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A QUASISTATIC CONTACT PROBLEM WITH COULOMB FRICTION IN ELECTRO-VISCOELASTICITY WITH LONG-TERM MEMORY BODY

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Abstract. We consider a quasistatic contact problem with coulomb friction in electro-viscoelasticity with long-term memory body. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable. We derive variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field, the damage field and the adhesion field. We prove the existence of a unique weak solution to the problem. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed points.

Keywords. Electro-viscoelastic Materials; Friction contact with adhesion, Fixed point; Variational inequality; Differential equation.

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1. Introduction

In this paper, we study a mathematical model which describes the adhesive contact problem with damage for an electro viscoelastic with long-term memory body, when the frictional tangential traction with the traction due to adhesion. We derive a variational formulation of the model and prove its unique solvability, which provides the existence of a unique weak solution to the adhesive contact problem.

The piezoelectric effect is the apparition of electric charges on surfaces of particular crystals after deformation. Its reverse effect consists of the generation of stress and strain in crystals under the action of the electric field on the boundary. Materials undergoing piezoelectric materials effects are called piezoelectric materials, and their study requires techniques and results from electromagnetic theory and continuum mechanics. Piezoelectric materials are used extensively as switches and, actually, in many engineering systems in radioelectronics, electroacoustics and measuring equipment. However, there

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are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [1, 2, 3, 4, 6, 7, 8] and the references therein.

Process of adhesion are important in industry where parts, usually non metallic, are glued together. Recently, composite materials reached prominence, since they are very strong and light, and therefore, of considerable importance in aviation, space exploration and in the automotive industry. However, composite materials my undergo delamination under stress, in which different layers debond and move relative to each other. To model the process when bonding is not permanent, and debonding may take place, we need to describe the adhesion together with the contact. A number of recent publications deal with such models, see, e.g., [9, 10, 11, 12] and the references therein.

The subject of damage is extremely important in design engineering since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [13, 14] from the virtual power principle. The mathematical analysis of one-dimensional problems can be found in [15]. In all these results, the damage of the material is described by a damage function α restricted to have values between zero and one. If $\alpha = 1$, there is no damage in the material. If $\alpha = 0$, then the material is completely damaged. If $0 < \alpha < 1$, there is a partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [5, 16, 17, 18] and the references therein. In this paper, the inclusion describing the evolution of damage field is

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K\ (\alpha) \ni S(\varepsilon(\mathbf{u}), \alpha),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\zeta \in H^1(\Omega)\ \mid\ 0 \le \zeta \le\ 1 \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of set K and S is a given constitutive function which describes the sources of the damage in the system.

We use an electro viscoelastic constitutive law with long-term memory given by

$$\sigma = \mathscr{A}\varepsilon(\dot{\mathbf{u}}) + \mathscr{G}(\varepsilon(\mathbf{u}), \alpha) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla(\varphi),$$
$$\mathbf{D} = -B\nabla(\varphi) + \mathscr{E}\varepsilon(\mathbf{u}),$$

where $\mathscr{A}$ is a given nonlinear function, $\mathscr{M}$ is the relaxation tensor, and $\mathscr{G}$ represents the elasticity operator where α is an internal variable describing the damage of the material caused by elastic deformations. $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathscr{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathscr{E}^*$ is its transposition and $\mathbf{B}$ denotes the electric permittivity tensor.

The paper is organized as follows. In Section 2, we introduce some essential preliminaries. In Section 3, we present the mechanical problem, list the assumptions on the data, and give the variational formulation of the problem. In Section 4, the last section, we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on the theory of evolution equations with monotone operators, and a classical existence-uniqueness result for parabolic inequalities.

2. PRELIMINARIES

In this section, we present some essential tools for our main results. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d=2,3$), while $(\cdot,\cdot)$ and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\mathbf{u}\cdot\mathbf{v}=u_iv_i, \qquad \|\mathbf{v}\|=(\mathbf{v}\cdot\mathbf{v})^{1/2}, \quad \forall \mathbf{u},\mathbf{v}\in\mathbb{R}^d$$
$$\sigma\cdot\tau=\sigma_{ij}\tau_{ij}, \qquad \|\tau\|=(\tau\cdot\tau)^{1/2}, \quad \forall \sigma,\tau\in\mathbb{S}^d.$$

respectively. Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation

$$H=L^2(\Omega)^d=\{\mathbf{u}=(u_i)\mid u_i\in L^2(\Omega)\}, \qquad H_1(\Omega)^d=\{\mathbf{u}=(u_i)\mid u_i\in H^1(\Omega)\}$$
$$\mathscr{H}=\{\sigma=(\sigma_{ij})\mid \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\}, \qquad \mathscr{H}_1=\{\sigma\in\mathscr{H}\mid Div\sigma\in H\}.$$

we consider that $\varepsilon: H_1(\Omega)^d\to\mathscr{H}$ and Div : $\mathscr{H}_1\to H$ are the deformation and the divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u})=(\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u})=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad \operatorname{Div}\sigma=(\sigma_{ij,i}).$$

The spaces H, $H^1(\Omega)^d$, $\mathscr{H}$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H=\int_\Omega \mathbf{u}\cdot\mathbf{v}dx, \quad \forall\mathbf{u},\mathbf{v}\in H, \qquad (\mathbf{u},\mathbf{v})_{H^1(\Omega)^d}=(\mathbf{u},\mathbf{v})_H+(\nabla\mathbf{u},\nabla\mathbf{v})_H,$$

where $\nabla\mathbf{v}=(v_{i,j})$, $\forall\mathbf{v}\in H^1(\Omega)^d$

$$(\sigma,\tau)_{\mathscr{H}}=\int_\Omega \sigma\cdot\tau dx, \quad \forall\sigma,\tau\in\mathscr{H},$$
$$(\sigma,\tau)_{\mathscr{H}_1}=(\sigma,\tau)_{\mathscr{H}}+(\operatorname{Div}\sigma,\operatorname{Div}\tau)_H, \quad \forall\sigma,\tau\in\mathscr{H}_1.$$

The associated norms on H, $H^1(\Omega)^d$, $\mathscr{H}$ and $\mathscr{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathscr{H}}$, and $\|\cdot\|_{\mathscr{H}_1}$, respectively. Let $H_\Gamma=H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma: H^1(\Omega)^d\to H_\Gamma$ be the trace map. For every element $\mathbf{v}\in H^1(\Omega)^d$, we also write $\mathbf{v}$ for the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu=\mathbf{v}\cdot\nu, \qquad \mathbf{v}_\tau=\mathbf{v}-v_\nu\nu. \tag{2.1}$$

Similarly, for a regular tensor field $\sigma:\Omega\to\mathbb{S}^d$, we define its normal and tangential components by

$$\sigma_\nu=(\sigma\nu).\nu, \qquad \sigma_\tau=\sigma\nu-\sigma_\nu\nu. \tag{2.2}$$

We recall that the following Green's formula holds

$$(\sigma,\varepsilon(\mathbf{v}))_{\mathscr{H}}+(\operatorname{Div}\sigma,\mathbf{v})_H=\int_\Gamma \sigma\nu.\mathbf{v}da, \quad \forall\mathbf{v}\in H_1. \tag{2.3}$$

Finally, for the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g.,[5] ,p.48)

Theorem 2.1. *Assume that* $(X,\|\cdot\|_X)$ *is a real Banach space and* $T>0$*. Let* $F(t,\cdot):X\to X$ *be an operator defined* a.e. *on* $(0,T)$ *satisfying the following conditions*

$$\begin{cases} \text{There exists } L_F>0 \text{ such that} \\ \|F(t,x)-F(t,y)\|_X \le L_F\|x-y\|_X \qquad \forall x,y\in X, \text{a.e.} t\in(0,T), \end{cases}$$

and there exists $1\le p\le\infty$ *such that* $F(\cdot,x)\in L^p(0,T;X)$, $\forall x\in X$*. Then, for any* $x_0\in X$*, there exists a unique function* $x\in W^{1,p}(0,T;X)$ *such that*

$$\begin{cases} \dot{x}(t)=F(t,x(t)) \quad a.e.\ t\in(0,T), \\ x(0)=x_0. \end{cases}$$

Theorem 2.1 will be used in Section 4 to prove the unique solvability of the intermediate problem involving the bonding field. The following existence, uniqueness and regularity result is carried out in the next theorem and is based on the following abstract result for evolutionary variational inequalities

Theorem 2.2. *Let* X *be a Hilbert space. Assume that the operator* $A:X\to X$ *satisfies*

$$\begin{cases} \text{(a) } A:X\to X \text{ is strongly monotone and Lipschitz continuous, i.e.,} \\ \text{(b)there exists } m_A>0 \text{ such that} \\ \qquad (Au_1-Au_2,u_1-u_2)_X \ge m_A\|u_1-u_2\|_X^2 \quad \forall u_1,u_2\in X; \\ \text{(c)there exists } L_A \text{ such that} \\ \qquad \|Au_1-Au_2\|_X \le L_A\|u_1-u_2\|_X \quad \forall u_1,u_2\in X, \end{cases} \tag{2.4}$$

and that $j:X\to\bar{\mathbb{R}}$ *is a proper, convex, and lower semi continuous functional*

$$\begin{cases} \text{(a) } j(u,\cdot) \text{ is convex and lower semi continuous on } X \text{ for all } u\in X. \\ \text{(b) There exists } m_A>0 \text{ such that} \\ \qquad j(u_1,v_2)-j(u_1,v_1)+j(u_2,u_1)-j(u_2,u_2) \\ \qquad \le m_A\,\|u_1-u_2\|_X\,\|v_1-v_2\|_X \quad \forall u_1,u_2,v_1,v_2\in X. \end{cases} \tag{2.5}$$

Then, for each $f\in X$*, the elliptic variational inequality of the second kind,*

$$(Au,v-u)_X+j(v)-j(u)\ge(f(t),v-u(t))_X, \tag{2.6}$$

has a unique solution. Moreover, the solution depends Lipschitz continuously on f*.*

Theorem 2.3. *Let* $V\subset H\subset V'$ *be a Gelfand triple. Let* K *be a nonempty closed, and convex set of* V*. Assume that* $a(\cdot,\cdot):V\times V\to\mathbb{R}$ *is a continuous and symmetric bilinear form such that, for some constants* C_0 *and* $C_1>0$*,*

$$a(\mathbf{v},\mathbf{v})+C_0|\mathbf{v}|_H^2\ge C_1|\mathbf{v}|_V^2 \quad \forall\mathbf{v}\in V.$$

Then, for every $\mathbf{u}_0\in \mathrm{K}$ *and* $\mathbf{f}\in L^2(0,T;H)$*, there exists a unique function* $\mathbf{u}\in W^{1,2}(0,T;H)\cap L^2(0,T;V)$ *such that* $\mathbf{u}(0)=\mathbf{u}_0$*,* $\mathbf{u}(t)\in\mathrm{K}\quad\forall t\in[0,T]$*, and for almost all* $t\in(0,T)$*,*

$$(\dot{\mathbf{u}}(t),\mathbf{v}-\mathbf{u}(t))_{\acute{V}\times V}+a(\mathbf{u}(t),\mathbf{v}-\mathbf{u}(t))\ge(\mathbf{f}(t),\mathbf{v}-\mathbf{u}(t))_H, \quad \forall\mathbf{v}\in\mathrm{K}.$$

3. Mechanical and variational formulations

We describe the model for the process and present its variational formulation. The physical setting is the following. An electro viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $(d = 2,3)$ with outer Lipschitz surface Γ. The body submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also submitted the mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and in two measurable parts Γ_a and Γ_b, on the other hand, such that $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there. A surface traction of density f_2 acts on $\Gamma_2 \times (0,T)$ and a body force of density f_0 acts in $\Omega \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3. Moreover, the process is quasistatic, and thus the inertial terms are neglected in the equation of motion. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor.

To simplify the notations, we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. Then, the classical formulation of the mechanical problem of electro-viscoelastic material, frictional, adhesive contact may be stated as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$, a damage field β : $\Omega \times [0,T] \to \mathbb{R}$, and a bonding field $\alpha : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}\sigma(t) = \mathscr{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathscr{G}(\varepsilon(\mathbf{u}(t)),\alpha)\\ + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla\varphi(t)\end{aligned} \qquad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\mathbf{D} = \mathscr{E}\varepsilon(\mathbf{u}) - B\nabla(\varphi) \qquad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni S(\varepsilon(\mathbf{u}),\alpha), \qquad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$\text{Div}\,\sigma + \mathbf{f}_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{3.4}$$

$$\text{div}\,\mathbf{D} - q_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{3.5}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{3.6}$$

$$\sigma\nu = \mathbf{f}_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{3.7}$$

$$\begin{cases} -\sigma_\nu = p_\nu(u_\nu - g), \\ \|\sigma_\tau\| \le p_\tau(u_\nu - g), \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\dot{u}_\tau \neq 0 \Rightarrow \sigma_\tau = -p_\tau(u_\nu - g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu))^2 + \gamma_\tau\|\mathbf{R}_\tau(\mathbf{u}_\tau)\|^2) - \varepsilon_a)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.10}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{3.11}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{3.12}$$

$$\mathbf{D}\cdot\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{3.13}$$

$$\mathbf{D}\cdot\nu = \psi(u_\nu - g)\phi_l(\varphi-\varphi_0) \qquad \text{on } \Gamma_3\times(0,T), \tag{3.14}$$

$$\mathbf{u}(0)=\mathbf{u}_0, \qquad \alpha(0)=\alpha_0 \qquad \text{in } \Omega, \tag{3.15}$$

$$\beta(0)=\beta_0 \qquad \text{on } \Gamma_3. \tag{3.16}$$

(3.1) and (3.2) represent the electro-viscoelastic constitutive law with long term-memory and damage. The evolution of the damage field is governed by the inclusion of parabolic type given by relation (3.3), where S is the mechanical source of the damage growth. $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Equations (3.4) and (3.5) represent the equilibriumequations for the stress and electric displacement fields while (3.6) and (3.7) are the displacement and traction boundary condition, respectively.

We turn to boundary condition (3.8)-(3.9)and (3.14) which describe the mechanical conditions on the potential contact surface Γ_3. The normal compliance function p_ν in (3.8) is described below, and g represents the gap in the reference configuration between Γ_3 and the foundation, and measured along the direction of ν. When positive, $u_\nu - g$ represents the interpenetration of the surface asperities into those of the foundation. Conditions (3.9) is the associated friction law where p_τ is a given function. According to (3.9), the tangential shear cannot exceed the maximum frictional resistance $p_\tau(u_\nu - g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion.

Equation (3.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [19] (see also [5] for more details). In addition to γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2R_\nu(u_\nu)$. The adhesive traction is tensile and proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s)=\begin{cases} L, & \text{if} \quad s<-L,\\ -s, & \text{if} \quad -L\le s\le 0,\\ 0, & \text{if} \quad s>0.\end{cases}$$

Here $L>0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (3.10) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v})=\begin{cases} \mathbf{v}, & \text{if} \quad |\mathbf{v}|\le L,\\ L\dfrac{\mathbf{v}}{|\mathbf{v}|}, & \text{if} \quad |\mathbf{v}|>L.\end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. The introduction of operator R_ν, together with operator R_τ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of L is made in what follows. Relation (3.11) describes a homogeneous Neumann boundary condition, where $\partial\alpha/\partial\nu$ is the normal derivative of β. (3.12) and (3.13) represent the electric boundary conditions. Next, (3.14) is the electrical contact condition on Γ_3, introduced in [9, 20]. It may be obtained as follows.

We assume that the foundation is electrically conductive and its potential is maintained at φ_0. When there is no contact at a point on the surface (i.e., $u_\nu < g$), the gap is assumed to be an insulator (say, it is filled with air), there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$u_\nu < g \;\Rightarrow\; \mathbf{D}\cdot\nu = 0. \tag{3.17}$$

During the process of the contact (i.e., when $u_\nu \geq g$) the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with k as the proportionality factor. Thus,

$$u_\nu \geq g \;\Rightarrow\; \mathbf{D}\cdot\nu = k\,(\varphi-\varphi_0). \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\mathbf{D}\cdot\nu = k\,\chi_{[0,\infty)}(u_\nu - g)\,(\varphi-\varphi_0), \tag{3.19}$$

where $\chi_{[0,\infty)}$ is the characteristic function of $[0,\infty)$, that is,

$$\chi_{[0,\infty)}(r) = \begin{cases} 0, & \text{if } r<0,\\ 1, & \text{if } r\geq 0.\end{cases}$$

Condition (3.19) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications. To make it more realistic, we regularize condition (3.19) and write it as (3.14) in which $k\,\chi_{[0,\infty)}(u_\nu - g)$ is replaced with ψ, which is a regular function and ϕ_l is the truncation function

$$\phi_l(s) = \begin{cases} -l, & \text{if } s<-l,\\ s, & \text{if } -l\leq s\leq l,\\ l, & \text{if } s>l,\end{cases}$$

where l is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since l may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\phi_l(\varphi-\varphi_0) = \varphi-\varphi_0$. The reasons for regularization (3.14) of (3.19) are mathematical. First, we need to avoid the discontinuity in the free electric charge when the contact is established and, therefore, we regularize function $k\,\chi_{[0,\infty)}$ in (3.19) with a Lipschitz continuous function ψ. A possible choice is

$$\psi(r) = \begin{cases} 0, & \text{if } r<0,\\ k\delta r, & \text{if } 0\leq r\leq 1/\delta,\\ k, & \text{if } r>\delta,\end{cases} \tag{3.20}$$

where $\delta > 0$ is a small parameter. This choice means that during the process of the contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when $u_\nu - g$

reaches δ. Second, we need $\phi_l(\varphi-\varphi_0)$ to control the boundednes of $\varphi-\varphi_0$. If $\psi\equiv 0$ in (3.14), then

$$\mathbf{D}\cdot\nu=0,\quad \text{on } \Gamma_3\times(0,T), \tag{3.21}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (3.21) models the case when the obstacle is a perfect insulator and was used in [21, 22]. Condition (3.14), instead of (3.21), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model. Because of friction condition (3.9), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. In equation (3.15), $\mathbf{u}_0$ is the initial displacement, and β_0 is the initial damage. Finally, in equation (3.16), α_0 denotes the initial bonding.

To obtain the variational formulation of (3.1)-(3.16), we introduce

$$\mathscr{Z}=\left\{\ \theta\in L^\infty(0,T;L^2(\Gamma_3))\quad :0\le\theta(t)\le 1,\quad \text{a.e. } t\in[0,T],\quad \text{on } \Gamma_3\ \right\},$$

and for the displacement field we need the closed subspace of $H^1(W)^d$ defined by

$$V=\left\{v\in H^1(\Omega)^d\mid v=0 \text{ on } \Gamma_1\right\}.$$

Since $\text{meas}(\Gamma_1)>0$, Korn's inequality holds and there exists a constant $C_k>0$, that depends only on W and Γ_1 such that

$$\|\varepsilon(v)\|_{\mathscr{H}}\ge C_k\|v\|_{H^1(\Omega)^d},\quad \forall v\in V.$$

On V, we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v})_V=(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathscr{H}},\quad \|\mathbf{v}\|_V=\|\varepsilon(\mathbf{v})\|_{\mathscr{H}},\quad \forall\mathbf{u},\mathbf{v}\in V. \tag{3.22}$$

It follows that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V,(\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant C_0, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d}\le\tilde{C}_0\|\mathbf{v}\|_V,\quad \forall\mathbf{v}\in V. \tag{3.23}$$

We also introduce the spaces

$$W=\left\{\phi\in H^1(\Omega)\mid\phi=0 \text{ on } \Gamma_a\right\},$$

$$\mathscr{W}=\{\ \mathbf{D}=(D_i)\mid D_i\in L^2(\Omega),\ \text{div}\,\mathbf{D}\in L^2(\Omega)\},$$

where $\text{div}\,D=(D_{i,i})$. The spaces W and $\mathscr{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi,\phi)_W=\int_\Omega\nabla\varphi\cdot\nabla\phi\,dx,$$

$$(\mathbf{D},\mathbf{E})_{\mathscr{W}}=(\mathbf{D},\mathbf{E})_H+(\text{div}\,\mathbf{D},\text{div}\,\mathbf{E})_{L^2(\Omega)},$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathscr{W}}$, respectively. Moreover, by the Sobolev trace theorem, there exists a constant c_0, depending only on Ω, Γ_a and Γ_3 such that

$$\|\phi\|_{L^2(\Gamma_3)}\le c_0\|\phi\|_H, \tag{3.24}$$

If $\mathbf{D}\in\mathscr{W}$ is a regular function, then the following Green's type formula holds

$$(\mathbf{D},\nabla\zeta)_H+(\text{div}\,\mathbf{D},\zeta)_{L^2(\Omega)}=\int_\Gamma\mathbf{D}\cdot\nu\zeta\,da,\quad \forall\zeta\in H^1(\Omega). \tag{3.25}$$

Notice also that, since meas$(\Gamma_a) > 0$, the following Friedrichs-Poincar inequality holds

$$\|\nabla\zeta\|_H \geq C_F \|\zeta\|_{H^1(\Omega)}, \quad \forall \zeta \in W, \tag{3.26}$$

where $C_F > 0$ is a constant which depends only on W and Γ_a.

In the study of mechanical problem (3.1)-(3.16), we assume that viscosity function $\mathscr{A} : \Omega \times \mathscr{S}^d \to \mathscr{S}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists constants } \mathscr{C}_1^{\mathscr{A}}, \mathscr{C}_2^{\mathscr{A}} > 0 \text{ such that} \\
\qquad \|\mathscr{A}(\mathbf{x},\varepsilon)\| \leq \mathscr{C}_1^{\mathscr{A}} \|\varepsilon\| + \mathscr{C}_2^{\mathscr{A}} \qquad \forall \varepsilon \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(b) There exists } m_{\mathscr{A}} > 0 \text{ such that} \\
\qquad (\mathscr{A}(\mathbf{x},\varepsilon_1) - \mathscr{A}(\mathbf{x},\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathscr{A}} \|\varepsilon_1 - \varepsilon_2\|^2 \\
\qquad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) The mapping } \mathbf{x} \mapsto \mathscr{A}(\mathbf{x},\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \text{for any } \varepsilon \in \mathbb{S}^d. \\
\text{(d) The mapping } \varepsilon \mapsto \mathscr{A}(\mathbf{x},\varepsilon) \text{ is continuous on } \mathbb{S}^d \text{a.e. } \mathbf{x} \in \Omega.
\end{array}\right. \tag{3.27}$$

The elasticity operator $\mathscr{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists } L_{\mathscr{G}} > 0 \text{ such that} \\
\qquad \|\mathscr{G}(\mathbf{x},\xi_1,\alpha_1) - \mathscr{G}(\mathbf{x},\xi_2,\alpha_2)\| \leq L_{\mathscr{G}}(\|\xi_1 - \xi_2\| + \|\alpha_1 - \alpha_2\|) \\
\qquad \forall \xi_1, \xi_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(b) The mapping } \mathbf{x} \mapsto \mathscr{G}(\mathbf{x},\xi,\alpha) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \text{for any } \xi \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}. \\
\text{(c) The mapping } \mathbf{x} \mapsto \mathscr{G}(\mathbf{x},0,0) \text{ belongs to } \mathscr{H}.
\end{array}\right. \tag{3.28}$$

The damage source function $S : \Omega \times S^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists a constant } L_S > 0 \text{ such that} \\
\qquad \|S(\mathbf{x},\varepsilon_1,\alpha_1) - S(\mathbf{x},\varepsilon_2,\alpha_2)\| \leq L_S(\|\varepsilon_1 - \varepsilon_2\| + \|\alpha_1 - \alpha_2\|) \\
\qquad \forall \varepsilon_1, \varepsilon_2 \in S^d, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e.} \mathbf{x} \in \Omega. \\
\text{(b) } \forall \varepsilon \in \mathbb{S}^d, \text{ and } \alpha \in \mathbb{R}, \quad S(\mathbf{x},\varepsilon,\alpha) \text{ the function is measurable in } \Omega. \\
\text{(c) The mapping } \mathbf{x} \mapsto S(\mathbf{x},0,0) \text{ belongs to } L^2(\Omega).
\end{array}\right. \tag{3.29}$$

The electric permittivity operator $B = (B_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) } B(x,E) = (B_{ij}(x)E_j) \;\; \forall E = (E_i) \in \mathbb{R}^d, \text{ p.p.} x \in \Omega. \\
\text{(b) } B_{ij} = B_{ji} \in L^\infty(\Omega), \; 1 \leq i,j \leq d. \\
\text{(c) There exists a constant } M_B > 0 \text{ such that} \\
\qquad BE.E \geq M_B |E|^2 \;\; \forall E = (E_i) \in \mathbb{R}^d, \text{ a.a.in } \Omega.
\end{array}\right. \tag{3.30}$$

The piezoelectric operator $\mathscr{E} : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) } \mathscr{E} = (e_{ijk}), \; e_{ijk} \in L^\infty(\Omega), \; 1 \leq i,j,k \leq d. \\
\text{(b) } \mathscr{E}(\mathbf{x})\sigma.\tau = \sigma.\mathscr{E}^*\tau, \quad \forall \sigma \in S^d, \quad \forall \tau \in \mathbb{R}^d.
\end{array}\right. \tag{3.31}$$

The normal compliance functions $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, $(r = \nu, \tau)$ satisfy

$$\begin{cases} \text{(a) } \exists L_r > 0 \text{ such that } \|p_r(\mathbf{x}, u_1) - p_r(\mathbf{x}, u_2)\| \le L_r \|u_1 - u_2\| \\ \quad \forall u_1, u_2 \in \mathbb{R}, \text{a.e.} \mathbf{x} \in \Gamma_3. \\ \text{(b) } \mathbf{x} \mapsto p_r(\mathbf{x}, u) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}. \\ \text{(c) } \mathbf{x} \mapsto p_r(\mathbf{x}, u) = 0, \text{ for all } u \le 0. \end{cases} \tag{3.32}$$

The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists L_\psi > 0 \text{ such that } \|\psi(\mathbf{x}, u_1) - \psi(\mathbf{x}, u_2)\| \le L_\psi \|u_1 - u_2\| \\ \quad \forall u_1, u_2 \in \mathbb{R}, \text{ a.e.} \mathbf{x} \in \Gamma_3. \\ \text{(b) } \exists M_\psi > 0 \text{ such that } \|\psi(\mathbf{x}, u)\| \le M_\psi \ \forall u \in \mathbb{R}, \text{ a.e.} \mathbf{x} \in \Gamma_3. \\ \text{(c) } \mathbf{x} \mapsto \psi(\mathbf{x}, u) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}. \\ \text{(d) } \mathbf{x} \mapsto \psi(\mathbf{x}, u) = 0, \text{ for all } u \le 0. \end{cases} \tag{3.33}$$

The relaxation tensor $\mathscr{M}$ satisfies

$$M \in C(0, T; \mathscr{H}). \tag{3.34}$$

The adhesion coefficients and the limit bound satisfies

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \varepsilon_a \ge 0, \text{ a.e. on } \Gamma_3. \tag{3.35}$$

The initial bonding field satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1, \text{ a.e. on } \Gamma_3 \tag{3.36}$$

and the initial damage field satisfies

$$\alpha_0 \in K. \tag{3.37}$$

Finally, we assume that the gap function, the given potential and the initial displacement satisfy

$$g \in L^2(\Gamma_3), \quad g \ge 0, \quad \text{a.e. on } \Gamma_3, \tag{3.38}$$

$$\varphi_0 \in L^2(\Gamma_3), \tag{3.39}$$

$$\mathbf{u}_0 \in V. \tag{3.40}$$

The forces, tractions, volume and surface free charge densities satisfy

$$\mathbf{f}_0 \in C(0, T; L^2(\Omega)^d), \qquad \mathbf{f}_2 \in C(0, T; L^2(\Gamma_2)^d), \tag{3.41}$$

$$q_0 \in C(0, T; L^2(\Omega)), \qquad q_2 \in C(0, T; L^2(\Gamma_b)). \tag{3.42}$$

Here, $1 \le p \le \infty$. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla \xi . \nabla \varphi dx, \tag{3.43}$$

and the microcrack diffusion coefficient verifies

$$k > 0. \tag{3.44}$$

Next, we use the Riesz representation theorem to define $\mathbf{f} : [0, T] \to V$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \, da, \tag{3.45}$$

for all $v \in V$, $t \in [0, T]$. Then conditions (3.41) and (3.45) imply

$$\mathbf{f} \in C(0, T; V), \tag{3.46}$$

and we denote by $q : [0,T] \to W$ the function defined by

$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta\,dx - \int_{\Gamma_b} q_2(t)\zeta\,da, \tag{3.47}$$

for all $\zeta \in W$, $t \in [0,T]$. Then conditions (3.42) and (3.47) imply

$$\mathbf{q} \in C(0,T;W). \tag{3.48}$$

Next, we denote by $j : V \times V \to \mathbb{R}$ the functional

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} (p_\nu(u_\nu - g)v_\nu + p_\tau(u_\nu - g)\|\mathbf{v}_\tau\|)da. \tag{3.49}$$

By the assumptions on p_ν and p_τ, we obtain that, for $v \in V$,

$$p_\nu(u_\nu - g),\ p_\tau(u_\nu - g) \in L^2(\Gamma_3), \tag{3.50}$$

and, thus, $j(\cdot,\cdot)$ is well defined on $V \times V$.

Next, we define the mapping $h : V \times W \to W$ by

$$(h(\mathbf{u},\varphi),\zeta)_W = \int_{\Gamma_3} \psi(u_\nu - g)\phi_l(\varphi - \varphi_0)\zeta\,da. \tag{3.51}$$

Using standard arguments, we obtain the variational formulation of (3.1)-(3.16).

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathscr{H}$, an electric potential field $\varphi : [0,T] \to W$, a damage field $\alpha : [0,T] \to H^1(\Omega)$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathscr{G}\varepsilon(\mathbf{u}(t),\alpha) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla\varphi(t), \tag{3.52}$$

$$(\sigma(t),\varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t))_{\mathscr{H}} + j(\mathbf{u}(t),\mathbf{v}) - j(\mathbf{u}(t),\dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t),\mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{3.53}$$

for all $\mathbf{v} \in V$ and $t \in [0,T]$,

$$(B\nabla\varphi(t),\nabla\zeta)_H - (\mathscr{E}\varepsilon(\mathbf{u}(t)),\nabla\zeta)_H + (h(\mathbf{u}(t),\varphi(t)),\zeta)_W = (q(t),\zeta)_W, \tag{3.54}$$

for all $\zeta \in W$ and $t \in [0,T]$,

$$\begin{aligned} \alpha(t) \in K,\ (\dot{\alpha}(t),\xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t),\xi - \alpha(t)) \\ \geq (S(\varepsilon(\mathbf{u}(t)),\alpha(t)),\xi - \alpha(t))_{L^2(\Omega)}, \end{aligned} \tag{3.55}$$

for all $\xi \in K$ and $t \in [0,T]$,

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu))^2 + \gamma_\tau\|\mathbf{R}_\tau(\mathbf{u}_\tau)\|^2) - \varepsilon_a)_+ \ \text{ on } \Gamma_3 \times (0,T), \tag{3.56}$$

and

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \beta(0) = \beta_0, \qquad \alpha(0) = \alpha_0. \tag{3.57}$$

To study problem **PV**, we make the following assumption

$$M_\psi < \frac{m_{\mathbf{B}}}{c_0^2}, \tag{3.58}$$

where M_ψ, c_0 and $m_{\mathbf{B}}$ are given in (3.33), (3.23) and (3.29), respectively. We note that only the trace constant, the coercivity constant of $\mathbf{B}$ and the bound of are involved in (3.58). Therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated. So, $M_\psi = 0$. We notice that variational problem **PV** is formulated in terms of displacement field, an electrical potential field, damage field and bonding field. The existence of the unique solution of problem **PV** is stated and proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 3.1. We note that, in problem **P** and in problem **PV**, we do not need to impose explicitly the restriction $0<\beta<1$. Indeed, equation (3.57) guarantees that $\beta(x,t)\leq\beta_0(x)$ and, therefore, assumption (3.36) shows that $\beta(x,t)\leq 1$ for $t\geq 0$, a.e. $x\in\Gamma_3$. On the other hand, if $\beta(x,t_0)=0$ at time t_0, then it follows from (3.57) that $\dot{\beta}(x,t)=0$ for all $t\geq t_0$ and therefore, $\beta(x,t)=0$ for all $t\geq t_0$, a.e. $x\in\Gamma_3$. We conclude that $0\leq\beta(x,t)\leq 1$ for all $t\in[0,T]$, a.e. $x\in\Gamma_3$.

4. Existence and Uniqueness

Our main existence and uniqueness result for Problem $\mathscr{PV}$ is the following.

Theorem 4.1. *Assume that (3.7)-(3.23) hold. Then there exists $\mu_0>0$ depending only on Ω, Γ_1, $\mathscr{A}$ and p_r such that, if $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0$, then Problem $\mathscr{PV}$ has a unique solution $(\mathbf{u},\varphi)$. Moreover, the solution satisfies*

$$\mathbf{u}\in C^1(0,T;V), \tag{4.1}$$

$$\sigma\in C(0,T;\mathscr{H}_1), \tag{4.2}$$

$$\varphi\in C(0,T;W), \tag{4.3}$$

$$\alpha\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \tag{4.4}$$

$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap\mathscr{Z}. \tag{4.5}$$

The functions $u,\sigma,\varphi,\mathbf{D},\alpha$ and β which satisfy (3.1)-(3.2) and (3.50)-(3.55) are called weak solutions of contact problem **P**. We conclude that, under the assumptions (3.27)-(3.40), mechanical problem (3.1)-(3.16) has a unique weak solution satisfying (4.1)-(4.5). The regularity of the weak solution is given by (4.1)-(4.5) and, in term of electric displacement,

$$D\in C(0,T;\mathscr{W}). \tag{4.6}$$

Indeed, it follows from (3.53) that $\operatorname{div}D=q_0(t)$ for all $t\in[0,T]$. Therefore the regularity (4.1) and (4.2) of φ combined with (3.30), (3.31) and (3.42) implies (4.6).

The proof of Theorem 4.1 will be split into several steps. From now on, in this section, we always suppose that the assumptions of Theorem 4.1 hold, and we always assume that C is a generic positive constant which depends on Ω, Γ_1, Γ_3, p_ν, p_τ, γ_ν, γ_τ and L may change from place to place. Let $\eta\in C([0,T];\mathscr{H})$ and $\theta\in C([0,T];L^2(\Omega))$ be given.

Problem $\mathscr{PV}^u_\eta$. Find a displacement field $\mathbf{u}_\eta:[0,T]\to V$ and a stress field $\sigma_\eta:[0,T]\to\mathscr{H}_1$ such that for all $t\in[0,T]$,

$$\sigma_\eta=\mathscr{A}\varepsilon(\dot{\mathbf{u}}_\eta)+\eta, \tag{4.7}$$

$$(\sigma_\eta,\varepsilon(\omega-\dot{\mathbf{u}}_\eta)_{\mathscr{H}}+j(\mathbf{u}_\eta,\omega)-j(\mathbf{u}_\eta,\dot{\mathbf{u}}_\eta)\geq(\mathbf{f},\omega-\dot{\mathbf{u}}_\eta)_V,\quad\forall\omega\in V, \tag{4.8}$$

$$\mathbf{u}_\eta(0)=\mathbf{u}_0. \tag{4.9}$$

To study Problem $\mathscr{PV}^u_\eta$, we need the following lemma.

Lemma 4.2. *Let $\mathbf{g} \in C(0,T;V)$. Then there exists a unique function $\mathbf{v}_{\eta g} \in C(0,T;V)$ such that, for all $t \in [0,T]$,*

$$(\mathscr{A}\varepsilon(v_{\eta\mathbf{g}}), \varepsilon(\omega - v_{\eta\mathbf{g}}))_{\mathscr{H}} + j(\mathbf{g}, \omega) - j(\mathbf{g}, v_{\eta g}) \\ \geq (\mathbf{f}, \omega - v_{\eta\mathbf{g}})_V - (\eta, \varepsilon(\omega - v_{\eta\mathbf{g}}))_H \quad \forall \omega \in V. \tag{4.10}$$

Proof. It follows from Theorem 2.2 that there exists a unique function $v_{\eta\mathbf{g}} : [0,T] \to V$ solving the elliptic variational inequality (4.10). To establish its regularity by showing that $v_{\eta\mathbf{g}} \in C([0,T];V)$, we let $t_1, t_2 \in [0,T]$ and denote by $\eta_i = \eta(t_i)$, $\mathbf{g}_i = \mathbf{g}(t_i)$, $f_i = f(t_i)$, and $v_i = v_{\eta\mathbf{g}}(t_i)$, $i = 1,2$. We choose $\omega = v_2$ in (4.10) at $t = t_1$, $\omega = v_1$ in (4.10) at $t = t_2$, and add the two inequalities to obtain

$$(\mathscr{A}\varepsilon(\mathbf{v}_1) - \mathscr{A}\varepsilon(\mathbf{v}_2), \varepsilon(v_1 - v_2))_{\mathscr{H}} \leq (f_1 - f_2, v_1 - v_2)_V + \\ + (\eta_1 - \eta_2, \varepsilon(v_1 - v_2))_{\mathscr{H}} + j(\mathbf{g}_1, v_2) - j(\mathbf{g}_1, v_1) + j(\mathbf{g}_2, v_1) - j(\mathbf{g}_2, v_2). \tag{4.11}$$

The left-hand side is bounded from below by (3.27). Thus,

$$(\mathscr{A}\varepsilon(\mathbf{v}_1) - \mathscr{A}\varepsilon(\mathbf{v}_2), \varepsilon(v_1 - v_2))_{\mathscr{H}} \geq m_{\mathscr{A}} \|v_1 - v_2\|_V^2. \tag{4.12}$$

The last line of (4.12) is bounded by the property (3.32) as follows

$$j(\mathbf{g}_1, v_2) - j(\mathbf{g}_1, v_1) + j(\mathbf{g}_2, v_1) - j(\mathbf{g}_2, v_2) \leq c\|\mathbf{g}_1 - \mathbf{g}_2\|_V \|v_1 - v_2\|_V. \tag{4.13}$$

Using these bounds in (4.12), we obtain

$$\|v_1 - v_2\|_V \leq c\Big(\|f_1 - f_2\|_V + \|\eta_1 - \eta_2\|_{\mathscr{H}} + \|g_1 - g_2\|_V\Big). \tag{4.14}$$

Then the conclusion that $v_{\eta\mathbf{g}} \in C([0,T];V)$ follows from the continuity of f, η and $\mathbf{g}$ in their respective spaces V, $\mathscr{H}$ and V. □

With the help of Lemma 4.2, we are in a position to show the following existence and uniqueness result for Problem $\mathscr{P}\mathscr{V}_\eta^1$.

Lemma 4.3. *There exists a unique solution to Problem $\mathscr{P}\mathscr{V}_\eta^1$ such that $\mathbf{u}_\eta \in C^1(0,T;V)$ and $\boldsymbol{\sigma}_\eta \in C(0,T;\mathscr{H}_1)$.*

Proof. We consider an operator $\Lambda_\eta : C(0,T;V) \to C(0,T;V)$ defined by

$$\Lambda_\eta \mathbf{g}(t) = u_0 + \int_0^t v_{\eta\mathbf{g}}(s)ds, \qquad \mathbf{g} \in C(0,T;V),\ t \in [0,T], \tag{4.15}$$

where $v_{\eta\mathbf{g}}$ is the solution of (4.10). We will show that this operator has a unique fixed point $\mathbf{g}_\eta \in C([0,T];V)$. To this end, let $\mathbf{g}_1, \mathbf{g}_2 \in C([0,T];V)$ and denote by $v_i = v_{\eta\mathbf{g}_i}$, $i = 1,2$, the corresponding solutions of (4.10). Using the definition (4.15), we obtain

$$\|\Lambda_\eta \mathbf{g}_1(t) - \Lambda_\eta \mathbf{g}_2(t)\|_V \leq \int_0^t \|v_1(s) - v_2(s)\|_V ds, \quad \forall t \in [0,T]. \tag{4.16}$$

Moreover, using estimates similar to those leading to (4.14) in the proof of Lemma 4.2, we have

$$\|v_1(s) - v_2(s)\|_V \leq c\|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|_V, \quad s \in [0,T].$$

It follows from (4.16) that

$$\|\Lambda_\eta \mathbf{g}_1(t) - \Lambda_\eta \mathbf{g}_2(t)\|_V \leq \int_0^t \|\mathbf{g}_1(s) - \mathbf{g}_2(s)\|_V ds, \quad \forall t \in [0,T]. \tag{4.17}$$

Reiterating this inequality m times, we obtain

$$\|\Lambda_\eta^m \mathbf{g}_1 - \Lambda_\eta^m \mathbf{g}_2\|_{C(0,T;V)} \leq \frac{c^m T^m}{m!} \|\mathbf{g}_1 - \mathbf{g}_2\|_{C(0,T;V)} ds, \quad \forall t \in [0,T].$$

This shows that for m large enough Λ_η^m is a contraction in space $C([0,T],V)$. Thus, Λ_η has a unique fixed point $\mathbf{g}_\eta \in C([0,T],V)$.

Next, let $v_\eta \in C([0,T];V)$, $u_\eta \in C^1([0,T];V)$ and $\sigma_\eta \in C([0,T];\mathscr{H})$ be given by

$$v_\eta = v_{\eta g_\eta}, \tag{4.18}$$

$$\mathbf{u}_\eta(t) = u_0 + \int_0^t v_\eta(s) ds, \qquad \forall t \in [0,T], \tag{4.19}$$

$$\sigma_\eta = \mathscr{A}\varepsilon(v_\eta) + \eta. \tag{4.20}$$

Clearly, (4.7) and (4.9) are satisfied. Moreover, by (4.19), (4.18) and (4.15), it follows that $u_\eta = \mathbf{g}_\eta$ and $\dot{u}_\eta = v_\eta$. Therefore, if $\mathbf{g} = \mathbf{g}_\eta$ in (4.10), then we obtain (4.8).

To prove the regularity of σ_η, we choose $\omega = \dot{u}_\eta \pm \varphi$ in ((4.8) with $\varphi \in C_0^\infty(\Omega)^d$ to obtain

$$(\sigma_\eta, \varepsilon(\varphi-))_{\mathscr{H}} = (\mathbf{f}, \varphi)_V, \quad \forall \varphi \in C_0^\infty(\Omega)^d, \text{ on } [0,T]. \tag{4.21}$$

From the definition of $(f,\varphi)_V$ in (3.44), we find

$$\operatorname{Div} \sigma_\eta + \mathbf{f}_0 = 0, \qquad \text{on } [0,T]. \tag{4.22}$$

Now, assumption (3.41) and equation (4.22) imply that $\sigma_\eta \in C([0,T];\mathscr{H}_1)$. This establishes the existence part in Lemma 4.3. From (3.27), (3.33) and Gronwall's inequality, we find the uniqueness of the solution follows from (4.7) immediately. □

Next, we use $\mathbf{u}_\eta \in C^1([0,T],V)$, which is obtained in Lemma 4.2, to construct the following variational problem for the electrical potential.

Problem $\mathscr{PV}_\eta^\varphi$. Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathscr{E}\varepsilon(\mathbf{u}_\eta(t)), \nabla\zeta)_H + (h(\mathbf{u}_\eta(t), \varphi_\eta(t)), \zeta)_W = (q(t), \zeta)_W, \tag{4.23}$$

for all $\zeta \in W$, $t \in [0,T]$.

The well-posedness of problem $\mathscr{PV}_\eta^\varphi$ follows.

Lemma 4.4. *There exists a unique solution $\varphi_\eta \in C(0,T;W)$ satisfying (4.23). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (4.23) corresponding to η_1, $\eta_2 \in C([0,T];\mathscr{H})$, then there exists $c > 0$ such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq c\,\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V \quad \forall t \in [0,T]. \tag{4.24}$$

We use an abstract existence and unique result which may be found in [20].

In the third step, we let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage filed.

Problem $\mathscr{PV}_\theta$. Find the damage field $\alpha_\theta : [0,T] \to H^1(\Omega)$ such that

$$\alpha_\theta(t) \in K, \quad (\dot{\alpha}_\theta(t), \xi - \alpha_\theta)_{L^2(\Omega)} + a(\alpha_\theta(t), \xi - \alpha_\theta(t)) \\ \geq (\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T), \tag{4.25}$$

$$\alpha_\theta(0) = \alpha_0. \tag{4.26}$$

We apply Theorem 2.3 to problem PV_θ.

Lemma 4.5. *There exists a unique solution β_θ to the auxiliary problem $\mathscr{PV}_\theta$ satisfying* (4.4).

To solve $\mathscr{PV}_\theta$, we recall the following standard result for parabolic variational inequalities (see, e.g.,[5]).

In the fourth step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and consider the following initial-value problem.

Problem $\mathscr{PV}^\beta$. Find the adhesion field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta_\eta} = -(\beta_\eta(\gamma_\nu R_\nu(u_{\eta\nu})^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau})\|^2) - \varepsilon_a)_+, \tag{4.27}$$

$$\beta_\eta(0) = \beta_0 \in \Omega. \tag{4.28}$$

We have the following result.

Lemma 4.6. *There exists a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathscr{Z}$ to Problem $\mathscr{PV}^\beta$.*

Proof. . For the sake of simplicity, we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -(\beta(\gamma_\nu R_\nu(u_{\eta\nu}(t))^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau})(t)\|^2) - \varepsilon_a)_+, \ \forall t \in [0,T]. \tag{4.29}$$

It follows from the properties of R_ν and $\mathbf{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of the classical Cauchy-Lipschitz theorem 2.1, we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to Problem$\mathscr{PV}^\beta$. Also, the arguments used in Remark 3.1 show that $0 \leq \beta_\eta(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathscr{Z}$, we find that $\beta_\eta \in \mathscr{Z}$, which concludes the proof of Lemma 4.6. □

Finally, as a consequence of these results and using the properties of operator $\mathscr{G}$ operator $\mathscr{E}$, and function S, we consider the operator

$$\Lambda : C(0,T;\mathscr{H} \times L^2(\Omega)) \to C(0,T;\mathscr{H} \times L^2(\Omega)), \tag{4.30}$$

which maps every element $(\eta,\theta) \in C(0,T;\mathscr{H} \times L^2(\Omega))$ to $\Lambda(\eta,\theta) \in C(0,T;\mathscr{H} \times L^2(\Omega))$ defined by

$$\Lambda(\eta,\theta)(t) = (\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)) \in \mathscr{H} \times L^2(\Omega), \tag{4.31}$$

and

$$(\Lambda^1(\eta,\theta)(t), v)_{\mathscr{H} \times V} = (\mathscr{G}(\varepsilon(\mathbf{u}_\eta(t)), \alpha_\theta(t)), \varepsilon(v))_{\mathscr{H}} + (\mathscr{E}^* \nabla \varphi_\eta(t), \varepsilon(v))_{\mathscr{H}} \\ + \Big(\int_0^t \mathscr{M}(t-s), \varepsilon(\mathbf{u}_\eta(s)) ds, \varepsilon(v) \Big)_{\mathscr{H}}, \quad \forall \mathbf{v} \in V. \tag{4.32}$$

$$\Lambda^2(\eta,\theta)(t) = S\big(\varepsilon(\mathbf{u}_\eta(t)), \alpha_\theta(t)\big). \tag{4.33}$$

Here, for every $(\eta,\theta)\in C(0,T;\mathscr{H}\times L^2(\Omega))$ u_η, φ_η,β_η and α_θ represent the displacement field, the potential electric field, the bonding field and the damage field obtained in Lemmas 4.2, 4.4, 4.5 and 4.6 respectively. We have the following result.

Lemma 4.7. *Λ has a unique fixed point $(\eta^*,\theta^*)\in C(0,T;\mathscr{H}\times L^2(\Omega))$ such that $\Lambda(\eta^*,\theta^*)=(\eta^*,\theta^*)$.*

Proof. Let $t\in(0,T)$ and $(\eta_1,\theta_1),(\eta_2,\theta_2)\in C(0,T;\mathscr{H}\times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i}=\mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i}=\dot{\mathbf{u}}_i$, $\beta_{\eta_i}=\beta_i$, $\varphi_{\eta_i}=\varphi_i$ et $\alpha_{\theta_i}=\alpha_i$, for $i=1,2$. Using (3.28) (3.31), (3.32), (3.34), the definition of R_ν, $\mathbf{R}_\tau$ and Remark 3.1, we have

$$\begin{aligned}
&\|\Lambda^1(\eta_1,\theta_1)(t)-\Lambda^1(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}}\\
&\quad\le\|\mathscr{G}(\varepsilon(\mathbf{u}_1(t)),\alpha_1(t))-\mathscr{G}(\varepsilon(\mathbf{u}_2(t)),\alpha_2(t))\|^2_{\mathscr{H}}\\
&\quad+\|\mathscr{E}^*\nabla\varphi_1(t)-\mathscr{E}^*\nabla\varphi_2(t)\|^2_{\mathscr{H}}\\
&\quad+\int_0^t\|\mathscr{M}(t-s)\varepsilon(u_1(s)-u_2(s))\|^2_{\mathscr{H}}ds\\
&\quad\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds\\
&\quad+\|\varphi_1(t)-\varphi_2(t)\|^2_{\mathscr{H}}+\|\alpha_1-\alpha_2(t))\|^2_{L^2(\Gamma_3)}\Big).
\end{aligned}\tag{4.34}$$

By a similar argument, we conclude from (4.33) and (3.29) that

$$\begin{aligned}
&\|\Lambda^2(\eta_1,\theta_1)(t)-\Lambda^2(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}}\\
&\quad\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big).
\end{aligned}\tag{4.35}$$

Therefore,

$$\begin{aligned}
&\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)}\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V\\
&+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds+\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big).
\end{aligned}\tag{4.36}$$

Moreover, from (4.19), we obtain

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\le\int_0^t\|v_1(s)-v_2(s)\|_V ds.\tag{4.37}$$

Using (4.7), (4.8), and estimates similar to those in the proof of Lemma 4.2 (see (4.14)), we find that, for $s\in[0,T]$,

$$\|v_1(s)-v_2(s)\|_V\le c\Big(\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}}+\|u_1(s)-u_2(s)\|_V\Big).\tag{4.38}$$

Combining (4.36) and (4.37), and using the Gronwall's inequality, we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\le c\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathscr{H}}ds,\quad t\in[0,T],$$

which implies that

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V\le c\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{\mathscr{H}}ds,\quad t\in[0,T].\tag{4.39}$$

From (4.25), we obtain that, a.e. on $(0,T)$,

$$(\dot\alpha_1-\dot\alpha_2,\alpha_1-\alpha_2)_{L^2(\Omega)}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\le(\theta_1-\theta_2,\beta_1-\alpha_2)_{L^2(\Omega)}\quad\forall t\in[0,T].$$

Integrating the previous inequality with respect to time, and using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, one finds that

$$\frac{1}{2}\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \leq \int_0^t (\theta_1(s) - \theta_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} ds, \quad \forall t \in [0,T],$$

which in turn implies that

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \leq \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|^2_{L^2(\Omega)} ds,$$

for all $t \in [0,T]$. This inequality combined with the Gronwall's inequality leads to

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds, \quad \forall t \in [0,T]. \tag{4.40}$$

We use now (4.23), (3.30), (3.31) and (3.26) to find

$$\|\varphi_1(t) - \varphi_2(t)\|^2_W \leq C\|u_1(t) - u_2(t)\|^2_V ds. \tag{4.41}$$

Using (4.36), (4.39), (4.40) and (4.42), we find that, for all $t \in [0,T]$,

$$\|\Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)} \leq C \int_0^t \|(\eta_1,\theta_1)(s) - (\eta_2,\theta_2)(s)\|^2_{\mathscr{H}\times L^2(\Omega)} ds.$$

Then, as in the proof of Lemma 4.7, we obtain

$$\|\Lambda^n(\eta_1,\theta_1) - \Lambda^n(\eta_2,\theta_2)\|_{C(0,T;\mathscr{H}\times L^2(\Omega))} \leq \left(\frac{C^n T^n}{n!}\right)^{\frac{1}{2}} \|(\eta_1,\theta_1) - (\eta_2,\theta_2)\|_{C(0,T;\mathscr{H}\times L^2(\Omega))},$$

for all $n \in \mathbb{N}$. This inequality and the Banach fixed-point theorem imply that Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem (4.1).

Existence. Let $(\mathbf{u}_\eta, \sigma_\eta)$ be the solution of problem $\mathbf{PV}^u_\eta$. Let φ_η be the solution of problem $\mathbf{PV}^\varphi_\eta$. Let β_η be the solution of problem $\mathbf{PV}^\beta_\eta$ for $\eta = \eta^*$, and let α^*_θ be the solution of problem $\mathbf{PV}_\theta$ for $\theta = \theta^*$. Since $\eta^* = \mathscr{G}\varepsilon(\mathbf{u}_{\eta^*}, \alpha_{\theta^*}) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}_{\eta^*}(s))ds + \mathscr{E}^*\nabla\varphi_{\eta^*}$ and $\theta^* = S(\varepsilon(\mathbf{u}_{\eta^*}), \alpha_{\theta^*})$, we see that $(\mathbf{u}_{\eta^*}, \sigma_{\eta^*}, \varphi_{\eta^*}, \beta_{\eta^*}, \alpha_{\theta^*})$ is a solution of problem (3.52) through (3.57) and it satisfies (4.1) through (4.5)

Finally, we conclude that the weak solution $(\mathbf{u}_{\eta^*}, \sigma_{\eta^*}, \varphi_{\eta^*}, \mathbf{D}_{\eta^*}, \beta_{\eta^*}, \alpha_{\theta^*})$ of piezoelectric contact problem $\mathbf{P}$ has the regularity (4.1)-(4.6), which concludes the existence part of Theorem 4.1.

Uniqueness. Let $(\mathbf{u}_{\eta^*}, \sigma_{\eta^*}, \varphi_{\eta^*}, \beta_{\eta^*}, \alpha_{\theta^*})$ be the solution of (3.52)-(3.57) obtained above and let $(\mathbf{u}, \sigma, \varphi, \beta, \alpha)$ be another solution of the problem, which satisfies (4.1)-(4.5). We denote by $\eta \in C([0,T], \mathscr{H})$ and $\theta \in C([0,T]; L^2(\Omega))$ the functions

$$\eta^*(t) = \mathscr{G}\varepsilon((\mathbf{u}(t)), \alpha) + \int_0^t \mathscr{M}(t-s)\varepsilon(\mathbf{u}(s))ds + \mathscr{E}^*\nabla\varphi(t), \tag{4.42}$$

$$\theta^*(t) = S(\varepsilon(\mathbf{u}(t)), \alpha). \tag{4.43}$$

Now, (3.51)-(3.54), (3.56) and (3.57) imply that $(\mathbf{u}, \sigma, \varphi, \beta)$ is a solution of Problems $\mathbf{PV}^u_\eta$, $\mathbf{PV}^\varphi_\eta$ and $\mathbf{PV}^\beta_\eta$. From Lemma 4.3, it follows that this problem has a unique solution $\mathbf{u}_\eta \in C^1([0,T]; V)$, $\varphi_\eta C([0,T]; W)$ and $\sigma_\eta C([0,T]; \mathscr{H}_1)$. It follows that

$$\mathbf{u} = \mathbf{u}_\eta, \quad \sigma = \sigma_\eta, \quad \varphi = \varphi_\eta, \qquad \beta = \beta_\eta \tag{4.44}$$

From (3.55) and (3.57), we can obtain that

$$\alpha = \alpha_\theta. \tag{4.45}$$

Using (4.32), (4.44) , (4.45) and (4.42)-(4.43), we obtain $\Lambda(\eta,\theta) = (\eta,\theta)$. By the uniqueness of the fixed point of the operator Λ, guaranteed by Lemma 4.7, it follows that

$$\eta = \eta^*, \qquad \theta = \theta^*. \tag{4.46}$$

The solution uniqueness is now a consequence of (4.44).

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