongly converges to* $x^* = P_\Omega x_0$*.*

Remark 3.10. In [11, 14, 15, 16], the parameter γ is chosen from $(0, 1/\|A^*\|^2)$ or $(0, 1/L^2)$, where L is the spectral radius of the operator A^*A. However, it is sometime hard to compute the spectral radius of A^*A and the norm $\|A^*\|$. That is difficult to choose the parameter γ. In our paper, the parameter γ is chosen from $(0, 1/M^2)$, where M is a boundedness above of $\|A\|$. Usually, the boundedness above of $\|A\|$ is easy to estimate and so it is easier to choose the parameter γ.

4. Numerical examples

In this section, we give an example in infinite dimensional space to illustrate Algorithm 3.3 ($T = I$) and compare the results with [22, Algorithm 1] and [31, Algorithm 3.1]. The program is performed by Matlab R2008a running on a PC Desktop with Core(TM) i3CPU M550 3.20GHz and 4GB Ram.

Example 4.1. Let $H_1 = L^1[0,1]$, $H_2 = l^2$, $C_1 = \{x \in L^2[0,1] : x = \sum_{i=1}^n a_i t^i + b : a_i \geq 0$ for each $i = 1, \cdots, n, b \in [0, 10], n \geq 1\}$ and $C_2 = \{x = (x_1, x_2, \cdots) \in l^2 : x_1 = 0, x_i \geq 0, i \geq 2, \|x\| \leq 30\}$. Define the bifunctions $f_1 : C_1 \times C_1 \to \mathbb{R}$ by

$$f_1(x,y) = \int_0^1 x(t)(y(t) - x(t))\mathrm{d}t$$

for all $x = x(t), y = y(t) \in C_1$ and $f_2 : C_2 \times C_2 \to \mathbb{R}$ by $f_2(x,y) = \sum_{i=1}^{\infty} x_i(y_i - x_i)$ for all $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots) \in C_2$. It is easy to prove that f_1 and f_2 are Lipschitz-type continuous with the common constants $c_1 = c_2 = \frac{1}{2}$. So, f_1 and f_2 satisfy the conditions (b1)-(b4).

Let $A : H_1 \to H_2$ be a bounded linear operator defined by

$$Ax = \Big(\int_0^1 x(t)\mathrm{d}t, \frac{1}{2}\int_0^1 x(t)\mathrm{d}t, \frac{1}{4}\int_0^1 x(t)\mathrm{d}t, \cdots \Big), \forall x(t) \in H_1.$$

It follows that

$$A^* y = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} y_i, \ \forall y = (y_1, y_2, \cdots) \in H_2.$$

Obviously, $x^* = x^*(t) = 0$ for all $t \in [0,1]$ is the unique solution of split equilibrium problem.

It is easy to check that f_2 satisfies the following conditions:

(a1) f_2 is monotone on C_2 and $f_2(x,x) = 0$ for all $x \in C_2$;

(a2) for all $x, y, z \in C_i$,

$$\lim_{t \to 0^+} \sup f_2(tz + (1-t)x, y) \leq f_2(x,y);$$

(a3) for all $x \in C_i$, $f_2(x, \cdot)$ is convex and lower semicontinuous.

For each $r > 0$ and $x \in H_2$, it follows from [4] that there exists the unique $z \in C_2$ such that

$$f_2(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C_2.$$

Denote z by $T_r^{f_2}(x)$, that is, $T_r^{f_2}(x) = z$. It is easy to show that $T_r^{f_2} x = \frac{x}{1+r}$ for each $x \in H_2$. Note that f_1 satisfies the Condition 1 and Condition A with $\varepsilon = 0$ in [22, Algorithm 1] and [31, Algorithm 3.1], respectively. Hence the Algorithm 1 in [22] (HIEUP) and Algorithm 3.1 in [31] (HIEUT) can be applied to this example.

All the optimization problems in Algorithm 3.3 are reduced to the projections on C_1 and C_2 which can be explicitly computed. The parameters chosen for Algorithm 3.3, HIEUT and HIEUP are as follow:

(i) $\alpha_n = \frac{n+1}{4n}$, $\beta_n = \frac{n+1}{40n}$ for Algorithm 3.3;
(ii) $\lambda = \mu = \lambda_n = \gamma = \frac{1}{3}$ for HIEUP and Algorithm 3.3;
(iii) $\rho_n = 1$, $\mu = \frac{1}{3}$, $\varepsilon = 0$ and $\beta_n = \frac{1}{n}$ for HIEUT;
(iv) $r_n = 1$ for HIEUP and HIEUT.

We use the sequence $\{\|x_n - x^*\|\}$ to check the convergence of Algorithm 3.3, HIEUP and HIEUT with the initial point $x_1(t) = t$, $x_1(t) = 2t^2 + t + 1/6$ and $x_1(t) = 8t^5 + 5t^4 + t$, where the sequence $\{x_n\}$ is generated by the algorithms. The convergence of $\{\|x_n - x^*\|\}$ implies that $\{x_n\}$ converges to the solution of the split equilibrium problem. The following figures are the iterations and elapsed times for the algorithms with the different initial points. The figures show that the rates of convergence of $\{x_n\}$ are very fast for the three algorithms.

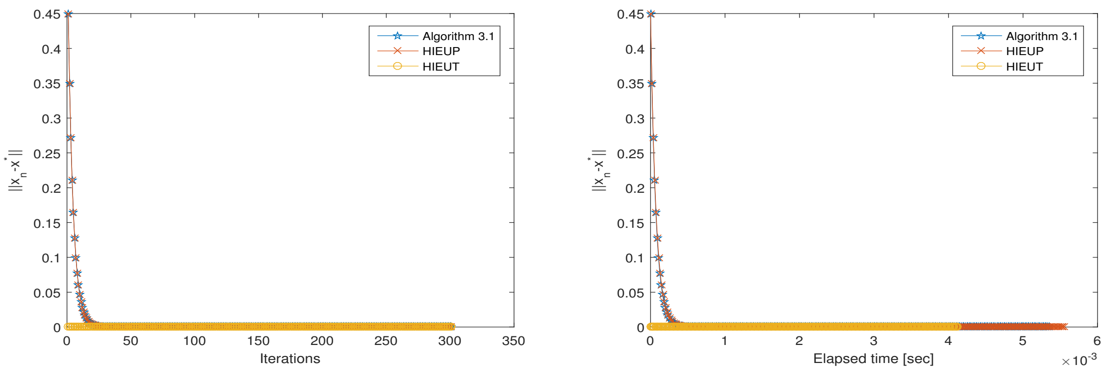

FIGURE 1. Iteration and elapsed time with $x_1(t) = t$

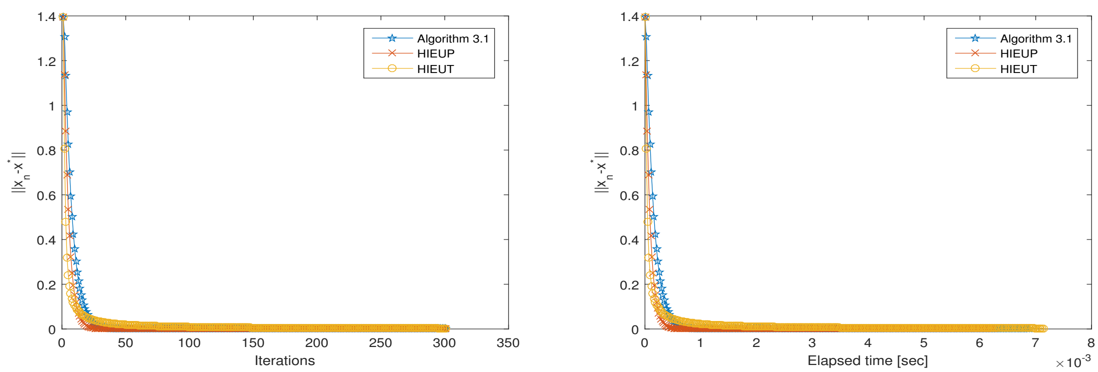

FIGURE 2. Iteration and elapsed time with $x_1(t) = 2t^2 + t + \frac{1}{6}$

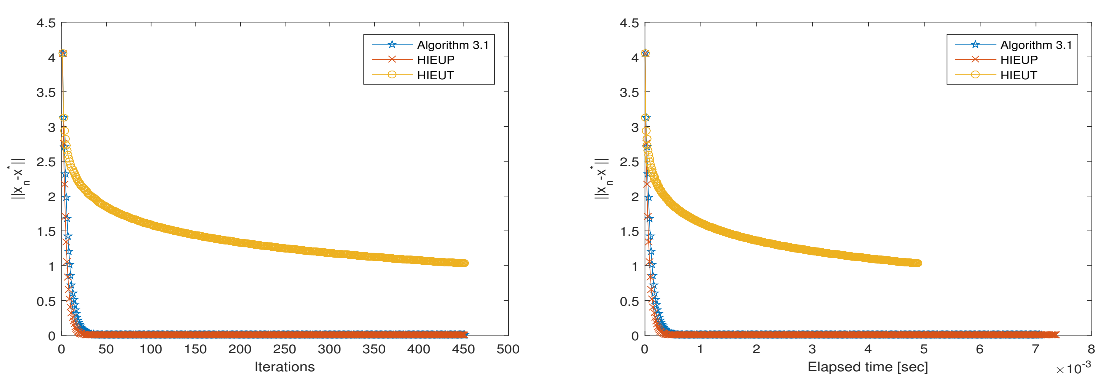

FIGURE 3. Iteration and elapsed time with $x_1(t) = 8t^5 + 5t^4 + t$

REFERENCES

[1] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007), 506-515.

[2] A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, J. Optim. Theory Appl. 133 (2007), 359-370.

[3] X. Qin, S.Y. Cho, S.M. Kang, Strong convergence of shrinking projection methods for quasi-phi-nonexpansive mappings and equilibrium problems, J. Comput. Appl. Math. 234 (2010), 750-760.

[4] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming using proximal like algorithms, Math. Program. 78 (1997), 29-41.

[5] J.W. Peng, J.C. Yao, A modified CQ method for equilibrium problems, fixed points and variational inequality, Fixed Point Theory, 9 (2008), 515-531.

[6] S.Y. Cho, X. Qin, On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems, Appl. Math. Comput. 235 (2014), 430-438.

[7] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008), 1025–1033.

[8] X. Qin, S.Y. Cho, S.M. Kang, On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings, Appl. Math. Comput. 215 (2010), 3874-3883.

[9] Y. Censor, T.A. Elving, A multiprojections algorithm using Bregman projections in a product spaces, Numer. Algo. 8 (1994), 221-239.

[10] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numer. Algo. 59 (2012), 301-323.

[11] Z. He, The split equilibrium problem and its convergence algorithms, J. Inequal. Appl. 2012 (2012), Article ID 162.

[12] A. Moudafi, Split monotone variational inclusions, J. Optim Theory Appl. 150 (2011), 275-283.

[13] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018), 157-165.

[14] K.R. Kazmi, S.H. Rizvi, Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem, J. Egyptian Math. Soc. 21 (2013), 44–51.

[15] S. Wang, X. Gong, A. Abdou, Y.J. Cho, Iterative algorithm for a family of split equilibrium problems and fixed point problems in Hilbert spaces with applications. Fixed Point Theory Appl. 2016 (2016), Article ID 4.

[16] S. Wang, X. Gong, S.M. Kang, Strong convergence theorem on split equilibrium and fixed point problems in Hilbert spaces, Bull. Malays. Math. Sci. Soc. 41 (2018), 1309-1326.

[17] D.J. Wen, Y.A. Chen, General iterative methods for generalized equilibrium problems and fixed point problems of k-strict pseudo-contractions, Fixed Point Theory Appl. 2012: 125.

[18] L. Liu, A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings, J Nonlinear Convex Anal. 20 (2019), 471-488.

[19] L.Q. Thuy, P.K. Anh, L.D. Muu, T.N. Hai, Novel hybrid methods for pseudomonotone equilibrium problems and common fixed point problems, Numer. Funct. Anal. Optim. 38 (2017), 443-465.

[20] P.N. Anh, H.A. Le Thi, An Armijo-type method for pseudomonotone equilibrium problems and its applications, J. Global Optim. 57 (2013), 803-820.

[21] D.V. Hieu, L.D. Muu, P.K. Anh, Parallel hybrid extragradient methods for pseudomotone equilibrium problems and nonexpansive mappings, Numer. Algor. 73 (2016), 197-217.

[22] D.V. Hieu, New extragradient method for a class of equilibrium problems in Hilbert spaces, Appl. Anal. 97 (2017), 811-824.

[23] B.V. Dinh, L.D. Muu, A projection algorithm for solving pseudomonotone equilibrium problems and its application to a class of bilievel equilibria, Optimization. 64 (2015), 559-575.

[24] D.Q. Tran, L.D. Muu, V.H. Nguyen, Extragradient algorithms extended to equilibrium problems, Optimization. 57 (2008), 749-776.

[25] P.T. Vuong, J.J. Strodiot, V.H. Nguyen, Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems, J. Optim. Theory Appl. 155 (2012), 605-627.

[26] P.N. Anh, A hybrid extragradient method extended to fixed point problems and equilibrium problems, Optimization 62 (2013), 271-283.

[27] L. Mathiesen, An algorithm based on a sequence of linear complementarity problems applied to a Walrasian equilibrium model: an example, Math. Program. 37 (1987), 1-18.

[28] S. Wang, M. Zhao, P. Kumam, Y.J. Cho, A viscosity extragradient method for an equilibrium problem and fixed point problem in Hilbert space, J. Fixed Point Theory Appl. 20 (2018), Article ID 19.
[29] H. Zhou, Minimum-norm fixed point of nonexpansive mappings with applications, Optimization. 64 (2013), 799-814.
[30] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 561-597.
[31] D.V. Hieu, Two hybrid algorithms for solving split equilibrium problems, Int. J. Comput. Math. 95 (2018), 561-583.