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SEVERAL ITERATIVE ALGORITHMS FOR SOLVING THE MULTIPLE-SET SPLIT COMMON FIXED-POINT PROBLEM OF AVERAGED OPERATORS

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Abstract. In this paper, we use the dual variable to propose a new cyclic iterative algorithm for solving the multiple-set split common fixed-point problem of averaged operators in Hilbert spaces. Inspired by the idea of parallel iterative algorithms, we also introduce two mixed iterative algorithms which combine the process of cyclic and parallel together. Under mild assumptions, we prove weak convergence of the proposed iterative sequences in Hilbert spaces. As applications, we obtain several iterative algorithms to solve the multiple-set split feasibility problem.

Keywords. Multiple-set split common fixed point problem; Multiple-set split feasibility problem; Averaged operators; Weak convergence; Iterative algorithms.

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1. INTRODUCTION

The multiple-set split feasibility problem (MSFP), first introduced by Censor *et al* in [1], is a general way to characterize various linear inverse problems which arise in many real-world problems, such as, medical image reconstruction [2] and intensity-modulated radiation therapy [3, 4]. The multiple-set split feasibility problem is required to find a point in the intersection of a family of nonempty closed convex subsets in one space such that its image under a certain operator is in the intersection of another family of nonempty closed convex subsets in the image space. Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given integers $p, r \geq 1$, the MSFP is formulated as finding a point x^* satisfying the property:

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ such that } Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

where $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of H_1 and H_2, respectively.

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The MSFP (1.1) with $p = r = 1$ is known as the split feasibility problem (SFP) originally introduced by Censor and Elfving [5] which is formulated as finding a point x^* satisfying the property:

$$x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{1.2}$$

where C and Q are nonempty closed convex subset of H_1 and H_2, respectively. The iterative methods for solving the SFP and MSFP have been paid much attention in recent years.

Note that, if the SFP (1.2) is consistent (i.e., it has a solution), the SFP (1.2) can be formulated as a fixed point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*, \tag{1.3}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. That is, x^* solves the SFP (1.2) if and only if x^* solves the fixed point equation (1.3) (see [6] for the details). This implies that we can use fixed point algorithms (see [7, 8, 9, 10, 11, 12, 13, 14]) to solve the SFP. Byrne [15, 16] proposed the so-called CQ algorithm which generates a sequence $\{x_k\}$ by

$$x_{k+1} = P_C(x_k - \gamma A^*(I - P_Q)Ax_k), \tag{1.4}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A. The CQ algorithm only involves the computations of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions.

Since every closed convex subset of a Hilbert space is the fixed point set of its associated projection, problems (1.1) and (1.2) are special cases of the so-called multiple-set split common fixed-point problem (MSCFP). Let $p, r \geq 1$ be integers, $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ and $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ be nonlinear operators and $A : H_1 \to H_2$ be a bounded linear operator. The MSCFP can be formulated as the problem of finding

$$x^* \in \bigcap_{i=1}^{p} F(U_i) \ \text{ such that } \ Ax^* \in \bigcap_{j=1}^{r} F(T_j), \tag{1.5}$$

where, for $1 \leq i \leq p$ and $1 \leq j \leq r$, $F(U_i)$ and $F(T_j)$ stand for the fixed point sets of $U_i : H_1 \to H_1$ and $T_j : H_2 \to H_2$, respectively. In particular, if $p = r = 1$, then MSCFP (1.5) is reduced to finding a point x^* with the property:

$$x^* \in F(U) \ \text{ such that } \ Ax^* \in F(T), \tag{1.6}$$

which is usually called the split common fixed-point problem (SCFP). The concept of the SCFP in finite-dimensional Hilbert spaces was originally introduced by Censor and Segal [17] who proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm for the SCFP of directed operator with nonempty fixed point sets:

$$x_{k+1} = U(x_k + \gamma A^t(T - I)Ax_k), \ \ k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix $A^t A (A^t$ stands for matrix transposition). Some authors have introduced some algorithms to solve SCFP (1.2) (see [18, 19, 20, 21]).

Many iterative algorithms have been developed to solve the MSCFP and the MSFP. See, for example [22, 23, 24, 25, 26, 27, 28, 29, 30] and references therein. For solving the constrained MSFP, Censor et.

al. [1] introduced the proximity function:

$$g(x) := \frac{1}{2} \sum_{i=1}^{p} \alpha_i \|x - P_{C_i} x\|^2 + \frac{1}{2} \sum_{j=1}^{r} \beta_j \|Ax - P_{Q_j}(Ax)\|^2, \tag{1.7}$$

where $\alpha_i > 0$ $(1 \le i \le p)$, $\beta_j > 0$ $(1 \le j \le r)$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{r} \beta_j = 1$. Then

$$\nabla g(x) = \sum_{i=1}^{p} \alpha_i(x - P_{C_i}x) + \sum_{j=1}^{r} \beta_j A^*(Ax - P_{Q_j}(Ax))$$

and they proposed the following projection method:

$$x_{k+1} = P_\Omega(x_k - \gamma \nabla g(x_k)), \tag{1.8}$$

where Ω is the constrained set, $0 < \gamma_L \le \gamma \le \gamma_U < \frac{2}{L}$ and L is the Lipschitz constant of ∇g.

Wang and Xu [31] proposed the following cyclic iterative algorithm for the MSCFP (1.5) of directed operators:

$$x_{k+1} = U_{[k]_1}(x_k + \gamma A^*(T_{[k]_2} - I)Ax_k), \tag{1.9}$$

where $0 < \gamma < 2/\rho(A^*A)$, $[k]_1 := k(\mod p)$ and $[k]_2 := k(\mod r)$. They proved the weak convergence of the sequence $\{x_k\}$ generated by (1.9).

Assume that $\{\omega_i\}_{i=1}^{p} \subset (0,1)$, $\{\eta_j\}_{j=1}^{r} \subset (0,1)$ with $\sum_{i=1}^{p} \omega_i = 1$ and $\sum_{j=1}^{r} \eta_j = 1$, and $\lambda = \rho(A^*A)$, $0 < a \le \gamma_k \le b < 2/\lambda$. For solving the MSCFP (1.5) of directed operators, Tang and Liu [32] introduced the parallel iterative algorithm by the following way:

$$x_{k+1} = \sum_{i=1}^{p} \omega_i U_i \Big(x_k + \gamma_k \sum_{j=1}^{r} \eta_j A^*(T_j - I)Ax_k\Big). \tag{1.10}$$

They also proposed inner parallel and outer cyclic iterative algorithm:

$$x_{k+1} = U_{[k]_1}(x_k + \gamma_k \sum_{j=1}^{r} \eta_j A^*(T_j - I)Ax_k)$$

and outer parallel and inner cyclic iterative algorithm:

$$x_{k+1} = \sum_{i=1}^{p} \omega_i U_i(x_k + \gamma_k A^*(T_{[k]_2} - I)Ax_k),$$

where $[k]_1 = k(\mod p)$ and $[k]_2 = k(\mod r)$. They obtained the weak convergence of the above three iterative sequences to solve the MSCFP (1.5) of directed operators.

In this paper, we use the dual variable to propose several iterative algorithms for solving the MSCFP (1.5) governed by averaged operators. In [33], Chen, Huang and Zhang designed and discussed an efficient algorithm framework for minimizing the sum of two proper lower semi-continuous convex functions, i.e.,

$$x^* = arg \min_{x \in R^n} f_1(x) + f_2(x), \tag{1.11}$$

where $f_1, f_2 \in \Gamma_0(\mathbb{R}^n)$ (all proper lower semi-continuous convex functions from $\mathbb{R}^n$ to $(-\infty, +\infty]$) and f_2 is differentiable on R^n with $1/\beta$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$. To solve convex

separable problem (1.11), they obtained the following fixed point formulation: the point x^* is a solution of (1.11) if and only if there exists $v^* \in \mathbb{R}^n$ such that

$$
\begin{cases}
v^* = (I - prox_{\frac{\gamma}{\lambda} f_1})(x^* - \gamma \nabla f_2(x^*) + (1 - \lambda)v^*), \\
x^* = x^* - \gamma \nabla f_2(x) - \lambda v^*,
\end{cases}
$$

where λ and γ are two positive numbers. They introduced the following Picard iterative sequence:

$$
\begin{cases}
v_{k+1} = (I - prox_{\frac{\gamma}{\lambda} f_1})((x_k - \gamma \nabla f_2(x_k)) + (1 - \lambda)v_k), \\
x_{k+1} = x_k - \gamma \nabla f_2(x_k) - \lambda v_{k+1}.
\end{cases}
\tag{1.12}
$$

It was shown [33] that under appropriate conditions $\{x_k\}$ converges to a solution of problem (1.11). Since x is the primal variable related to (1.11), it is very natural to ask what role the variable v plays in above algorithm. After a thorough study, they found out that v is actually the dual variable of the primal-dual form related to (1.11).

Motivated by [33], by optimization approach, Zhao and Zong [34] proposed the following algorithm to solve SFP (1.2)

$$
\begin{cases}
y_k = x_k - \gamma_k A^*(I - P_Q)Ax_k, \\
v_{k+1} = (I - P_C)(y_k + (1 - \lambda)v_k), \\
x_{k+1} = y_k - \lambda v_{k+1},
\end{cases}
\tag{1.13}
$$

where $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < \frac{2}{\|A\|^2}$ and $0 < \lambda \leq 1$. When $\lambda = 1$, algorithm (1.13) becomes the CQ algorithm (1.6). They obtained that the sequence $\{x_k\}$ weakly converges to a solution of SFP (1.2).

Inspired and motivated by the works mentioned above, we use the dual variable to propose a new cyclic iterative algorithm for MSCFP (1.5) governed by averaged operators. We also propose two mixed iterative algorithms which combine the process of cyclic and parallel iteration together. As applications, we obtain several iterative algorithms to solve MSFP (1.1).

The contents of this paper are as follows. First, we give some useful definitions and results for the convergence analysis of the iterative algorithms in next section. Second, we prove weak convergence theorems of the proposed cyclic iterative algorithm with the dual variable. Finally, we prove weak convergence theorems of two mixed iterative algorithms.

2. PRELIMINARIES

Next, we denote the inner product by $\langle \cdot, \cdot \rangle$ and the norm by $\|\cdot\|$. Let I denote the identity operator on Hilbert space H. We denote the fixed point set of an operator T by $F(T)$. We denote by $\to$ the strong convergence and by $\rightharpoonup$ the weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$.

Definition 2.1. An operator $T : H \to H$ is said to be
(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$,
(ii) firmly nonexpansive if $2T - I$ is nonexpansive or, equivalently, for all $x, y \in H$,

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2,
$$

(iii) firmly quasi-nonexpansive (also directed) if $F(T) \neq \emptyset$ and

$$\langle x - q, Tx - q \rangle \geq \|Tx - q\|^2, \quad \forall x \in H, \ \forall q \in F(T)$$

or, equivalently,

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2, \quad \forall x \in H, \ \forall q \in F(T).$$

Definition 2.2. An operator $T : H \to H$ is said to be an averaged operator if it can be written as the average of I and a nonexpansive operator, that is,

$$T = (1 - \alpha)I + \alpha S,$$

where α is a number in $(0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when the above equality holds, we say that T is α-averaged.

Remark 2.3. T is α-averaged if and only if the following inequality holds:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1 - \alpha}{\alpha} \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in H$.

Definition 2.4. An operator $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Definition 2.5. Given a number $\alpha > 0$, $T : H \to H$ is said to be α-inverse strongly monotone (α-ism) if

$$\langle x - y, Tx - Ty \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Recall that the metric projection from H onto a nonempty closed convex subset C of H, denoted by P_C, is defined as follows: for each $x \in H$,

$$P_C(x) = arg \min_{y \in C} \{ \|x - y\| \}.$$

It is well known that $P_C x$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x \rangle \leq 0, \quad z \in C.$$

The following lemmas will be helpful for the convergence analysis of the proposed iterative algorithms.

Lemma 2.6. *Let H be a real Hilbert space. Then*

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in H. \tag{2.1}$$

Lemma 2.7. [35] *Let H be a real Hilbert space. Then for all $t \in [0,1]$ and $x, y \in H$,*

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

Lemma 2.8. [36] *Let H be a real Hilbert space. Then*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_r x_r\|^2 \leq \Sigma_{i=0}^{r} \alpha_i \|x_i\|^2 - \alpha_s \alpha_t \|x_s - x_t\|^2$$

for any $s, t \in \{0, 1, 2, \cdots, r\}$ and for $x_i \in H, i = 0, 1, 2, \cdots, r$ with $\alpha_0 + \alpha_1 + \cdots + \alpha_r = 1$ and $0 \leq \alpha_i \leq 1$.

Lemma 2.9. [16] *T is averaged if and only if the complement $I - T$ is v-ism for some $v > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 2.10. [6] *Let K be a nonempty closed convex subset of real Hilbert space H. Let $\{x_k\}$ be a bounded sequence which satisfies the following properties:*

(i) every weak limit point of $\{x_k\}$ lies in K;

(ii) $\lim_{k \to \infty} \|x_k - x\|$ exists for every $x \in K$.

Then $\{x_k\}$ converges weakly to a point in K.

Lemma 2.11. [37] *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demi-closed at origin.*

3. THE CYCLIC ITERATIVE ALGORITHM

In this section, we introduce a new cyclic iterative algorithm and prove the weak convergence for solving the MSCFP (1.5) of averaged operators with the dual variable.

In what follows, we adopt the following assumptions:

(A1) Γ denotes the solution set of the MSCFP (1.5) and Γ is nonempty;

(A2) $U_i : H_1 \to H_1$ $(1 \le i \le p)$ is s_i-averaged operators and $T_j : H_2 \to H_2$ $(1 \le j \le r)$ is t_j-averaged operators, $s = \max\{s_i : 1 \le i \le p\}$ and $t = \max\{t_j : 1 \le j \le r\}$;

(A3) $\forall k \ge 0$, $j(k) = k(\mod r) + 1$, $i(k) = k(\mod p) + 1$.

Algorithm 3.1. Let $x_0, v_0 \in H_1$ be arbitrary. For $k \ge 0$, let

$$\begin{cases} y_k = x_k - \gamma_k A^*(I - T_{j(k)})Ax_k, \\ v_{k+1} = (I - U_{i(k)})(y_k + (1 - \lambda)v_k), \\ x_{k+1} = y_k - \lambda v_{k+1}, \end{cases} \tag{3.1}$$

where $\gamma_k \in (0, \frac{1}{t\|A\|^2})$, $0 < \lambda \le 1$.

Theorem 3.2. *Assume that $0 < s \le \frac{1}{2}$ and $0 < \liminf_{k \to \infty} \gamma_k \le \limsup_{k \to \infty} \gamma_k < \frac{1}{t\|A\|^2}$. Then the sequence $\{x_k\}$ generated by Algorithm 3.1 weakly converges to a solution x^* of MSCFP(1.5). Moreover, $\|x_{k+1} - x_k\| \to 0$ and $\|Ax_{k+1} - Ax_k\| \to 0$ as $k \to \infty$.*

Proof. Firstly, we show that $\lim_{k \to \infty} \|x_k - x^*\|$ exists for any $x^* \in \Gamma$. Taking $x^* \in \Gamma$, we have $x^* \in \bigcap_{i=1}^p F(U_i)$ and $Ax^* \in \bigcap_{j=1}^r F(T_j)$. For $1 \le i \le p$ and $1 \le j \le r$, as U_i and T_j are s_i-averaged operators and t_j-averaged operators, respectively, it follows from Lemma 2.9 that $I - U_i$ and $I - T_j$ are $\frac{1}{2s_i}$-ism and $\frac{1}{2t_j}$-ism, respectively. Thus, from Algorithm 3.1, we have

$$\begin{aligned} \|v_{k+1}\|^2 &= \|(I - U_{i(k)})(y_k + (1 - \lambda)v_k) - (I - U_{i(k)})x^*\|^2 \\ &\le 2s_i\langle (I - U_{i(k)})(y_k + (1 - \lambda)v_k), y_k - x^* + (1 - \lambda)v_k \rangle \\ &\le 2s\langle v_{k+1}, y_k - x^* + (1 - \lambda)v_k \rangle \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|y_k - \lambda v_{k+1} - x^*\|^2 \\ &= \|y_k - x^*\|^2 - 2\lambda\langle y_k - x^*, v_{k+1} \rangle + \lambda^2\|v_{k+1}\|^2. \end{aligned} \tag{3.3}$$

Then we can get

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \lambda \|v_{k+1}\|^2 \\
=&\|y_k - x^*\|^2 - 2\lambda \langle y_k - x^*, v_{k+1} \rangle + \lambda^2 \|v_{k+1}\|^2 + \lambda \|v_{k+1}\|^2 \\
=&\|y_k - x^*\|^2 - 2\lambda \langle y_k - x^*, v_{k+1} \rangle + \frac{\lambda}{s} \|v_{k+1}\|^2 - \lambda (\frac{1}{s} - 1 - \lambda) \|v_{k+1}\|^2 \\
\leq&\|y_k - x^*\|^2 - 2\lambda \langle y_k - x^*, v_{k+1} \rangle + 2\lambda \langle y_k - x^* + (1-\lambda)v_k, v_{k+1} \rangle - \lambda (\frac{1}{s} - 1 - \lambda) \|v_{k+1}\|^2 \\
=&\|y_k - x^*\|^2 + 2\lambda (1-\lambda) \langle v_k, v_{k+1} \rangle - \lambda (\frac{1}{s} - 1 - \lambda) \|v_{k+1}\|^2.
\end{aligned} \tag{3.4}$$

Since

$$\lambda (1-\lambda) \|v_{k+1} - v_k\|^2 = \lambda (1-\lambda)(\|v_{k+1}\|^2 - 2\langle v_{k+1}, v_k \rangle + \|v_k\|^2), \tag{3.5}$$

we obtain

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \lambda \|v_{k+1}\|^2 \\
\leq&\|y_k - x^*\|^2 + \lambda (1-\lambda) \|v_k\|^2 - \lambda (1-\lambda) \|v_{k+1} - v_k\|^2 - \lambda (\frac{1}{s} - 2) \|v_{k+1}\|^2.
\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}
\langle x_k - x^*, A^*(I - T_{j(k)})Ax_k \rangle =&\langle Ax_k - Ax^*, (I - T_{j(k)})Ax_k - (I - T_{j(k)})Ax^* \rangle \\
\geq&\frac{1}{2t_{j(k)}} \|(I - T_{j(k)})Ax_k - (I - T_{j(k)})Ax^*\|^2 \\
\geq&\frac{1}{2t} \|(I - T_{j(k)})Ax_k\|^2,
\end{aligned} \tag{3.7}$$

we have

$$\begin{aligned}
&\|y_k - x^*\|^2 \\
=&\|x_k - \gamma_k A^*(I - T_{j(k)})Ax_k - x^*\|^2 \\
\leq&\|x_k - x^*\|^2 - 2\gamma_k \langle x_k - x^*, A^*(I - T_{j(k)})Ax_k \rangle + \gamma_k^2 \|A\|^2 \|(I - T_{j(k)})Ax_k\|^2 \\
\leq&\|x_k - x^*\|^2 - \frac{\gamma_k}{t} \|(I - T_{j(k)})Ax_k\|^2 + \gamma_k^2 \|A\|^2 \|(I - T_{j(k)})Ax_k\|^2 \\
=&\|x_k - x^*\|^2 - \gamma_k (\frac{1}{t} - \gamma_k \|A\|^2) \|(I - T_{j(k)})Ax_k\|^2.
\end{aligned} \tag{3.8}$$

From (3.6) and (3.8), we obtain

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \lambda \|v_{k+1}\|^2 \\
\leq&\|x_k - x^*\|^2 - \gamma_k (\frac{1}{t} - \gamma_k \|A\|^2) \|(I - T_{j(k)})Ax_k\|^2 + \lambda (1-\lambda) \|v_k\|^2 \\
&- \lambda (1-\lambda) \|v_{k+1} - v_k\|^2 - \lambda (\frac{1}{s} - 2) \|v_{k+1}\|^2 \\
=&\|x_k - x^*\|^2 + \lambda \|v_k\|^2 - \gamma_k (\frac{1}{t} - \gamma_k \|A\|^2) \|(I - T_{j(k)})Ax_k\|^2 - \lambda^2 \|v_k\|^2 \\
&- \lambda (1-\lambda) \|v_{k+1} - v_k\|^2 - \lambda (\frac{1}{s} - 2) \|v_{k+1}\|^2.
\end{aligned} \tag{3.9}$$

Let $c_k = \|x_k - x^*\|^2 + \lambda \|v_k\|^2$. By the assumptions on $\{\gamma_k\}$, λ, s, and from (3.9), we can get $c_{k+1} \leq c_k$. That is the sequence $\{c_k\}$ is non-increasing and lower bounded by 0, so $\lim_{k \to \infty} c_k$ exists. Then we can

get $\{c_k\}$ is bounded and hence $\{x_k\}$ is bounded. Moreover, from (3.9), we also have

$$\gamma_k(\frac{1}{t} - \gamma_k\|A\|^2)\|(I - T_{j(k)})Ax_k\|^2 + \lambda^2\|v_k\|^2 \leq c_k - c_{k+1},$$

which implies that

$$\lim_{k \to \infty} \|(I - T_{j(k)})Ax_k\| = 0 \tag{3.10}$$

and

$$\lim_{k \to \infty} \|v_k\| = 0. \tag{3.11}$$

So $\lim_{k\to\infty} \|x_k - x^*\|^2 = \lim_{k\to\infty}(c_k - \lambda\|v_k\|^2) = \lim_{k\to\infty} c_k$ exists. By using Algorithm 3.1 and (3.10), we have

$$\lim_{k \to \infty} \|x_k - y_k\| = \lim_{k \to \infty} \gamma_k\|A^*(I - T_{j(k)})Ax_k\| = 0. \tag{3.12}$$

It follows from (3.11), (3.12) and

$$\begin{aligned}
&\|x_k - U_{i(k)}x_k\| \\
&= \|x_k - y_k - (1-\lambda)v_k + (I - U_{i(k)})(y_k + (1-\lambda)v_k) + U_{i(k)}(y_k + (1-\lambda)v_k) - U_{i(k)}x_k\| \\
&\leq \|x_k - y_k - (1-\lambda)v_k\| + \|(I - U_{i(k)})(y_k + (1-\lambda)v_k)\| + \|U_{i(k)}(y_k + (1-\lambda)v_k) - U_{i(k)}x_k\| \\
&\leq 2\|x_k - y_k - (1-\lambda)v_k\| + \|(I - U_{i(k)})(y_k + (1-\lambda)v_k)\| \\
&\leq 2\|x_k - y_k\| + 2(1-\lambda)\|v_k\| + \|v_{k+1}\|
\end{aligned} \tag{3.13}$$

that

$$\lim_{k \to \infty} \|x_k - U_{i(k)}x_k\| = 0. \tag{3.14}$$

By (3.11) and (3.12), we also have

$$\lim_{k \to \infty} \|x_{k+1} - x_k\| = \lim_{k \to \infty} \|y_k - \lambda v_{k+1} - x_k\| = 0, \tag{3.15}$$

which infers that $\{x_k\}$ is asymptotically regular. Then we have

$$\lim_{k \to \infty} \|Ax_{k+1} - Ax_k\| = 0. \tag{3.16}$$

So $\{Ax_k\}$ is asymptotically regular.

Next, we prove that $\omega_w(x_k) \subseteq \Gamma$. Assume that $\bar{x} \in \omega_w(x_k)$, i.e., there exists a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ such that $x_{k_l} \rightharpoonup \bar{x}$, then $Ax_{k_l} \rightharpoonup A\bar{x}$. Noting that the pool of indexes is finite and $\{x_k\}$ is asymptotically regular, for any $i \in \{1, 2, \cdots, p\}$, we can choose a subsequence $\{k_{i_m}\} \subset \{k\}$ such that $x_{k_{i_m}} \rightharpoonup \bar{x}$ as $m \to \infty$ and $i(k_{i_m}) = i$ for all m. It turns out that

$$\lim_{m \to \infty} \|(I - U_i)x_{k_{i_m}}\| = \lim_{m \to \infty} \|(I - U_{i(k_{i_m})})x_{k_{i_m}}\| = 0. \tag{3.17}$$

By the same reason, for any $j \in \{1, 2, \cdots, r\}$, we can choose a subsequence $\{k_{j_n}\} \subset \{k\}$ such that $Ax_{k_{j_n}} \rightharpoonup A\bar{x}$ as $n \to \infty$ and $j(k_{j_n}) = j$ for all n. It turns out that

$$\lim_{n \to \infty} \|(I - T_j)Ax_{k_{j_n}}\| = \lim_{n \to \infty} \|(I - T_{j(k_{j_n})})Ax_{k_{j_n}}\| = 0. \tag{3.18}$$

It follows from (3.17), (3.18) and Lemma 2.11 that $\bar{x} \in \bigcap_{i=1}^p F(U)$ and $A\bar{x} \in \bigcap_{j=1}^r F(T)$. So $\bar{x} \in \Gamma$, and hence $\omega_w(x_k) \subseteq \Gamma$.

Finally, by Lemma 2.10, we can get $x_k \rightharpoonup x^*$, where x^* is a solution of the MSCFP (1.5). And by $v_k \rightharpoonup 0$, we have $(v_k, x_k) \rightharpoonup (0, x^*)$ as $k \to \infty$. $\qquad\square$

Now, we give an application of Theorem 3.2 to solve MSFP (1.1). Let $\{C_i\}_{i=1}^p$ and $\{Q_j\}_{j=1}^r$ are nonempty closed convex subsets of real Hilbert space H_1 and H_2, respectively. It is well known that the orthogonal projection operator is firmly nonexpansive, and a firmly nonexpansive operator is $\frac{1}{2}$-averaged. Assume that the solution set Φ of MSFP (1.1) is nonempty. From Theorem 3.2, we can deduce easily the following results for solving MSFP (1.1).

Corollary 3.3. *Let $\{(v_k, x_k)\}$ be the sequence generated by*

$$
\begin{cases}
x_0, v_0 \in H_1 \text{ chosen arbitrarily,} \\
y_k = x_k - \gamma_k A^*(I - P_{Q_{j(k)}})Ax_k, \\
v_{k+1} = (I - P_{C_{i(k)}})(y_k + (1 - \lambda)v_k), \\
x_{k+1} = y_k - \lambda v_{k+1},
\end{cases}
\tag{3.19}
$$

where $0 < \lambda \le 1$ and $0 < \liminf_{k \to \infty} \gamma_k \le \limsup_{k \to \infty} \gamma_k < \frac{2}{\|A\|^2}$. Then the sequence $\{x_k\}$ weakly converges to a solution x^*, where $x^* \in \Phi$. Moreover, $\|x_k - x_{k+1}\| \to 0$ and $\|Ax_k - Ax_{k+1}\| \to 0$ as $k \to \infty$.

Remark 3.4. When $\lambda = 1$, for solving MSFP (1.1), algorithm (3.19) becomes the following cyclic iterative algorithm:

$$
x_{k+1} = P_{C_{i(k)}}(x_k - \gamma_k A^*(I - P_{Q_{j(k)}})Ax_k).
\tag{3.20}
$$

So, Theorem 3.2 extends the related results of Wang and Xu [31] for solving the MSFP.

4. THE MIXED CYCLIC AND PARALLEL ITERATIVE ALGORITHMS

In this section, for solving the MSCFP (1.5) of averaged operators, we introduce two mixed cyclic and parallel iterative algorithms which make full use of the two methods' structure, and prove the corresponding convergence of the mixed iterative algorithms. We go on making use of the assumptions (A1)-(A3) and adopt the following assumption:

(A4) for all $1 \le i \le p$, $1 \le j \le r$, the sequences $\{\alpha_k^i\}$, $\{\beta_k^j\} \subset [0, 1]$ such that $\Sigma_{i=1}^p \alpha_k^i = 1$ and $\Sigma_{j=1}^r \beta_k^j = 1$ ($\forall k \ge 0$).

Algorithm 4.1. (Inner cyclic and outer parallel iterative algorithm) Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary. For $k \ge 1$,

$$
\begin{cases}
y_k = x_k - \gamma_k A^*(I - T_{j(k)})Ax_k, \\
v_{k+1} = (I - \Sigma_{i=1}^p \alpha_k^i U_i)(y_k + (1 - \lambda)v_k), \\
x_{k+1} = y_k - \lambda v_{k+1},
\end{cases}
\tag{4.1}
$$

where $\gamma_k \in (0, \frac{1}{t\|A\|^2})$, $0 < \lambda \le 1$.

Theorem 4.2. *Assume that $0 < s \le \frac{1}{2}$, $0 < \liminf_{k \to \infty} \gamma_k \le \limsup_{k \to \infty} \gamma_k < \frac{1}{t\|A\|^2}$ and $\liminf_{k \to \infty} \alpha_k^i > 0 (1 \le i \le p)$. Then the sequence $\{x_k\}$ generated by Algorithm 4.1 weakly converges to a solution x^* of the MSCFP (1.5). Moreover, $\|x_k - x_{k+1}\| \to 0$ and $\|Ax_k - Ax_{k+1}\| \to 0$ as $k \to \infty$.*

Proof. Firstly, we show that $\lim_{k \to \infty} \|x_k - x^*\|$ exists for any $x^* \in \Gamma$. Taking $x^* \in \Gamma$, we have $x^* \in \bigcap_{i=1}^p F(U_i)$ and $Ax^* \in \bigcap_{j=1}^r F(T_j)$. For $1 \le i \le p$ and $1 \le j \le r$, as U_i and T_j are s_i-averaged operator and t_j-averaged operator, respectively, by Lemma 2.9, it follows that $I - U_i$ and $I - T_j$ are $\frac{1}{2s_i}$-ism and

$\frac{1}{2t_j}$-ism, respectively. Thus, from Algorithm 4.1, we have

$$
\begin{aligned}
\|v_{k+1}\|^2 =& \|[I - (\alpha_k^1 U_1 + \alpha_k^2 U_2 + \cdots + \alpha_k^p U_p)](y_k + (1-\lambda)v_k)\|^2 \\
=& \|\alpha_k^1 (I - U_1)(y_k + (1-\lambda)v_k) + \cdots + \alpha_k^p (I - U_p)(y_k + (1-\lambda)v_k)\|^2 \\
\leq& \alpha_k^1 \|(I - U_1)(y_k + (1-\lambda)v_k)\|^2 + \cdots + \alpha_k^p \|(I - U_p)(y_k + (1-\lambda)v_k)\|^2 \\
=& \alpha_k^1 \|(I - U_1)(y_k + (1-\lambda)v_k) - (I - U_1)x^*\|^2 + \cdots \\
&+ \alpha_k^p \|(I - U_p)(y_k + (1-\lambda)v_k) - (I - U_p)x^*\|^2 \\
\leq& 2s_1 \alpha_k^1 \langle (I - U_1)(y_k + (1-\lambda)v_k), y_k - x^* + (1-\lambda)v_k \rangle + \cdots \\
&+ 2s_p \alpha_k^p \langle (I - U_p)(y_k + (1-\lambda)v_k), y_k - x^* + (1-\lambda)v_k \rangle \\
\leq& 2s \langle v_{k+1}, y_k - x^* + (1-\lambda)v_k \rangle
\end{aligned}
\tag{4.2}
$$

and (3.3) holds. From Theorem 3.2, we get that inequality (3.6) holds, i.e.,

$$
\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \lambda \|v_{k+1}\|^2 \\
&\leq \|y_k - x^*\|^2 + \lambda(1-\lambda)\|v_k\|^2 - \lambda(1-\lambda)\|v_{k+1} - v_k\|^2 - \lambda(\frac{1}{s} - 2)\|v_{k+1}\|^2.
\end{aligned}
$$

By the proof of Theorem 3.2, we obtain that inequalities (3.7) and (3.8) hold. So, we can get

$$
\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \lambda \|v_{k+1}\|^2 \\
&\leq \|x_k - x^*\|^2 + \lambda \|v_k\|^2 - \gamma_k(\frac{1}{t} - \gamma_k\|A\|^2)\|(I - T_{j(k)})Ax_k\|^2 - \lambda^2 \|v_k\|^2 \\
&\quad - \lambda(1-\lambda)\|v_{k+1} - v_k\|^2 - \lambda(\frac{1}{s} - 2)\|v_{k+1}\|^2.
\end{aligned}
\tag{4.3}
$$

Let $c_k = \|x_k - x^*\|^2 + \lambda \|v_k\|^2$. Similar to the proof of Theorem 3.2, we have

$$
\lim_{k \to \infty} \|(I - T_{j(k)})Ax_k\| = 0,
\tag{4.4}
$$

$$
\lim_{k \to \infty} \|v_k\| = 0
\tag{4.5}
$$

and $\lim_{k \to \infty} \|x_k - x^*\|^2 = \lim_{k \to \infty} c_k$ exists. Then we have that $\{x_k\}$ and $\{y_k\}$ are bounded.

Now, we prove that

$$
\lim_{k \to \infty} \|x_k - U_i x_k\| = 0
\tag{4.6}
$$

for all $1 \leq i \leq p$. By Algorithm 4.1, we have

$$
\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|y_k - \lambda v_{k+1} - x^*\|^2 \\
&= \|y_k - x^*\|^2 - 2\lambda \langle v_{k+1}, y_k - x^* \rangle + \lambda^2 \|v_{k+1}\|^2,
\end{aligned}
\tag{4.7}
$$

which implies that

$$
\|y_k - x^*\|^2 = \|x_{k+1} - x^*\|^2 + 2\lambda \langle v_{k+1}, y_k - x^* \rangle - \lambda^2 \|v_{k+1}\|^2.
\tag{4.8}
$$

It follows from (4.5), the boundedness of $\{y_k\}$ and

$$
|\langle v_{k+1}, y_k - x^* \rangle| \leq \|v_{k+1}\| \cdot \|y_k - x^*\|
$$

that

$$
\lim_{k \to \infty} \|y_k - x^*\| = \lim_{k \to \infty} \|x_{k+1} - x^*\| = \lim_{k \to \infty} \|x_k - x^*\|
\tag{4.9}
$$

exists. Similarly, from

$$\|y_k - \lambda v_k - x^*\|^2 = \|y_k - x^*\|^2 - 2\lambda \langle v_k, y_k - x^* \rangle + \lambda^2 \|v_k\|^2$$

and

$$\|y_k + (1-\lambda)v_k - x^*\|^2 = \|y_k - x^*\|^2 + 2(1-\lambda)\langle v_k, y_k - x^* \rangle + (1-\lambda)^2 \|v_k\|^2,$$

we get

$$\lim_{k\to\infty} \|y_k - \lambda v_k - x^*\| = \lim_{k\to\infty} \|y_k + (1-\lambda)v_k - x^*\| = \lim_{k\to\infty} \|x_{k+1} - x^*\|. \tag{4.10}$$

On the other hand, we have

$$\begin{aligned}
x_{k+1} &= y_k - \lambda v_{k+1}\\
&= y_k - \lambda \left[y_k + (1-\lambda)v_k - \sum_{i=1}^{p} \alpha_k^i U_i(y_k + (1-\lambda)v_k) \right]\\
&= (1-\lambda)y_k - \lambda(1-\lambda)v_k + \lambda \sum_{i=1}^{p} \alpha_k^i U_i(y_k + (1-\lambda)v_k)\\
&= (1-\lambda)(y_k - \lambda v_k) + \lambda \sum_{i=1}^{p} \alpha_k^i U_i(y_k + (1-\lambda)v_k).
\end{aligned} \tag{4.11}$$

It follows from (4.11) and Remark 2.3 that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2\\
&\leq (1-\lambda)\|y_k - \lambda v_k - x^*\|^2 + \lambda \left\| \sum_{i=1}^{p} \alpha_k^i U_i(y_k + (1-\lambda)v_k) - x^* \right\|^2\\
&\leq (1-\lambda)\|y_k - \lambda v_k - x^*\|^2 + \lambda \sum_{i=1}^{p} \alpha_k^i \|U_i(y_k + (1-\lambda)v_k) - x^*\|^2\\
&\leq (1-\lambda)\|y_k - \lambda v_k - x^*\|^2\\
&\quad + \lambda \sum_{i=1}^{p} \alpha_k^i \left(\|y_k + (1-\lambda)v_k - x^*\|^2 - \frac{1-s_i}{s_i} \|(I-U_i)(y_k + (1-\lambda)v_k)\|^2 \right)\\
&\leq (1-\lambda)\|y_k - \lambda v_k - x^*\|^2 + \lambda \|y_k + (1-\lambda)v_k - x^*\|^2\\
&\quad - \frac{\lambda(1-s)}{s} \sum_{i=1}^{p} \alpha_k^i \|(I-U_i)(y_k + (1-\lambda)v_k)\|^2,
\end{aligned} \tag{4.12}$$

which implies that

$$\begin{aligned}
&\frac{\lambda(1-s)}{s} \sum_{i=1}^{p} \alpha_k^i \|(I-U_i)(y_k + (1-\lambda)v_k)\|^2\\
&\leq (1-\lambda)\|y_k - \lambda v_k - x^*\|^2 + \lambda \|y_k + (1-\lambda)v_k - x^*\|^2 - \|x_{k+1} - x^*\|^2.
\end{aligned} \tag{4.13}$$

From (4.10), (4.13) and the conditions on $\{\alpha_k^i\}$ ($1 \leq i \leq p$), we have

$$\lim_{k\to\infty} \|(I-U_i)(y_k + (1-\lambda)v_k)\| = 0 \tag{4.14}$$

for all $1 \leq i \leq p$. By Algorithm 4.1, we have

$$\|x_k - y_k\| \leq |\gamma_k| \cdot \|A\| \cdot \|(I - T_{j(k)})Ax_k\|. \tag{4.15}$$

It follows from (4.4) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{4.16}$$

From

$$\|x_k - U_i x_k\|$$
$$=\|x_k - y_k - (1-\lambda)v_k + (I-U_i)(y_k + (1-\lambda)v_k) + U_i(y_k + (1-\lambda)v_k) - U_i x_k\|$$
$$\leq \|x_k - y_k - (1-\lambda)v_k\| + \|(I-U_i)(y_k + (1-\lambda)v_k)\| + \|U_i(y_k + (1-\lambda)v_k) - U_i x_k\| \qquad (4.17)$$
$$\leq 2\|x_k - y_k - (1-\lambda)v_k\| + \|(I-U_i)(y_k + (1-\lambda)v_k)\|$$
$$\leq 2\|x_k - y_k\| + 2\|(1-\lambda)v_k\| + \|(I-U_i)(y_k + (1-\lambda)v_k)\|,$$

combining (4.5), (4.14) and (4.16), we can get

$$\lim_{k\to\infty} \|x_k - U_i x_k\| = 0 \qquad (4.18)$$

for all $1 \leq i \leq p$. It follows from (4.5), (4.16) and

$$\|x_{k+1} - x_k\| = \|y_k - \lambda v_{k+1} - x_k\| \leq \|y_k - x_k\| + \lambda\|v_{k+1}\|$$

that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \qquad (4.19)$$

So, $\{x_k\}$ is asymptotically regular. Then we have

$$\lim_{k\to\infty} \|Ax_{k+1} - Ax_k\| = 0,$$

which infers that $\{Ax_k\}$ is asymptotically regular.

Next, we prove that $\omega_w(x_k) \subseteq \Gamma$. Assume that $\bar{x} \in \omega_w(x_k)$, i.e., there exists a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ such that $x_{k_l} \rightharpoonup \bar{x}$, then $Ax_{k_l} \rightharpoonup A\bar{x}$. It follows from (4.18) and Lemma 2.11 that $\bar{x} \in \bigcap_{i=1}^{p} F(U_i)$. Since the pool $\{1, 2, \cdots, r\}$ is finite, by the same reason of Theorem 3.2, we have $A\bar{x} \in \bigcap_{j=1}^{r} F(T_j)$. So $\bar{x} \in \Gamma$, and hence $\omega_w(x_k) \subseteq \Gamma$. By Lemma 2.10, we can get $x_k \rightharpoonup x^*$, and hence $(v_k, x_k) \rightharpoonup (0, x^*)$, where $x^* \in \Gamma$. $\quad\square$

Next, we propose another mixed cyclic and parallel iterative algorithm for solving the MSCFP (1.5) of averaged operators.

Algorithm 4.3. (Inner parallel and outer cyclic iterative algorithm) Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary. For $k \geq 1$,

$$\begin{cases} y_k = x_k - \gamma_k \sum_{j=1}^{r} \beta_k^j A^*(I-T_j)Ax_k, \\ v_{k+1} = (I-U_{i(k)})(y_k + (1-\lambda)v_k), \\ x_{k+1} = y_k - \lambda v_{k+1}, \end{cases} \qquad (4.20)$$

where $\gamma_k \in (0, \frac{1}{t\|A\|^2})$, $0 < \lambda \leq 1$.

Theorem 4.4. *Assume that $0 < s \leq \frac{1}{2}$, $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < \frac{1}{t\|A\|^2}$ and $\liminf_{k\to\infty} \beta_k^j > 0 (1 \leq j \leq r)$. Then the sequence $\{x_k\}$ generated by Algorithm 4.3 weakly converges to a solution x^* of MSCFP (1.5). Moreover, $\|x_k - x_{k+1}\| \to 0$ and $\|Ax_k - Ax_{k+1}\| \to 0$ as $k \to \infty$.*

Proof. Taking $x^* \in \Gamma$, by the same reason of Theorem 3.2, we know (3.6) holds. Since, for all $1 \leq j \leq r$,

$$\langle x_k - x^*, A^*(I-T_j)Ax_k \rangle = \langle Ax_k - Ax^*, (I-T_j)Ax_k - (I-T_j)Ax^* \rangle$$
$$\geq \frac{1}{2t_j}\|(I-T_j)Ax_k - (I-T_j)Ax^*\|^2$$
$$\geq \frac{1}{2t}\|(I-T_j)Ax_k\|^2,$$

we have

$$\|y_k - x^*\|^2$$

$$= \|x_k - \gamma_k \sum_{j=1}^{r} \beta_k^j A^*(I - T_j)Ax_k - x^*\|^2$$

$$= \|x_k - x^*\|^2 - 2\gamma_k \langle x_k - x^*, \sum_{j=1}^{r} \beta_k^j A^*(I - T_j)Ax_k \rangle + \gamma_k^2 \|\sum_{j=1}^{r} \beta_k^j A^*(I - T_j)Ax_k\|^2$$

$$\leq \|x_k - x^*\|^2 - 2\gamma_k \sum_{j=1}^{r} \beta_k^j \langle x_k - x^*, A^*(I - T_j)Ax_k \rangle + \gamma_k^2 \sum_{j=1}^{r} \beta_k^j \|A^*(I - T_j)Ax_k\|^2 \qquad (4.21)$$

$$\leq \|x_k - x^*\|^2 - \frac{\gamma_k}{t} \sum_{j=1}^{r} \beta_k^j \|(I - T_j)Ax_k\|^2 + \gamma_k^2 \|A\|^2 \left(\sum_{j=1}^{r} \beta_k^j \|(I - T_j)Ax_k\|^2 \right)$$

$$= \|x_k - x^*\|^2 - \gamma_k \left(\frac{1}{t} - \gamma_k \|A\|^2 \right) \left(\sum_{j=1}^{r} \beta_k^j \|(I - T_j)Ax_k\|^2 \right).$$

It follows from (3.6) and (4.21) that

$$\|x_{k+1} - x^*\|^2 + \lambda \|v_{k+1}\|^2$$

$$\leq \|x_k - x^*\|^2 - \gamma_k \left(\frac{1}{t} - \gamma_k \|A\|^2 \right) \left(\sum_{j=1}^{r} \beta_k^j \|(I - T_j)Ax_k\|^2 \right)$$

$$+ \lambda(1 - \lambda)\|v_k\|^2 - \lambda(1 - \lambda)\|v_{k+1} - v_k\|^2 - \lambda \left(\frac{1}{s} - 2 \right) \|v_{k+1}\|^2 \qquad (4.22)$$

$$= \|x_k - x^*\|^2 + \lambda \|v_k\|^2 - \gamma_k \left(\frac{1}{t} - \gamma_k \|A\|^2 \right) \left(\sum_{j=1}^{r} \beta_k^j \|(I - T_j)Ax_k\|^2 \right)$$

$$- \lambda^2 \|v_k\|^2 - \lambda(1 - \lambda)\|v_{k+1} - v_k\|^2 - \lambda \left(\frac{1}{s} - 2 \right) \|v_{k+1}\|^2.$$

Similar to the proof of Theorem 3.2, we have

$$\lim_{k \to \infty} \sum_{j=1}^{r} \beta_k^j \|(I - T_j)Ax_k\|^2 = \lim_{k \to \infty} \|v_k\| = 0,$$

which implies that

$$\lim_{k \to \infty} \|(I - T_j)Ax_k\| = 0$$

for all $1 \leq j \leq r$. And we can get $\lim_{k \to \infty} \|x_k - x^*\|^2$ exists. By the same proof of Theorem 3.2, we have $\omega_w(x_k) \subseteq \Gamma$ and $x_k \rightharpoonup x^*$, where x^* is a solution of MSCFP (1.5). $\qquad \square$

Finally, similar to Corollary 3.3 we obtain two mixed iterative algorithms to solve MSFP (1.1). Assume that the solution set Φ of MSFP (1.1) is nonempty.

Corollary 4.5. *For any given* $x_0 \in H_1, v_0 \in H_2$, *define a sequence* $\{(v_k, x_k)\}$ *by the following procedure*

$$\begin{cases} y_k = x_k - \gamma_k A^*(I - P_{Q_{j(k)}})Ax_k, \\ v_{k+1} = (I - \sum_{i=1}^{p} \alpha_k^i P_{C_i})(y_k + (1 - \lambda)v_k), \\ x_{k+1} = y_k - \lambda v_{k+1}, \end{cases} \qquad (4.23)$$

where $0 < \lambda \leq 1$, $0 < \liminf_{k \to \infty} \gamma_k \leq \limsup_{k \to \infty} \gamma_k < \frac{2}{\|A\|^2}$ *and* $\liminf_{k \to \infty} \alpha_k^i > 0 (1 \leq i \leq p)$. *Then the sequence* $\{x_k\}$ *weakly converges to a solution* $x^* \in \Phi$. *Moreover,* $\|x_k - x_{k+1}\| \to 0$ *and* $\|Ax_k - Ax_{k+1}\| \to 0$ *as* $k \to \infty$.

Corollary 4.6. *For any given $x_0 \in H_1, v_0 \in H_2$, define a sequence $\{(v_k, x_k)\}$ by the following procedure*

$$\begin{cases} y_k = x_k - \gamma_k \sum_{j=1}^r \beta_k^j A^*(I - P_{Q_j}) A x_k, \\ v_{k+1} = (I - P_{C_{i(k)}})(y_k + (1-\lambda)v_k), \\ x_{k+1} = y_k - \lambda v_{k+1}, \end{cases} \tag{4.24}$$

where $0 < \lambda \le 1$, $0 < \liminf_{k\to\infty} \gamma_k \le \limsup_{k\to\infty} \gamma_k < \frac{2}{\|A\|^2}$ and $\liminf_{k\to\infty} \beta_k^j > 0 (1 \le j \le r)$. Then the sequence $\{x_k\}$ weakly converges to a solution $x^* \in \Phi$. Moreover, $\|x_k - x_{k+1}\| \to 0$ and $\|Ax_k - Ax_{k+1}\| \to 0$ as $k \to \infty$.

Remark 4.7. When $\lambda = 1$, for solving MSFP (1.1), algorithms (4.23) and (4.24) become the following mixed cyclic and parallel iterative algorithm:

$$x_{k+1} = \sum_{i=1}^p \alpha_k^i P_{C_i}(x_k - \gamma_k A^*(I - P_{Q_{j(k)}}) A x_k)$$

and

$$x_{k+1} = P_{C_{i(k)}}(x_k - \gamma_k \sum_{j=1}^r \beta_k^j A^*(I - P_{Q_j}) A x_k).$$

So, Theorem 4.2 and Theorem 4.4 extend the related results of Tang and Liu [32] (Corollary 4.3 and 4.4) for solving the MSFP.

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