J. Nonlinear Funct. Anal. 2019 (2019), Article ID 4 https://doi.org/10.23952/jnfa.2019.4

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

NEW ITERATIVE DESIGNS FOR AN INFINITE FAMILY OF d-ACCRETIVE MAPPINGS IN A BANACH SPACE

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Abstract. In this paper, iterative algorithms for common zero points of an infinite family of d-accretive mappings are designed. Strong convergence theorems are proved in a real uniformly convex and uniformly smooth Banach space. Applications to elliptic systems and parabolic systems with p-Laplacian and curvature systems with the Neumann boundary value are provided to support our main results.

Keywords. d-accretive mapping; Elliptic system; Lyapunov functional; Metric projection; Zero point.

2010 Mathematics Subject Classification. 47H05, 47H09.

1. Introduction and Preliminaries

Suppose that E is a real Banach space with E^* being its dual space. Let K be a nonempty closed and convex subset of E. The symbol $\langle x, f\rangle$ denotes the value of $f \in E^*$ at $x \in E$. And, "$\to$" and "$\rightharpoonup$" denote strong and weak convergence, respectively.

Recall that a Banach space E is strictly convex [1] if for any two points x and y in E, which are linearly independent, one has $\|x+y\| < \|x\|+\|y\|$. E is said to be uniformly convex [1] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|x_n+y_n\| = 2$, one has $\lim_{n\to\infty}\|x_n-y_n\| = 0$.

Let $\eta_E : [0,+\infty) \to [0,+\infty)$ be a function. Recall that η_E is said to be the modulus of smoothness of E [1] if it is defined as follows:

$$\eta_E(t) = \sup\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : x,y \in E,\ \|x\| = 1,\ \|y\| \le t\}.$$

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Received August 8, 2018; Accepted January 19, 2019.

Recall that a Banach space E is said to be uniformly smooth [1] if $\lim_{t\to 0}\frac{\eta_E(t)}{t}\to 0$, as $t\to 0$. E is said to have Property (H) if for any sequence $\{x_n\}\subset E$ which satisfies both $x_n\rightharpoonup x$ and $\|x_n\|\to\|x\|$ as $n\to\infty$, one has $x_n\to x$, as $n\to\infty$.

Recall that the normalized duality mapping $J_E : E\to 2^{E^*}$ is defined by

$$J_E(x)=\{f\in E^* : \langle x,f\rangle=\|x\|^2, \|f\|=\|x\|\},\ x\in E.$$

Lemma 1.1. [1] *The normalized duality mapping $J_E : E\to 2^{E^*}$ has the following properties:*

(i) if E is a real reflexive and smooth Banach space, then J_E is single-valued;

(ii) if E is reflexive, then J_E is surjective;

(iii) if E is uniformly smooth and uniformly convex, then $J_{E^*}=J_E^{-1}$ is the duality mapping from E^* into E. Moreover, both J_E and J_{E^*} are uniformly continuous on each bounded subset of E or E^*, respectively;

(iv) for any $x\in E$ and $k\in(0,+\infty)$, $J_E(kx)=kJ_E(x)$.

Definition 1.2. [2] Suppose that $A : D(A)\sqsubseteq E\to E$ is a mapping. Then

(1) A is called d-accretive if for all $x,y\in D(A)$, $\langle Ax-Ay, j_E(x)-j_E(y)\rangle\geq 0$, where $j_E(x)\in J_E(x)$, $j_E(y)\in J_E(y)$;

(2) A is called m-d-accretive if A is d-accretive and $R(I+\lambda A)=E$ for $\forall\lambda>0$;

(3) A is called accretive if for all $x,y\in D(A)$, $\langle Ax-Ay, j_E(x-y)\rangle\geq 0$, where $j_E(x-y)\in J_E(x-y)$;

(4) A is called m-accretive if A is accretive and $R(I+\lambda A)=E$ for $\forall\lambda>0$.

It is easy to see that in non-Hilbertian Banach space, d-accretive mappings and accretive mappings are two different types of nonlinear mappings. Let $T : D(T)\sqsubseteq E\to E$ be a mapping. For mapping T, we use $T^{-1}0$ to denote the set of zero points of T, that is, $T^{-1}0=\{x\in D(T) : Tx=0\}$. We use $Fix(T)$ to denote the set of fixed points of T. That is, $Fix(T)=\{x\in D(T) : Tx=x\}$.

Definition 1.3. [3] A mapping $B\subset E\times E^*$ is said to be monotone if $\langle x_1-x_2, y_1-y_2\rangle\geq 0$, $\forall y_i\in Bx_i, i=1,2$. The monotone mapping B is called a maximal monotone mapping if $R(J_E+\lambda B)=E^*$, for $\forall\lambda>0$.

Lemma 1.4. [3] *Let $B\subset E\times E^*$ be maximal monotone. Then*

(1) *$B^{-1}0$ is convex and closed subset of E;*

(2) *if $x_n\to x$ and $y_n\in Bx_n$ with $y_n\rightharpoonup y$, or $x_n\rightharpoonup x$ and $y_n\in Bx_n$ with $y_n\to y$, then $x\in D(B)$ and $y\in Bx$.*

Definition 1.5. [4] The Lyapunov functional $\omega : E\times E\to R^+$ is defined as follows:

$$\omega(x,y)=\|x\|^2-2\langle x, j_E(y)\rangle+\|y\|^2,\ \forall x,y\in E, j_E(y)\in J_E(y).$$

Similarly, the Lyapunov functional $\overline{\omega} : E^*\times E^*\to R^+$ is denoted as follows:

$$\overline{\omega}(x,y)=\|x\|^2-2\langle j_{E^*}(y),x\rangle+\|y\|^2,\ \forall x,y\in E^*, j_{E^*}(y)\in J_{E^*}(y).$$

Definition 1.6. Let $C : E\to E$ be a mapping. Then

(1) C is said to be non-expansive if $\|Cx-Cy\|\leq\|x-y\|$ for $\forall x,y\in E$;

(2) C is said to be generalized non-expansive [5] if $Fix(C)\neq\emptyset$ and $\omega(Cx,p)\leq\omega(x,p)$, for $\forall x\in E$ and $p\in Fix(C)$.

It is easy to see that non-expansive and generalized non-expansive mappings are two different types of mappings.

Definition 1.7. [1, 6] (1) If E is a reflexive and strictly convex Banach space, then, for each $x \in E$, there exists a unique element $v \in K$ such that $\|x - v\| = \inf\{\|x - y\| : y \in K\}$. Such an element v is denoted by $P_K x$ and P_K is called the metric projection of E onto K.

(2) If E is a real reflexive, strictly convex and smooth Banach space, then, for $\forall x \in E$, there exists a unique element $x_0 \in K$ satisfying $\omega(x_0, x) = \inf\{\omega(z, x) : z \in K\}$. In this case, $\forall x \in E$, define $\pi_K : E \to K$ by $\pi_K x = x_0$. Then π_K is called the generalized projection from E onto K.

Definition 1.8. [7] Let E be a real smooth Banach space.

(1) Define $G : K \times E^* \to (0, +\infty]$ by

$$G(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 + 2\rho f(x), \forall x \in K, y \in E^*,$$

where $\rho > 0$ and $f : K \to (-\infty, +\infty]$ is a proper convex and lower-semi-continuous function.

(2) $\pi_K^f : E \to 2^E$ is called the generalized $f-$projection if $\pi_K^f(y) = \{z \in K : G(z, J_E y) \le G(x, J_E y), \forall x \in K\}, \forall y \in E$.

Definition 1.9. [5] Let Q be a mapping of E onto K. Then Q is said to be sunny if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \ge 0$. A mapping $Q : E \to K$ is said to be a retraction if $Q(z) = z$ for every $z \in K$. If E is a smooth and strictly convex Banach space, then the sunny generalized non-expansive retraction of E onto K is uniquely determined, which is denoted by R_K.

Definition 1.10. [8] Let $\{K_n\}$ be a sequence of nonempty closed and convex subsets of E. Then

(1) s-$\liminf K_n$, which is called a strong lower limit of $\{K_n\}$, is defined as the set of all $x \in E$ such that there exists $x_n \in K_n$ for almost all n and it tends to x as $n \to \infty$ in norm.

(2) w-$\limsup K_n$, which is called a weak upper limit of $\{K_n\}$, is defined as the set of all $x \in E$ such that there exists a subsequence $\{K_{n_m}\}$ of $\{K_n\}$ and $x_{n_m} \in K_{n_m}$ for every n_m and it tends to x as $n_m \to \infty$ in the weak topology;

(3) If s-$\liminf K_n = w - \limsup K_n$, then the common value is denoted by $\lim K_n$.

Lemma 1.11. [9] *Suppose that E is a real reflexive and strictly convex Banach space. If $\lim K_n$ exists and is not empty, then $\{P_{K_n} x\}$ converges weakly to $P_{\lim K_n} x$ for every $x \in X$. Moreover, if E has Property (H), then the convergence is in norm.*

Lemma 1.12. [8] *Let $\{K_n\}$ be a decreasing sequence of closed and convex subsets of E, i.e. $K_n \subset K_m$ if $n \ge m$. Then $\{K_n\}$ converges in E and $\lim K_n = \bigcap_{n=1}^{\infty} K_n$.*

Lemma 1.13. [10] *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\omega(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Recently, a lot of results on iterative designs for zero points of m-accretive mappings has been done during past 20 years, see, for examples, [11, 12, 13, 14, 15, 16] and the references therein. The d-accretive mapping is also a kind of important nonlinear accretive-type mappings. However, less research results on d-accretive mappings were established. The possible reasons are that they do not have "good" properties and "few" real word applications can be founded. In recent years, some work has been done to design iterative algorithms to approximate zero points of d-accretive mappings and to find their applications, see, for example, [2, 7, 17, 18, 19].

In 2014, Wei, Liu and Agarwal [2] presented the following block projection iterative algorithm for a finite family of m-d-accretive mappings $\{A_i\}_{i=1}^m \subset E \times E$:

$$\begin{cases} x_1 \in E, \\ u_n = \sum_{i=1}^m a_{n,i}[\alpha_{n,i}x_n + (1-\alpha_{n,i})(I + r_{n,i}A_i)^{-1}x_n], \\ v_{n+1} = \sum_{i=1}^m b_{n,i}[\beta_{n,i}x_n + (1-\beta_{n,i})(I + s_{n,i}A_i)^{-1}y_n], \\ H_1 = E, \\ H_{n+1} = \{z \in H_n : \omega(v_n, z) \le \omega(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}}x_1, \quad n \in N. \end{cases} \tag{1.1}$$

Under mild assumptions, They proved that $\{x_n\}$ generated by (1.1) strongly converges to an element in $\bigcap_{i=1}^m A_i^{-1}0$.

We also mention that they also studied m-d-accretive mappings based on the following nonlinear elliptic boundary value problem

$$\begin{cases} -div(\alpha(gradu)) + |u|^{p-2}u + g(x, u(x)) = f(x), \ a.e. \ in \ \Omega, \\ -\langle \vartheta, \alpha(gradu)\rangle \in \beta_x(u(x)), \ a.e. \ in \ \Gamma, \end{cases} \tag{1.2}$$

where Ω is a bounded conical domain of Ω with its boundary $\Gamma \in C^1$, ϑ is the exterior normal derivative of Γ and $\alpha : R^N \to R^N$ is a monotone and continuous function satisfying some conditions. This gives us an example of d-accretive mappings.

In 2016, by employing G-function and π_K^f in Definition 1.8, Wei and Liu [7] presented the following iterative algorithm with computational errors for a finite family of m-d-accretive mappings $\{A_i\}_{i=1}^m \subset E^* \times E^*$

$$\begin{cases} x_1 \in E, r_{1,i} > 0, i = 1, 2, \cdots, m, \\ y_{n,i} = (J_E + r_{n,i}A_iJ_E)^{-1}J_Ex_n, i = 1, 2, \cdots, m, \\ J_Eu_{n,i} = \beta_{n,i}J_Ey_{n,i} + (1-\beta_{n,i})J_Ee_n, i = 1, 2, \cdots, m, \\ J_Ez_{n,i} = \alpha_{n,i}J_Ex_n + (1-\alpha_{n,i})J_Eu_{n,i}, i = 1, 2, \cdots, m, \\ C_{1,i} = E, i = 1, 2, \cdots, m, \\ C_1 = \bigcap_{i=1}^m C_{1,i}, \\ C_{n+1,i} = \{v \in C_n : G(v, J_Ez_{n,i}) \le (\alpha_{n,i} + \beta_{n,i} - \alpha_{n,i}\beta_{n,i})G(v, J_Ex_n) \\ \qquad\qquad + (1-\alpha_{n,i})(1-\beta_{n,i})G(v, J_Ee_n)\}, i = 1, 2, \cdots, m, \\ C_{n+1} = \bigcap_{i=1}^m C_{n+1,i}, \\ x_{n+1} = \pi_{C_{n+1}}^f x_1, \quad n \in N. \end{cases} \tag{1.3}$$

Under mild assumptions, $\{x_n\}$ generated by (1.3) was proved to be strongly convergent to an element in $\bigcap_{i=1}^m A_i^{-1}0$.

Comparing the results announced in Wei, Liu and Agarwal [2] and Wei and Liu [7], we find new ideas from Wei and Agarwal [19] that (1) a finite family of m-d-accretive mappings is extended to the case of an infinite family. (2) New projection sets are constructed to avoid computing $\omega(v_n, z)$ (or $\omega(x_n, z)$) in

(1.1) or $G(v, J_E z_{n,i})$ (or $G(v, J_E x_n)$ or $G(v, J_E e_n)$) in (1.3), and $R_{H_{n+1}} x_1$ in (1.1) or $\pi^f_{C_{n+1}} x_1$ in (1.3). (3) The iterative element can be arbitrarily chosen in a set. There is only one choice made in (1.1) or (1.3).

Indeed, Wei and Agarwal [19] presented the following two iterative algorithms. One is for an infinite family of m-d-accretive mappings $\{A_i\}_{i=1}^{\infty} \subset E \times E$ as follows

$$\begin{cases} u_1 = v \in E^*, \\ w_{n,i} = (I + s_{n,i} J_E A_i J_{E^*})^{-1} u_n, \\ U_1 = E^*, \\ U_{n+1,i} = \{z \in E^* : \langle J_{E^*}(u_n - w_{n,i}), w_{n,i} - z\rangle \geq 0\}, \\ U_{n+1} = (\bigcap_{i=1}^{\infty} U_{n+1,i}) \bigcap U_n, \\ V_{n+1} = \{z \in U_{n+1} : \|v - z\|^2 \leq \|P_{U_{n+1}}(v) - v\|^2 + \tau_{n+1}\} \ n \in N, \\ u_{n+1} \in V_{n+1}, \\ \overline{u_n} = J_{E^*} u_n, \ n \in N. \end{cases} \tag{1.4}$$

The sequence $\{\overline{u_n}\}$ generated by (1.4) was proved to be strongly convergent to an element in $\bigcap_{i=1}^{\infty} A_i^{-1} 0$.

The other one is for an infinite family of infinite m-d-accretive mappings $\{A_i\}_{i=1}^{\infty} \subset E^* \times E^*$ as follows

$$\begin{cases} x_1 = u \in E, \\ y_{n,i} = (I + r_{n,i} J_{E^*} A_i J_E)^{-1} x_n, \\ X_1 = E, \\ X_{n+1,i} = \{z \in E : \langle y_{n,i} - z, J_E(x_n - y_{n,i})\rangle \geq 0\}, \\ X_{n+1} = (\bigcap_{i=1}^{\infty} X_{n+1,i}) \bigcap X_n, \\ Y_{n+1} = \{z \in X_{n+1} : \|u - z\|^2 \leq \|P_{X_{n+1}}(v) - v\|^2 + \delta_{n+1}\}, \\ x_{n+1} \in Y_{n+1}, \\ \overline{x_n} = J_E x_n, \ n \in N. \end{cases} \tag{1.5}$$

The sequence $\{\overline{x_n}\}$ generated by (1.5) was proved to be strongly convergent to an element in $\bigcap_{i=1}^{\infty} A_i^{-1} 0$.

In Wei and Agarwal [19], m-d-accretive mappings were also studied based on the following generalized (p,q)-Laplacian parabolic systems

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - div[(C_1(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon_1 |u|^{r-2} u \\ + g_1(x, u, \nabla u) = f_1(x,t), \ (x,t) \in \Omega \times (0,T), \\ \frac{\partial v(x,t)}{\partial t} - div[(C_2(x,t) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla u] + \varepsilon_2 |v|^{s-2} v \\ + g_2(x, v, \nabla v) = f_2(x,t), \ (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C_1(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\rangle \in \beta_x(u(x,t)), \ (x,t) \in \Gamma \times (0,T), \\ -\langle \vartheta, (C_2(x,t) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v\rangle \in \beta_x(v(x,t)), \ (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), v(x,0) = v(x,T), \ x \in \Omega, \end{cases} \tag{1.6}$$

where Ω, Γ and ϑ are as in (1.2). This provides another example of d-accretive mappings.

Remark 1.14. There are two skips in Step 2 of Algorithm 2.1 and Step 2 of Algorithm 2.2 in [19]. V_{n+1} should be $V_{n+1} = \{z \in U_{n+1} : \|v-z\|^2 \leq \|P_{U_{n+1}}(v) - z\|^2 + \tau_{n+1}\}$ and Y_{n+1} should be $Y_{n+1} = \{z \in X_{n+1} : \|u-z\|^2 \leq \|P_{X_{n+1}}(u) - z\|^2 + \delta_{n+1}\}$, respectively.

Notice that, in both (1.4) and (1.5), infinite sets of $U_{n+1,i}$ and $X_{n+1,i}$ should be evaluated, $\forall i \in N$. Can we borrow the ideas of (1.1) and transfer the evaluation of infinite sets to the evaluation of the iterative element? In this paper, we shall give an answer to this question.

To obtain the strong convergence, the following lemma is needed.

Lemma 1.15. [20] *Let E be a real uniformly convex Banach space and $r \in (0,+\infty)$. Then there exists a continuous, strictly increasing and convex function $g : [0,2r] \to [0,+\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|),$$

for $\alpha \in [0,1]$, $x,y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r$.

2. New Iterative Design for m-d-accretive mappings $\{A_i\}_{i=1}^{\infty} \subset E^* \times E^*$

Lemma 2.1. *Let E be a real uniformly smooth and uniformly convex Banach space and let $A \subset E^* \times E^*$ be an m-d-accretive mapping with $A^{-1}0 \neq \emptyset$. Then, for $\forall x \in E$, $\forall z \in A^{-1}0$ and $\forall r > 0$,*

$$\omega(J_{E^*}z, (J_E + rAJ_E)^{-1}J_E x) + \omega((J_E + rAJ_E)^{-1}J_E x, x) \leq \omega(J_{E^*}z, x).$$

Proof. From Lemma 1.1, $\forall x \in E$ and $\forall r > 0$, we have

$$J_E(J_E + rAJ_E)^{-1}J_E x + \lambda A[J_E(J_E + rAJ_E)^{-1}J_E x] = J_E x. \tag{2.1}$$

Since A is d-accretive, we have

$$\langle (J_E + rAJ_E)^{-1}J_E x - J_{E^*}z, AJ_E(J_E + rAJ_E)^{-1}J_E x\rangle \geq 0.$$

From the definition of Lyapunov functional and (2.1), one has

$$\begin{aligned} &\omega(J_{E^*}z, x) - \omega((J_E + rAJ_E)^{-1}J_E x, x) - \omega(J_{E^*}z, (J_E + rAJ_E)^{-1}J_E x) \\ &= 2\langle J_{E^*}z, J_E(J_E + rAJ_E)^{-1}J_E x - J_E x\rangle - 2\langle (J_E + rAJ_E)^{-1}J_E x, J_E(J_E + rAJ_E)^{-1}J_E x - J_E x\rangle \\ &= 2r\langle (J_E + rAJ_E)^{-1}J_E x - J_{E^*}z, AJ_E(J_E + rAJ_E)^{-1}J_E x\rangle \geq 0, \end{aligned}$$

which concludes the desired conclusion. This completes the proof. □

Theorem 2.2. *Let E be a real uniformly smooth and uniformly convex Banach space and let $\{A_i\}_{i=1}^{\infty} \subset E^* \times E^*$ be an infinite family of m-d-accretive mappings such that $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$. Suppose that $\{e_n\} \subset E$ is an error sequence, $\{\alpha_n\} \subset [0,1)$, $\{\mu_n\} \subset [0,+\infty)$, $\{a_{n,i}\} \subset (0,1)$ and $\{r_{n,i}\} \subset (0,+\infty)$. Let $\{x_n\}$ be a* *sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 \in E,\ e_1 \in E, \\ y_n = J_{E^*}[\alpha_n J_E x_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_E (J_E + r_{n,i} A_i J_E)^{-1} J_E (x_n + e_n)], \\ U_1 = E = V_1, \\ U_{n+1} = \{p \in U_n : \langle p, J_E y_n - \alpha_n J_E x_n - (1-\alpha_n) J_E (x_n + e_n)\rangle \geq \frac{\|y_n\|^2 - \alpha_n \|x_n\|^2 - (1-\alpha_n)\|x_n + e_n\|^2}{2}\}, \\ V_{n+1} = \{p \in U_{n+1} : \|x_1 - p\|^2 \leq \|P_{U_{n+1}}(x_1) - x_1\|^2 + \mu_{n+1}\}, \\ x_{n+1} \in V_{n+1}, \\ \overline{x_n} = J_E x_n,\ n \in N. \end{cases} \tag{2.2}$$

If $\sum_{i=1}^{\infty} a_{n,i} = 1$, $\liminf_n r_{n,i} > 0$, $\forall i \in N$, $0 \leq \sup_n \alpha_n < 1$, $\lim_{n\to\infty} e_n = 0$ *and* $\limsup_{n\to\infty} \mu_n = 0$, *then* $\overline{x_n} \to J_E P_{\bigcap_{n=1}^{\infty} U_n}(x_1) = J_E P_{\bigcap_{i=1}^{\infty} (A_i J_E)^{-1}0}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, *as* $n \to \infty$.

Proof. We split the proof into ten steps.

Step 1. Show $\bigcap_{i=1}^{\infty} (A_i J_E)^{-1}0 \neq \emptyset$.

Since $\bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset$, we find that there exists $q \in E^*$ such that $A_i q = 0$, for $i \in N$. In view of Lemma 1.1, there exists $p \in E$ such that $J_E p = q$. Thus $(A_i J_E)p = A_i(J_E p) = A_i q = 0$, for $i \in N$, which implies that $\bigcap_{i=1}^{\infty} (A_i J_E)^{-1}0 \neq \emptyset$.

Step 2. Show $\bigcap_{i=1}^{\infty} (A_i J_E)^{-1}0 \subset U_n$, for all $n \in N$.

To this end, we shall use inductive method. For $n = 1$, it is obvious that $\bigcap_{i=1}^{\infty} (A_i J_E)^{-1}0 \subset U_1 = E$. Now, $\forall p \in \bigcap_{i=1}^{\infty} (A_i J_E)^{-1}0$, we know from Lemma 2.1 that

$$\begin{aligned} \omega(p, y_1) &= \|p\|^2 - 2\langle p, \alpha_1 J_E x_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} J_E (J_E + r_{1,i} A_i J_E)^{-1} J_E (x_1 + e_1)\rangle \\ &+ \|\alpha_1 J_E x_1 + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} J_E (J_E + r_{1,i} A_i J_E)^{-1} J_E (x_1 + e_1)\|^2 \\ &\leq \|p\|^2 - 2\alpha_1 \langle p, J_E x_1\rangle + \alpha_1 \|x_1\|^2 - 2(1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} \langle p, J_E (J_E + r_{1,i} A_i J_E)^{-1} J_E (x_1 + e_1)\rangle \\ &+ (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} \|(J_E + r_{1,i} A_i J_E)^{-1} J_E (x_1 + e_1)\|^2 \\ &= \alpha_1 \omega(p, x_1) + (1-\alpha_1)\sum_{i=1}^{\infty} a_{1,i} \omega(p, (J_E + r_{1,i} A_i J_E)^{-1} J_E (x_1 + e_1)) \\ &\leq \alpha_1 \omega(p, x_1) + (1-\alpha_1)\omega(p, x_1 + e_1). \end{aligned}$$

It follows from the definition of Lyapunov functional that

$$\langle p, J_E y_1 - \alpha_1 J_E x_1 - (1-\alpha_1) J_E (x_1 + e_1)\rangle \geq \frac{\|y_1\|^2 - \alpha_1 \|x_1\|^2 - (1-\alpha_1)\|x_1 + e_1\|^2}{2}.$$

Thus $p \in U_2$. Suppose that the result is true for $n = k+1$. Now, if $n = k+2$, using Lemma 2.1 again yields that, $\forall p \in \bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0$,

$$\omega(p, y_{k+1}) = \|p\|^2 - 2\langle p, \alpha_{k+1}J_Ex_{k+1} + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}J_E(J_E + r_{k+1,i}A_iJ_E)^{-1}J_E(x_{k+1}+e_{k+1})\rangle$$

$$+ \|\alpha_{k+1}J_Ex_{k+1} + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}J_E(J_E + r_{k+1,i}A_iJ_E)^{-1}J_E(x_{k+1}+e_{k+1})\|^2$$

$$\leq \alpha_{k+1}\omega(p, x_{k+1}) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\omega(p, (J_E + r_{k+1,i}A_iJ_E)^{-1}J_E(x_{k+1}+e_{k+1}))$$

$$\leq \alpha_{k+1}\omega(p, x_{k+1}) + (1-\alpha_{k+1})\omega(p, x_{k+1}+e_{k+1}).$$

It follows from the definition of Lyapunov functional that

$$\langle p, J_Ey_{k+1} - \alpha_{k+1}J_Ex_{k+1} - (1-\alpha_{k+1})J_E(x_{k+1}+e_{k+1})\rangle$$
$$\geq \frac{\|y_{k+1}\|^2 - \alpha_{k+1}\|x_{k+1}\|^2 - (1-\alpha_{k+1})\|x_{k+1}+e_{k+1}\|^2}{2}.$$

Thus $p \in U_{k+2}$. By induction, $\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0 \subset U_n$, for all $n \in N$.

Step 3. Show that U_n is a closed and convex subset of E, for all $n \in N$.

The result follows immediately from the definition of U_n in (2.2).

Step 4. Show $P_{U_n}(x_1) \to P_{\bigcap_{n=1}^{\infty}U_n}(x_1)$, as $n \to \infty$.

In fact, from Step 1 to Step 3, and employing Lemmas 1.11 and 1.12, we know that $P_{U_n}(x_1) \to P_{\bigcap_{n=1}^{\infty}U_n}(x_1)$, as $n \to \infty$.

Step 5. Show $V_n \neq \emptyset$.

Since $\|P_{U_{n+1}}(x_1) - x_1\| = \inf_{v \in U_{n+1}}\|v - x_1\|$, for μ_{n+1}, there exists $d_{n+1} \in U_{n+1}$ such that

$$\|x_1 - d_{n+1}\|^2 \leq (\inf_{v \in U_{n+1}}\|v - x_1\|)^2 + \mu_{n+1} = \|P_{U_{n+1}}(x_1) - x_1\|^2 + \mu_{n+1}.$$

Then $V_n \neq \emptyset$, which implies that $\{x_n\}$ is well-defined.

Step 6. Show that $\{x_n\}$ is bounded.

Since $x_{n+1} \in V_{n+1}$, one has

$$\|x_1 - x_{n+1}\|^2 \leq \|P_{U_{n+1}}(x_1) - x_1\|^2 + \mu_{n+1}.$$

From Step 4 and $\mu_n \to 0$, we know that $\{x_n\}$ is bounded.

Step 7. Show $x_n \to P_{\bigcap_{n=1}^{\infty}U_n}(x_1)$ and $y_n \to P_{\bigcap_{n=1}^{\infty}U_n}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in V_{n+1} \subset U_{n+1}$ and U_n is convex, we have, $\forall k \in (0,1)$,

$$kP_{U_{n+1}}(x_1) + (1-k)x_{n+1} \in U_{n+1}.$$

Thus

$$\|P_{U_{n+1}}(x_1) - x_1\| \leq \|kP_{U_{n+1}}(x_1) + (1-k)x_{n+1} - x_1\|.$$

Using Lemma 1.15, we have

$$\|P_{U_{n+1}}(x_1) - x_1\|^2 \leq \|kP_{U_{n+1}}(x_1) + (1-k)x_{n+1} - x_1\|^2$$
$$\leq k\|P_{U_{n+1}}(x_1) - x_1\|^2 + (1-k)\|x_{n+1} - x_1\|^2 - k(1-k)g(\|P_{U_{n+1}}(x_1) - x_{n+1}\|).$$

Therefore, $kg(\|P_{U_{n+1}}(x_1)-x_{n+1}\|)\leq \|x_{n+1}-x_1\|^2-\|P_{U_{n+1}}(x_1)-x_1\|^2\leq \mu_{n+1}\to 0$, as $n\to\infty$. Then $x_{n+1}-P_{U_{n+1}}(x_1)\to\infty$, as $n\to\infty$. Step 4 ensures that $x_n\to P_{\bigcap_{n=1}^\infty U_n}(x_1)$, as $n\to\infty$. Since $x_{n+1}\in V_{n+1}\subset U_{n+1}$, we have

$$\langle x_{n+1},J_Ey_n-\alpha_nJ_Ex_n-(1-\alpha_n)J_E(x_n+e_n)\rangle\geq \frac{\|y_n\|^2-\alpha_n\|x_n\|^2-(1-\alpha_n)\|x_n+e_n\|^2}{2},$$

which is equivalent to

$$\omega(x_{n+1},y_n)\leq \alpha_n\omega(x_{n+1},x_n)+(1-\alpha_n)\omega(x_{n+1},x_n+e_n).$$

Thus $x_{n+1}-y_n\to 0$, which ensures that $y_n\to P_{\bigcap_{n=1}^\infty U_n}(x_1)$, as $n\to\infty$.

Step 8. Show $P_{\bigcap_{n=1}^\infty U_n}(x_1)\in\bigcap_{i=1}^\infty (A_iJ_E)^{-1}0$.

For $\forall q\in\bigcap_{i=1}^\infty (A_iJ_E)^{-1}0$, using Lemma 2.1, we have

$$\begin{aligned}\omega(q,y_n)&\leq \alpha_n\omega(q,x_n)+(1-\alpha_n)\sum_{i=1}^{\infty}a_{n,i}\omega(q,(J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n))\\ &\leq \alpha_n\omega(q,x_n)+(1-\alpha_n)\sum_{i=1}^{\infty}a_{n,i}[\omega(q,x_n+e_n)-\omega((J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n),x_n+e_n)].\end{aligned}$$

Then

$$\begin{aligned}&(1-\alpha_n)\sum_{i=1}^{\infty}a_{n,i}\omega((J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n),x_n+e_n)\\ &\leq \alpha_n\omega(q,x_n)-\omega(q,y_n)+(1-\alpha_n)\omega(q,x_n+e_n)\\ &=\alpha_n[\omega(q,x_n)-\omega(q,x_n+e_n)]+[\omega(q,x_n+e_n)-\omega(q,y_n)]\\ &\leq \|x_n\|^2-\|x_n+e_n\|^2+2\|q\|\|J_E(x_n+e_n)-J_Ex_n\|\\ &\quad+\|x_n+e_n\|^2-\|y_n\|^2+2\|q\|\|J_E(x_n+e_n)-J_Ey_n\|.\end{aligned}$$

Since $0\leq\sup_n\alpha_n<1$, $x_n-y_n\to 0$ and $e_n\to 0$, we have

$$\sum_{i=1}^{\infty}a_{n,i}\omega((J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n),x_n+e_n)\to 0,$$

which implies from Lemma 1.13 that

$$(J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n)-(x_n+e_n)\to 0,$$

as $n\to\infty$. So, Step 7 implies that

$$(J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n)\to P_{\bigcap_{n=1}^\infty U_n}(x_1),$$

as $n\to\infty$. Let $z_{n,i}=(J_E+r_{n,i}A_iJ_E)^{-1}J_E(x_n+e_n)$. Then

$$J_Ez_{n,i}+r_{n,i}A_iJ_Ez_{n,i}=J_E(x_n+e_n).$$

Note that $z_{n,i}\to P_{\bigcap_{n=1}^\infty U_n}(x_1)$, $x_n\to P_{\bigcap_{n=1}^\infty U_n}(x_1)$, $e_n\to 0$ and $\liminf_n r_{n,i}>0$. Then $A_iJ_Ez_{n,i}\to 0$, as $n\to\infty$. It is easy to see that $A_iJ_E\subset E\times E^*$ is maximal monotone. From Lemma 1.4, we imply that $P_{\bigcap_{n=1}^\infty U_n}(x_1)\in\bigcap_{i=1}^\infty (A_iJ_E)^{-1}0$.

Step 9. Show $P_{\bigcap_{n=1}^\infty U_n}(x_1)=P_{\bigcap_{i=1}^\infty (A_iJ_E)^{-1}0}(x_1)$.

From Step 8, $P_{\bigcap_{n=1}^\infty U_n}(x_1)\in\bigcap_{i=1}^\infty (A_iJ_E)^{-1}0$. It follows that

$$\|P_{\bigcap_{i=1}^\infty (A_iJ_E)^{-1}0}(x_1)-x_1\|\leq\|P_{\bigcap_{n=1}^\infty U_n}(x_1)-x_1\|.$$

From Step 2, $\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0 \subset U_n$. Then

$$\|P_{\bigcap_{n=1}^{\infty}U_n}(x_1) - x_1\| \leq \|P_{\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0}(x_1) - x_1\|.$$

Thus

$$\|P_{\bigcap_{n=1}^{\infty}U_n}(x_1) - x_1\| = \|P_{\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0}(x_1) - x_1\|.$$

Since $P_{\bigcap_{n=1}^{\infty}U_n}(x_1)$ is unique, we have $P_{\bigcap_{n=1}^{\infty}U_n}(x_1) = P_{\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0}(x_1)$.

Step 10. Show $\overline{x_n} \to J_E P_{\bigcap_{n=1}^{\infty}U_n}(x_1) = J_E P_{\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, as $n \to \infty$.

Since $\overline{x_n} = J_E x_n$, Lemma 1.1 implies that

$$\overline{x_n} \to J_E P_{\bigcap_{n=1}^{\infty}U_n}(x_1) = J_E P_{\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0,$$

as $n \to \infty$. This completes the proof. □

Theorem 2.3. *Let $\{\overline{x_n}\}$ be a sequence generated by algorithm (2.2). Set $\lambda_n = \frac{\sum_{i=1}^{n+1} c_i \overline{x_i}}{\sum_{i=1}^{n+1} c_i}$ for all $n \in N$. Let $\sum_{i=1}^{n} c_i \to \infty$, as $n \to \infty$. Under the assumptions of Theorem 2.2, we obtain the result of ergodic convergence in the sense that $\lambda_n \to J_E P_{\bigcap_{n=1}^{\infty}U_n}(x_1) = J_E P_{\bigcap_{i=1}^{\infty}(A_iJ_E)^{-1}0}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$ as $n \to \infty$.*

Proof. The proof is similar to that of Step 5 of theorem 2.2 in [18]. So, we omit the proof here. □

Remark 2.4. We remark that iterative algorithm (2.2) is different from iterative algorithms (1.1), (1.3), (1.4) and (1.5). We no longer need to evaluated infinite projection sets in (2.2). Moreover, to obtain the strong convergence of the iterative sequence, the techniques of employing the properties of Lyapunov functional are employed.

3. New Iterative Design for m-d-accretive mappings $\{B_i\}_{i=1}^{\infty} \subset E \times E$

Lemma 3.1. *Let E be a real uniformly smooth and uniformly convex Banach space and let $B \subset E \times E$ be an m-d-accretive mapping with $B^{-1}0 \neq \emptyset$. Then, for $\forall x \in E^*$, $\forall z \in B^{-1}0$ and $\forall r > 0$,*

$$\overline{\omega}(J_E z, (J_{E^*} + rBJ_{E^*})^{-1}J_{E^*}x) + \overline{\omega}((J_{E^*} + rBJ_{E^*})^{-1}J_{E^*}x, x) \leq \overline{\omega}(J_E z, x).$$

Proof. From the proof of Lemma 2.1, we can find the desired conclusion immediately. This completes the proof. □

Theorem 3.2. *Let E be a real uniformly smooth and uniformly convex Banach space and let $\{B_i\}_{i=1}^{\infty} \subset E \times E$ be an infinite family of m-d-accretive mappings such that $\bigcap_{i=1}^{\infty} B_i^{-1}0 \neq \emptyset$. Suppose that $\{\varepsilon_n\} \subset E$ is the error sequence, $\{\alpha_n\} \subset [0,1)$, $\{\lambda_n\} \subset [0,+\infty)$, $\{b_{n,i}\} \subset (0,1)$ and $\{s_{n,i}\} \subset (0,+\infty)$. Let $\{u_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} u_1 \in E^*, \ \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*}u_n + (1-\alpha_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i}B_iJ_{E^*})^{-1}J_{E^*}(u_n + \varepsilon_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{q \in U_n : \langle J_{E^*}v_n - \alpha_n J_{E^*}u_n - (1-\alpha_n)J_{E^*}(u_n+\varepsilon_n), q\rangle \geq \frac{\|v_n\|^2 - \alpha_n\|u_n\|^2 - (1-\alpha_n)\|u_n+\varepsilon_n\|^2}{2}\}, \\ V_{n+1} = \{q \in U_{n+1} : \|u_1 - q\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ u_{n+1} \in V_{n+1}, \\ \overline{u_n} = J_{E^*}u_n, \ n \in N. \end{cases} \tag{3.1}$$

If $\sum_{i=1}^{\infty} b_{n,i} = 1$, $0 \le \sup_n \alpha_n < 1$, $\liminf_n s_{n,i} > 0$ *for* $i \in N$, $\lim_{n\to\infty} \varepsilon_n = 0$ *and* $\limsup_{n\to\infty} \lambda_n = 0$, *then* $\overline{u_n} \to J_{E^*} P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = J_{E^*} P_{\bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0}(u_1) \in \bigcap_{i=1}^{\infty} B_i^{-1}0$, *as* $n \to \infty$.

Proof. We split the proof into ten steps.

Step 1. Show $\bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0 \ne \emptyset$.

Since $\bigcap_{i=1}^{\infty} B_i^{-1}0 \ne \emptyset$, there exists $p \in E$ such that $B_i p = 0$, for $i \in N$. In view of Lemma 1.1, there exists $q \in E^*$ such that $J_{E^*} q = p$. Thus $(B_i J_{E^*})q = B_i(J_{E^*} q) = B_i p = 0$, for $i \in N$, which implies that $\bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0 \ne \emptyset$.

Step 2. Show $\bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0 \subset U_n$, for all $n \in N$.

For this, we shall use inductive method. For $n = 1$, it is obvious that $\bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0 \subset U_1 = E^*$. Now, $\forall q \in \bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0$, we know from Lemma 3.1 that

$$\begin{aligned}
\overline{\omega}(q, v_1) &= \|q\|^2 - 2\langle \alpha_1 J_{E^*} u_1 + (1-\alpha_1) \sum_{i=1}^{\infty} b_{1,i} J_{E^*}(J_{E^*} + s_{1,i} B_i J_{E^*})^{-1} J_{E^*}(u_1 + \varepsilon_1), q\rangle \\
&+ \|\alpha_1 J_{E^*} u_1 + (1-\alpha_1) \sum_{i=1}^{\infty} b_{1,i} J_{E^*}(J_{E^*} + s_{1,i} B_i J_{E^*})^{-1} J_{E^*}(u_1 + \varepsilon_1)\|^2 \\
&\le \|q\|^2 - 2\alpha_1 \langle J_{E^*} u_1, q\rangle + \alpha_1 \|u_1\|^2 - 2(1-\alpha_1) \sum_{i=1}^{\infty} b_{1,i} \langle J_{E^*}(J_{E^*} + s_{1,i} B_i J_{E^*})^{-1} J_{E^*}(u_1 + \varepsilon_1), q\rangle \\
&+ (1-\alpha_1) \sum_{i=1}^{\infty} b_{1,i} \|(J_{E^*} + s_{1,i} B_i J_{E^*})^{-1} J_{E^*}(u_1 + \varepsilon_1)\|^2 \\
&= \alpha_1 \overline{\omega}(q, u_1) + (1-\alpha_1) \sum_{i=1}^{\infty} b_{1,i} \overline{\omega}(q, (J_{E^*} + s_{1,i} B_i J_{E^*})^{-1} J_{E^*}(u_1 + \varepsilon_1)) \\
&\le \alpha_1 \overline{\omega}(q, u_1) + (1-\alpha_1) \overline{\omega}(q, u_1 + \varepsilon_1).
\end{aligned}$$

It follows from the definition of Lyapunov functional that

$$\langle J_{E^*} v_1 - \alpha_1 J_{E^*} u_1 - (1-\alpha_1) J_{E^*}(u_1 + \varepsilon_1), q\rangle \ge \frac{\|v_1\|^2 - \alpha_1 \|u_1\|^2 - (1-\alpha_1)\|u_1 + \varepsilon_1\|^2}{2}.$$

Thus $q \in U_2$. Suppose the result is true for $n = k+1$. Now, if $n = k+2$, using Lemma 3.1 again, we have $\forall q \in \bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0$,

$$\begin{aligned}
\overline{\omega}(q, v_{k+1}) &= \|q\|^2 - 2\langle \alpha_{k+1} J_{E^*} u_{k+1} + (1-\alpha_{k+1}) \sum_{i=1}^{\infty} b_{k+1,i} J_{E^*}(J_{E^*} + s_{k+1,i} B_i J_{E^*})^{-1} J_{E^*}(u_{k+1} + \varepsilon_{k+1}), q\rangle \\
&+ \|\alpha_{k+1} J_{E^*} u_{k+1} + (1-\alpha_{k+1}) \sum_{i=1}^{\infty} b_{k+1,i} J_{E^*}(J_{E^*} + s_{k+1,i} B_i J_{E^*})^{-1} J_{E^*}(u_{k+1} + \varepsilon_{k+1})\|^2 \\
&\le \alpha_{k+1} \overline{\omega}(q, u_{k+1}) + (1-\alpha_{k+1}) \sum_{i=1}^{\infty} b_{k+1,i} \overline{\omega}(q, (J_{E^*} + s_{k+1,i} B_i J_{E^*})^{-1} J_{E^*}(u_{k+1} + \varepsilon_{k+1})) \\
&\le \alpha_{k+1} \overline{\omega}(q, u_{k+1}) + (1-\alpha_{k+1}) \overline{\omega}(q, u_{k+1} + \varepsilon_{k+1}).
\end{aligned}$$

It follows from the definition of Lyapunov functional that

$$\begin{aligned}
&\langle J_{E^*} v_{k+1} - \alpha_{k+1} J_{E^*} u_{k+1} - (1-\alpha_{k+1}) J_{E^*}(u_{k+1} + \varepsilon_{k+1}), q\rangle \\
&\ge \frac{\|v_{k+1}\|^2 - \alpha_{k+1}\|u_{k+1}\|^2 - (1-\alpha_{k+1})\|u_{k+1} + \varepsilon_{k+1}\|^2}{2}.
\end{aligned}$$

Thus $p \in U_{k+2}$. Then, by induction, $\bigcap_{i=1}^{\infty}(B_i J_{E^*})^{-1}0 \subset U_n$, for all $n \in N$.

Step 3. Show that U_n is a closed and convex subset of E, for all $n \in N$.

The result follows immediately from the definition of U_n in (2.3).

Step 4. Show $P_{U_n}(u_1) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$.

In fact, from Step 1 to Step 3, and employing Lemmas 1.11 and 1.12 , we know that $P_{U_n}(u_1) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$.

Step 5. Show $V_n \neq \emptyset$.

Since $\|P_{U_{n+1}}(u_1) - u_1\| = \inf_{v \in U_{n+1}} \|v - u_1\|$, for λ_{n+1} there exists $d_{n+1} \in U_{n+1}$ such that

$$\|u_1 - d_{n+1}\|^2 \leq (inf_{v \in U_{n+1}} \|v - u_1\|)^2 + \lambda_{n+1} = \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}.$$

Then $V_n \neq \emptyset$, which implies that $\{u_n\}$ is well-defined.

Step 6. Show that $\{u_n\}$ is bounded.

Since $u_{n+1} \in V_{n+1}$, we have

$$\|u_1 - u_{n+1}\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}.$$

From Step 4 and $\lambda_n \to 0$, we know that $\{u_n\}$ is bounded.

Step 7. Show $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$.

Since $u_{n+1} \in V_{n+1} \subset U_{n+1}$ and U_n is convex, we have, $\forall k \in (0,1)$, $kP_{U_{n+1}}(u_1) + (1-k)u_{n+1} \in U_{n+1}$. Thus

$$\|P_{U_{n+1}}(u_1) - u_1\| \leq \|kP_{U_{n+1}}(u_1) + (1-k)u_{n+1} - u_1\|.$$

Using Lemma 1.15, we have

$$\begin{aligned}
&\|P_{U_{n+1}}(u_1) - u_1\|^2 \leq \|kP_{U_{n+1}}(u_1) + (1-k)u_{n+1} - u_1\|^2 \\
&\leq k\|P_{U_{n+1}}(u_1) - u_1\|^2 + (1-k)\|u_{n+1} - u_1\|^2 - k(1-k)g(\|P_{U_{n+1}}(u_1) - u_{n+1}\|).
\end{aligned}$$

Therefore,

$$kg(\|P_{U_{n+1}}(u_1) - u_{n+1}\|) \leq \|u_{n+1} - u_1\|^2 - \|P_{U_{n+1}}(u_1) - u_1\|^2 \leq \lambda_{n+1} \to 0,$$

as $n \to \infty$. Step 4 ensures that $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$. Since $u_{n+1} \in V_{n+1} \subset U_{n+1}$, we have

$$\langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + \varepsilon_n), u_{n+1} \rangle \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + \varepsilon_n\|^2}{2},$$

which is equivalent to

$$\overline{\omega}(u_{n+1}, v_n) \leq \alpha_n \overline{\omega}(u_{n+1}, u_n) + (1-\alpha_n)\overline{\omega}(u_{n+1}, u_n + \varepsilon_n).$$

Thus $u_{n+1} - v_n \to 0$, which ensures that $v_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$.

Step 8. $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} (B_i J_{E^*})^{-1} 0$.

For $\forall p \in \bigcap_{i=1}^{\infty} (B_i J_{E^*})^{-1} 0$, from Lemma 3.1, one has

$$\begin{aligned}
&\overline{\omega}(p, v_n) \leq \alpha_n \overline{\omega}(p, u_n) + (1-\alpha_n) \sum_{i=1}^{\infty} b_{n,i} \overline{\omega}(p, (J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(u_n + \varepsilon_n)) \\
&\leq \alpha_n \overline{\omega}(p, u_n) + (1-\alpha_n) \sum_{i=1}^{\infty} b_{n,i} [\overline{\omega}(p, u_n + \varepsilon_n) - \overline{\omega}((J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(u_n + \varepsilon_n), u_n + \varepsilon_n)].
\end{aligned}$$

Then

$$(1-\alpha_n)\sum_{i=1}^{\infty} b_{n,i}\overline{\omega}((J_{E^*}+s_{n,i}B_iJ_{E^*})^{-1}J_{E^*}(u_n+\varepsilon_n),u_n+\varepsilon_n)$$
$$\leq \alpha_n[\overline{\omega}(p,u_n)-\overline{\omega}(p,u_n+\varepsilon_n)]+[\overline{\omega}(p,u_n+\varepsilon_n)-\overline{\omega}(p,v_n)]$$
$$\leq \|u_n\|^2-\|u_n+\varepsilon_n\|^2+2\|p\|\|J_{E^*}(u_n+\varepsilon_n)-J_{E^*}u_n\|$$
$$+\|u_n+\varepsilon_n\|^2-\|v_n\|^2+2\|p\|\|J_{E^*}(u_n+\varepsilon_n)-J_{E^*}v_n\|.$$

Since $0\leq \sup_n \alpha_n<1$, $u_n-v_n\to 0$ and $\varepsilon_n\to 0$, we have

$$\sum_{i=1}^{\infty} b_{n,i}\overline{\omega}((J_{E^*}+s_{n,i}B_iJ_{E^*})^{-1}J_{E^*}(u_n+\varepsilon_n),u_n+\varepsilon_n)\to 0,$$

which implies from Lemma 1.13 that

$$(J_{E^*}+s_{n,i}B_iJ_{E^*})^{-1}J_{E^*}(u_n+\varepsilon_n)-(u_n+\varepsilon_n)\to 0,$$

as $n\to\infty$. Thus from Step 7 and $\varepsilon_n\to 0$, we know that

$$(J_{E^*}+s_{n,i}B_iJ_{E^*})^{-1}J_{E^*}(u_n+\varepsilon_n)\to P_{\bigcap_{n=1}^{\infty}U_n}(u_1),$$

as $n\to\infty$. Let $w_{n,i}=(J_{E^*}+s_{n,i}B_iJ_{E^*})^{-1}J_{E^*}(u_n+\varepsilon_n)$. Then $J_{E^*}w_{n,i}+s_{n,i}B_iJ_{E^*}w_{n,i}=J_{E^*}(u_n+\varepsilon_n)$. Since $w_{n,i}\to P_{\bigcap_{n=1}^{\infty}U_n}(u_1)$, $u_n\to P_{\bigcap_{n=1}^{\infty}U_n}(u_1)$, $\varepsilon_n\to 0$ and $\liminf_n s_{n,i}>0$. From Lemma 1.1, $B_iJ_{E^*}w_{n,i}\to 0$, as $n\to\infty$. Since $B_iJ_{E^*}\subset E^*\times E$ is maximal monotone, Lemma 1.4 implies that

$$P_{\bigcap_{n=1}^{\infty}U_n}(u_1)\in\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0.$$

Step 9. Show $P_{\bigcap_{n=1}^{\infty}U_n}(u_1)=P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1)$.

From Step 8, $P_{\bigcap_{n=1}^{\infty}U_n}(u_1)\in\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0$, then

$$\|P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1)-u_1\|\leq\|P_{\bigcap_{n=1}^{\infty}U_n}(u_1)-u_1\|.$$

From Step 2, $\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0\subset U_n$. Hence

$$\|P_{\bigcap_{n=1}^{\infty}U_n}(u_1)-u_1\|\leq\|P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1)-u_1\|.$$

Thus

$$\|P_{\bigcap_{n=1}^{\infty}U_n}(u_1)-u_1\|=\|P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1)-u_1\|.$$

Since $P_{\bigcap_{n=1}^{\infty}U_n}(u_1)$ is unique, we have

$$P_{\bigcap_{n=1}^{\infty}U_n}(u_1)=P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1).$$

Step 10. Show $\overline{u_n}\to J_{E^*}P_{\bigcap_{n=1}^{\infty}U_n}(u_1)\in\bigcap_{i=1}^{\infty}B_i^{-1}0$, as $n\to\infty$.

Since $\overline{u_n}=J_{E^*}u_n$, we conclude form Lemma 1.1 that

$$\overline{u_n}\to J_{E^*}P_{\bigcap_{n=1}^{\infty}U_n}(u_1)=J_{E^*}P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1)\in\bigcap_{i=1}^{\infty}B_i^{-1}0,$$

as $n\to\infty$. This completes the proof. □

Theorem 3.3. *Let $\{\overline{u_n}\}$ be generated by Algorithm (3.1). Set $\eta_n=\frac{\sum_{i=1}^{n+1}c_i\overline{u_i}}{\sum_{i=1}^{n+1}c_i}$ for $n\in N$. Let $\sum_{i=1}^{n}c_i\to\infty$, as $n\to\infty$. Under the assumptions of Theorem 3.2, we obtain the result of ergodic convergence in the sense that $\eta_n\to J_{E^*}P_{\bigcap_{n=1}^{\infty}U_n}(u_1)=J_{E^*}P_{\bigcap_{i=1}^{\infty}(B_iJ_{E^*})^{-1}0}(u_1)\in\bigcap_{i=1}^{\infty}B_i^{-1}0$, as $n\to\infty$.*

Remark 3.4. We remark that iterative algorithm (3.1) is different from iterative algorithms (1.1), (1.3), (1.4) and (1.5). We no longer need to evaluated infinite projection sets in (3.1).

Remark 3.5. The normalized duality mapping J_E or J_{E^*} is no longer needed to be weakly sequentially continuous as that in [2, 7, 17]. The m-d-accretive mapping is no longer needed to be uniformly bounded and demi-continuous as that in [21], and it need not satisfy condition (1.4) as that in [17].

4. Applications

In this section, we provide some applications of our main results.

4.1. Elliptic systems with generalized p-Laplacian.

Remark 4.1. Recall that the following nonlinear elliptic boundary value problem in [2] is presented:

$$\begin{cases} -div(\alpha(gradu)) + |u|^{p-2}u + g(x,u(x)) = f(x),\ x \in \Omega, \\ - < \vartheta, \alpha(gradu) > \in \beta_x(u(x)),\ x \in \Gamma, \end{cases} \tag{4.1}$$

which contains the special case for the p-Laplacian problem if we take $\alpha : R^N \to R^N$ by $\alpha(\xi) = |\xi|^{p-2}\xi$ for $\xi \in R^N$. Based on (4.1), an m-d-accretive mapping is defined. Under some assumptions, the connection between zero points of an m-d-accretive mapping and the solution of (4.1) is set up, which admits the iterative sequence derived from the iterative algorithms to approximate the solution of (4.1).

Now, we consider an elliptic systems with generalized p-Laplacian to support our main results.

$$\begin{cases} -div(\alpha_i(gradu^{(i)})) + |u^{(i)}|^{p_i-2}u^{(i)} + g(x,u^{(i)}(x)) = f(x),\ x \in \Omega, \\ - < \vartheta, \alpha_i(gradu^{(i)}) > \in \beta_x(u^{(i)}(x)),\ x \in \Gamma,\ i \in N, \end{cases} \tag{4.2}$$

where Ω is a bounded conical domain of a Euclidean space R^N $(N \geq 1)$ with its boundary $\Gamma \in C^1$, ϑ is the exterior normal derivative of Γ, $g : \Omega \times R^N \to R$ is a given function satisfying Carathéodory's conditions such that $u \in L^s(\Omega) \to g(x,u(x)) \in L^s(\Omega)$ is well-defined and there exists a function $T(x) \in L^s(\Omega)$ such that $g(x,t)t \geq 0$ for $|t| \geq T(x)$ and $x \in \Omega$ where $1 < s < +\infty$, β_x is the subdifferential of φ_x, where $\varphi_x = \varphi(x,\cdot) : R \to R$ for $x \in \Gamma$ and $\varphi : \Gamma \times R \to R$ is a given function. Moreover, β_x and φ_x are supposed to satisfy the following assumptions:

(A1) For each $x \in \Gamma$, $\varphi_x = \varphi(x,\cdot) : R \to R$ is a proper, convex and lower-semi-continuous function with $\varphi_x(0) = 0$.

(A2) $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Assume that $\alpha_i : R^N \to R^N$ is monotone and continuous and there exist positive constants k_1, k_2 and k_3 such that, for $\forall \xi, \xi' \in R^N$, the following conditions are satisfied, for each $i \in N$:

(A3) $|\alpha_i(\xi)| \leq k_1|\xi|^{p_i-1}$;

(A4) $|\alpha_i(\xi) - \alpha_i(\xi')| \leq k_2||\xi|^{p_i-2}\xi - |\xi'|^{p_i-2}\xi'|$;

(A5) $\ll \alpha_i(\xi), \xi \gg \geq k_3|\xi|^{p_i}$, where $\ll \cdot,\cdot \gg$ denotes the inner-product in R^N.

From [2], we can find the following results immediately.

Theorem 4.2. *For $1 < p_i \leq 2$ and $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, define $B_i : W^{1,p_i'}(\Omega) \to (W^{1,p_i'}(\Omega))^*$ by*

$$\langle w, B_i u\rangle = \int_\Omega \ll \alpha(grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})), grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'}) \gg dx,$$

$\forall u, v \in W^{1,p_i'}(\Omega)$. And, define $\widetilde{B_i}$ as the maximal monotone expansion of B_i from $W^{1,p_i'}(\Omega)$ to $L^{p_i'}(\Omega)$. Then $\widetilde{B_i} : L^{p_i'}(\Omega) \to L^{p_i}(\Omega)$ is maximal monotone, $\forall i \in N$.

Theorem 4.3. *For $1 < p_i \leq 2$ and $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, define $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ by $A_i u = \widetilde{B_i} J_i^{-1} u$, for $\forall u \in L^{p_i}(\Omega)$, where $J_i : L^{p_i'}(\Omega) \to L^{p_i}(\Omega)$ is the normalized duality mapping defined by $J_i u = |u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'}$. Then A_i is m-d-accretive, $\forall i \in N$.*

Theorem 4.4. *$A_i^{-1}0 = \{u(x) \in L^{p_i}(\Omega) : u(x) \equiv Const\}$, $\forall i \in N$.*

Theorem 4.5. *If $f(x) \in \bigcap_{i=1}^\infty L^{p_i}(\Omega)$, then (4.2) has a unique solution $u^{(i)}(x) \in L^{p_i}(\Omega)$, for $i \in N$.*

Theorem 4.6. *Consider a special case that $g(x, u^{(i)}(x)) = u^{(i)}(x) - |u^{(i)}(x)|^{p_i-2}u^{(i)}(x)$, where $i \in N$. If $f(x) \equiv k$, where k is a constant, then $A_i^{-1}0$ coincides with the set of solution of (4.2). In this case, if $p = \inf_{i\in N}\{p_i, p_i'\}$ and $E = L^p(\Omega)$, then it is not difficult to see that the iterative algorithms (2.2) and (3.1) can be applied to approximate zero point of m-d-accretive mappings A_i or solution of (4.2).*

4.2. **Parabolic systems involving p-Laplacian.**

Remark 4.7. Another example of the m-d-accretive mapping is presented in [19] which involves the following (p,q)-Laplacian parabolic systems:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - div[(C_1(x,t) + |gradu|^2)^{\frac{p-2}{2}} gradu] + \varepsilon_1 |u|^{r-2}u + g_1(x,u,gradu) \\ = f_1(x,t),\ (x,t) \in \Omega \times (0,T), \\ \frac{\partial v(x,t)}{\partial t} - div[(C_2(x,t) + |gradv|^2)^{\frac{q-2}{2}} gradv] + \varepsilon_2 |v|^{s-2}v + g_2(x,v,gradv) \\ = f_2(x,t),\ (x,t) \in \Omega \times (0,T), \\ - < \vartheta, (C_1(x,t) + |gradu|^2)^{\frac{p-2}{2}} gradu > \in \beta_x(u),\ (x,t) \in \Gamma \times (0,T), \\ - < \vartheta, (C_2(x,t) + |gradv|^2)^{\frac{q-2}{2}} gradv > \in \beta_x(v),\ (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), v(x,0) = v(x,T),\ x \in \Omega. \end{cases} \tag{4.3}$$

Under some assumptions, the connection between zero point problems of an m-d-accretive mapping and solutions of (4.3) is set up, which admits the iterative sequence derived from the iterative algorithms to approximate solutions of (4.3).

Now, we extend (4.3) to the following case to support our results.

$$\begin{cases} \frac{\partial u^{(i)}(x,t)}{\partial t} - div[(C_i(x,t) + |gradu^{(i)}|^2)^{\frac{p_i-2}{2}} \nabla u^{(i)}] \\ +\varepsilon_i |u^{(i)}|^{r_i-2}u + g_i(x,u^{(i)},gradu^{(i)}) = f_i(x,t),\ (x,t) \in \Omega \times (0,T), \\ - < \vartheta, (C_i(x,t) + |gradu^{(i)}|^2)^{\frac{p_i-2}{2}} gradu^{(i)} > \in \beta_x(u^{(i)}),\ (x,t) \in \Gamma \times (0,T), \\ u^{(i)}(x,0) = u^{(i)}(x,T),\ x \in \Omega,\ i \in N, \end{cases} \tag{4.4}$$

where Ω, Γ , ϑ and β_x are the same as those in (4.2). We also need the following assumptions: $\frac{2N}{N+1} < r_i \leq min\{p_i, p_i'\}$, ε_i is a non-negative constant, T is a positive constant, $0 \leq C_i(x,t) \in V_i = L^{p_i}(0,T;W^{1,p_i}(\Omega))$, $f_i(x,t) \in W_i = L^{max\{p_i,p_i'\}}(0,T;L^{max\{p_i,p_i'\}}(\Omega))$ are given functions, where $i \in N$. The function $g_i : \Omega \times R^{N+1} \to R$ $(i \in N)$ is a given function satisfying Carathéodory's conditions such that the following are also satisfied:

(A6) For $i \in N$, $(g_i(x, r_1, \cdots, r_{N+1}) - g_i(x, t_1, \cdots, t_{N+1})) \geq (r_1 - t_1)$, where $x \in \Omega$ and $(r_1, \cdots, r_{N+1})$, $(t_1, \cdots, t_{N+1}) \in R^{N+1}$.

(A7) For $i \in N$, $|g_i(x, r_1, \cdots, r_{N+1})|^{max\{p_i, p_i'\}} \leq |h_i(x,t)|^{p_i} + b_i|r_1|^{p_i},$, where $(r_1, \cdots, r_{N+1}) \in R^{N+1}$, $h_i(x,t) \in W_i$ and b_i is a positive constant.

From [19], we can obtain the following results.

Theorem 4.8. *Let $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $\forall i \in N$. Define the mapping $B_{p_i,r_i} : W^{1,p_i'}(\Omega) \to (W^{1,p_i'}(\Omega))^*$ by*

$$\langle w, B_{p_i,r_i} u \rangle = \int_\Omega \ll (C_i(x,t) + |grad(|u|^{p_i'-1} sgnu \|u\|_{p_i'}^{2-p_i'})|^2)^{\frac{p_i-2}{2}} grad(|u|^{p_i'-1} sgnu \|u\|_{p_i'}^{2-p_i'}), \\ grad(|w|^{p_i'-1} sgnw \|w\|_{p_i'}^{2-p_i'}) \gg dx,$$

for any $u, w \in W^{1,p_i'}(\Omega)$, where $i \in N$. Then B_{p_i,r_i} is maximal monotone.

Theorem 4.9. *Let $1 < p_i \leq 2$ and $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ for $i \in N$. The mapping $\widetilde{\widetilde{B_{p_i,r_i}}} : L^{p_i'}(0,T; W^{1,p_i'}(\Omega)) \to (L^{p_i'}(0,T; W^{1,p_i'}(\Omega)))^*$ defined by*

$$\langle w, \widetilde{\widetilde{B_{p_i,r_i}}} u \rangle = \int_0^T \langle w, B_{p_i,r_i} u \rangle dt,$$

for $u, w \in L^{p_i'}(0,T; W^{1,p_i'}(\Omega))$ is maximal monotone.

Theorem 4.10. *Define $S_i : D(S_i) = \{u(x,t) \in L^{p_i'}(0,T; W^{1,p_i'}(\Omega)) : \frac{\partial u}{\partial t} \in (L^{p_i'}(0,T; W^{1,p_i'}(\Omega)))^*, u(x,0) = u(x,T)\} \to (L^{p_i'}(0,T; W^{1,p_i'}(\Omega)))^*$ by $S_i u = \frac{\partial u}{\partial t}$, for $u \in D(S_i)$. Then S_i is linear maximal monotone, $\forall i \in N$.*

Theorem 4.11. *Let $1 < p_i \leq 2$ and $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ for $i \in N$. The mapping $U_i : D(U_i) \subset L^{p_i'}(0,T; W^{1,p_i'}(\Omega)) \to (L^{p_i'}(0,T; W^{1,p_i'}(\Omega)))^*$ defined by*

$$U_i w = \widetilde{\widetilde{B_{p_i,r_i}}} w + S_i w,$$

for $w \in D(U_i)$ is maximal monotone where $i \in N$.

Theorem 4.12. *For $1 < p_i \leq 2$ and $i \in N$, $A_i : L^{p_i}(0,T; W^{1,p_i}(\Omega)) \to L^{p_i}(0,T; W^{1,p_i}(\Omega))$ defined by $A_i u = \widetilde{U}_i J_i^{-1} u$, is m-d-accretive, where $u \in L^{p_i}(0,T; W^{1,p_i}(\Omega))$, $J_i^{-1} : L^{p_i}(0,T; W^{1,p_i}(\Omega)) \to L^{p_i'}(0,T; W^{1,p_i'}(\Omega))$ is the normalized duality mapping and $\widetilde{U}_i$ is the maximal monotone extension of U_i from $L^{p_i'}(0,T; W^{1,p_i'}(\Omega))$ to $L^{p_i}(0,T; W^{1,p_i}(\Omega))$.*

Theorem 4.13. *For $i \in N$, $A_i^{-1} 0 = \{u \in L^{p_i}(0,T; W^{1,p_i}(\Omega)) : u(x,t) \equiv Const\}$.*

Theorem 4.14. *For a special case that $g_i(x, u^{(i)}(x), gradu^{(i)}(x)) = u^{(i)}(x)$, $\varepsilon_i \equiv 0$ and $\beta_x \equiv 0$, if $f_i(x,t) \equiv k$, where k is a constant, then $A_i^{-1} 0$ coincides with the set of solution of (4.4), where $i \in N$. In this case, if we set $p = inf_{i \in N}\{p_i, p_i'\}$ and $E = L^p(\Omega)$, then it is not difficult to see that the iterative algorithms (2.2) and (3.1) can be applied to approximate zero point of m-d accretive mappings A_i or solution of (4.4).*

4.3. **Curvature systems.** Using similar methods as those in Sections 4.1 and 4.2, we can discuss an example of curvature systems with the Neumann boundary value in our paper

$$\begin{cases} -div[(1+|gradu^{(i)}|^2)^{\frac{s_i}{2}}|gradu^{(i)}|^{m_i-1}gradu^{(i)}]+\varepsilon|u^{(i)}|^{r_i-2}u^{(i)}+u^{(i)}(x)=f(x),\ x\in\Omega, \\ -<\vartheta,(1+|gradu^{(i)}|^2)^{\frac{s_i}{2}}|gradu^{(i)}|^{m_i-1}gradu^{(i)}>\in\beta_x(u^{(i)}(x)),\ x\in\Gamma,\ i\in N. \end{cases} \tag{4.5}$$

where Ω , $\Gamma\in C^1$, ϑ and β_x satisfy the same assumptions of (4.2), ε is a non-negative constant, $m_i+s_i+1=q_i$, $m_i\geq 0$ and $\frac{2N}{N+1}<q_i\leq 2$. If $q_i\geq N$, we suppose $1\leq r_i<+\infty$, and if $q_i<N$, we suppose $1\leq r_i\leq\frac{Nq_i}{N-q_i}$, $\forall i\in N$.

Theorem 4.15. *For $i\in N$, define $B_i:W^{1,q_i'}(\Omega)\to(W^{1,q_i'}(\Omega))^*$ by*

$$\begin{aligned}&\langle w,B_iu\rangle\\ &=\int_\Omega\ll(1+|grad(|u|^{q_i'-1}sgnu\|u\|_{q_i'}^{2-q_i'})|^2)^{\frac{s_i}{2}}|grad(|u|^{q_i'-1}sgnu\|u\|_{q_i'}^{2-q_i'})|^{m_i-1}grad(|u|^{q_i'-1}sgnu\|u\|_{q_i'}^{2-q_i'}),\\ &\quad grad(|v|^{q_i'-1}sgnv\|v\|_{q_i'}^{2-q_i'})\gg dx,\end{aligned}$$

for $\forall u,v\in W^{1,q_i'}(\Omega)$. And, define $\widehat{B}_i$ as the maximal monotone expansion of B_i from $W^{1,q_i'}(\Omega)$ to $L^{q_i'}(\Omega)$. Then $\widehat{B}_i:L^{q_i'}(\Omega)\to L^{q_i}(\Omega)$ is maximal monotone, $\forall i\in N$.

Theorem 4.16. *For $i\in N$, define $A_i:L^{q_i}(\Omega)\to L^{q_i}(\Omega)$ by $A_iu=\widehat{B}_iJ_i^{-1}u$, $\forall u\in L^{q_i}(\Omega)$, where $J_i:L^{q_i'}(\Omega)\to L^{q_i}(\Omega)$ is the normalized duality mapping. Then A_i is m-d-accretive, $\forall i\in N$.*

Theorem 4.17. $A_i^{-1}0=\{u(x)\in L^{q_i}(\Omega):u(x)\equiv Const\}$*, $\forall i\in N$.*

Theorem 4.18. *Consider the special case that $\beta_x\equiv 0$ and $\varepsilon\equiv 0$, if $f(x)\equiv k$, where k is a constant, then $A_i^{-1}0$ coincides with the set of solution of (4.5). In this case, if $q=\inf_{i\in N}\{q_i,q_i'\}$ and $E=L^q(\Omega)$, then it is not difficult to see that iterative algorithms (2.2) and (3.1) can be applied to approximate zero points of m-d-accretive mappings A_i or solutions of (4.2).*

Acknowledgements

The first and the third authors were supported by Natural Science Foundation of Hebei Province under Grant No. A2014207010, Key Project of Science and Research of Hebei Educational Department under Grant No. ZD2016024, Key Project of Science and Research of Hebei University of Economics and Business under Grant No. 2018ZD06, Youth Project of Science and Research of Hebei University of Economics and Business under Grant No. 2017KYQ09 and Youth Project of Science and Research of Hebei Educational Department under Grant No. QN2017328.

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