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POPOV SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we study a modified subgradient extragradient iterative algorithm for the approximation of solutions of pseudomonotone equilibrium problems in the framework of 2-uniformly convex Banach spaces which are also uniformly smooth. Our algorithm is based on two popular iterative methods: the Popov extragradient algorithm and the subgradient extragradient algorithm. We further state and prove a weak convergence result under some mild conditions.

Keywords. Popov subgradient extragradient iterative algorithm; Pseudomonotone equilibrium problem; 2-uniformly convex Banach space; Uniformly smooth Banach space.

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1. INTRODUCTION

Let C be a nonempty closed and convex subset of a real Banach space E. We denote by E^* the dual of E. Let $f : E \times E \to \mathbb{R}$ be a bifunction. The Equilibrium Problem (EP) for bifunction f on set C is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the solution set of EP (1.1) by $EP(f,C)$. The EP (1.1) was first introduced by Blum and Oettli [1] and it is well known that EP (1.1) covers many important mathematical models, such as, nonlinear optimization problems, variational inequality problems, nonlinear complementary problems and fixed point problems [1, 2, 3, 4]. The EP (1.1) also generalizes the convex minimization problems which is of great importance in almost all branches of pure and applied sciences because it can be applied in solving many practical problems that arise in areas such as economics, transportation and engineering (see [1, 5] and some of the references therein). In recognition of its importance and numerous applications, EP (1.1) have been studied by many authors (see, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]) and many iterative

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methods, such as, the Proximal Point Method (PPM) [16], auxiliary problem principle method [17] and gap function methods [18] etc, have been developed to solve EP (1.1).

Tran *et al.* [14] in 2008 introduced the following proximal-like iterative method also referred as the Extragradient Methods (EGM) to solve EP (1.1) in a real Hilbert space under the assumptions that the equilibrium bifunction is pseudomonotone and satisfies the Lipschitz-type condition. For $x_0 \in C$, let $\{x_n\}$ and $\{y_n\}$ be two sequences generated as follows:

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda f(x_n,y) + \frac{1}{2}||x_n - y||^2\}, \\ x_{n+1} = \arg\min_{y\in C}\{\lambda f(y_n,y) + \frac{1}{2}||x_n - y||^2\}, \end{cases} \tag{1.2}$$

where $\lambda > 0$ is a suitable parameter. The EGMs have attracted much attention; see, for example, [8, 9, 11, 13, 15, 19]. The advantages of the EGM over the PPM are that it is numerically more easier to compute and it can be used to solve EPs involving a more general class of (pseudomonotone) bifunctions.

The EGM has its own setback as it is required at each step of the iteration to solve two optimization problems on feasible set C and to compute two values of bifunction f at two points x_n and y_n. These can be very costly and can also affect the efficiency of the method if the bifunction and the feasible set have complex structures. To overcome these drawbacks in EGM, Hieu [20] introduced a Modified Extragradient Method (MEGM) in Hilbert spaces for approximating the solution of EP (1.1) involving pseudomonotone bifunctions and obtained a weak convergence result. The feasible set C in the first optimization program in MEGM is replaced by a half space T_n and therefore can be solved effectively by using methods of convex quadratic programming ([21], Chapter 8). Also, unlike the EGM, the MEGM only requires to compute a value of the bifunction f at current approximation y_n.

The purpose of this paper is to introduce an iterative algorithm, which is suitable for finding an element of $EP(f,C)$ for a pseudomonotone bifunction that satisfies the Lipschitz type condition, and obtain a weak convergence result in the framework of 2-uniformly convex Banach spaces which are also uniformly smooth. The results presented in this paper extend the results of Hieu [20] from Hilbert spaces to 2-uniformly convex Banach spaces which are also uniformly smooth.

2. Preliminaries

Let $B_E = \{x \in E : ||x|| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in B_E$ and $x \neq y$, we have $\frac{||x+y||}{2} < 1$. E is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in B_E$, $||x-y|| \geq \varepsilon$ implies $\frac{||x+y||}{2} \leq 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : x,y \in B_E; \varepsilon = ||x-y||\right\}.$$

E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$ and p-uniformly convex if there is a $C_p > 0$ so that $\delta_E(\varepsilon) \geq C_p\varepsilon^p$ for any $\varepsilon \in (0,2]$. Clearly, every p-uniformly convex Banach space is uniformly convex; see, e.g., [22, 23] for more details.

A Banach space E is said to be smooth if the limit $\lim\limits_{t\to 0}\dfrac{||x+ty|| - ||x||}{t}$ exists for all $x, y \in B_E$. Moreover, E is said to be uniformly smooth if the limit is attained uniformly for $x, y \in B_E$. It is well known that Hilbert and the Lebesgue $L_p (1 < p \leq 2)$ spaces are 2-uniformly convex and uniformly smooth.

The normalised duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \forall x \in E.$$

The normalised duality mapping J possesses the following properties [24]:

(1) If E is a smooth Banach space, then J is single-valued.
(2) If E is a strictly convex Banach space, then J is one-to-one and strictly monotone.
(3) If E is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on each bounded subset of E.
(4) If E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Let E be a reflexive, strictly convex and smooth Banach space and let J the normalised duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and is the duality mapping from E^* into E.

Let E be a smooth Banach space, Alber [25] introduced the following Lyapunov functional

$$\phi(x,y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2. \tag{2.1}$$

It can be seen from the definition that ϕ satisfies the following conditions.

A1. $(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2$.
A2. $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, J(z) - J(y) \rangle$.
A3. $\phi(x,y) = \langle x, J(x) - J(y) \rangle + \langle y - x, J(y) \rangle \leq ||x|| ||J(x) - J(y)|| + ||y - x|| ||y||$.

If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y,x) = 0$ if and only if $x = y$ (see Remark 2.1 in [26]).

Lemma 2.1. [26] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$, and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. Then, for each $x \in E$ (see Alber [25]), there exists a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$. The mapping $\Pi_C : E \to C$, defined by $\Pi_C(x) = x_0$, is called the generalized projection operator from E onto C and x_0 is called the generalized projection of x. In a Hilbert space, $\Pi_C = P_C$ (the metric projection operator).

Lemma 2.2. [26, 27] *Let C be a nonempty closed and convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C(x)$ if and only if $\langle x_0 - y, J(x) - J(x_0) \rangle \geq 0, \forall y \in C$.*

Lemma 2.3. [26, 27] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x), \forall y \in C$.*

Lemma 2.4. [28] *Let E be a 2-uniformly convex and smooth Banach space. Then, for every $x, y \in E$, $\phi(x,y) \geq c||x - y||^2$, where $c > 0$ is the 2-uniformly convexity constant of E.*

Next we give an Opial-like inequality for the Lyapunov functional (see [29], Lemma 1). This Opial-like inequality was proved for the more general Bregman distance (see [30], Lemma 5.1 and [31], Lemma 3). For completeness, we state and prove it for our setting here.

Lemma 2.5. *Let E be a smooth Banach space with a weakly sequentially continuous normalised duality mapping J from E into E^*. Let $\{x_n\}$ be a sequence in E such that $x_n \to u$ weakly for some $u \in E$. Then*

$$\liminf_{n\to\infty} \phi(u,x_n) < \liminf_{n\to\infty} \phi(v,x_n) \ \forall v \in E \text{ with } v \neq u.$$

Proof.

$$\begin{aligned} \phi(u,x_n) - \phi(v,x_n) &= ||u||^2 - 2\langle u, J(x_n)\rangle + ||x_n||^2 - [||v||^2 - 2\langle v, J(x_n)\rangle + ||x_n||^2] \\ &= ||u||^2 - 2\langle u - v, J(x_n)\rangle - ||v||^2 \\ &= ||u||^2 - ||v||^2 + 2\langle v-u, J(u)\rangle - 2\langle v-u, J(u)\rangle - 2\langle u-v, J(x_n)\rangle \\ &= -\phi(v,u) + \langle u - v, J(u) - J(x_n)\rangle. \end{aligned}$$

Since $x_n \to u$ weakly and J is weakly sequentially continuous, we have

$$\lim_{n\to\infty}[\phi(u,x_n) - \phi(v,x_n)] = -\phi(v,u).$$

Consequently,

$$\begin{aligned} \liminf_{n\to\infty} \phi(u,x_n) &= \liminf_{n\to\infty}[\phi(u,x_n) - \phi(v,x_n) + \phi(v,x_n)] \\ &= \lim_{n\to\infty}(\phi(u,x_n) - \phi(v,x_n)) + \liminf_{n\to\infty} \phi(v,x_n) \\ &= -\phi(v,u) + \liminf_{n\to\infty} \phi(v,x_n). \end{aligned}$$

Since $\phi(v,u) > 0,$ for $v \neq u$, we obtain

$$\liminf_{n\to\infty} \phi(u,x_n) < \liminf_{n\to\infty} \phi(v,x_n).$$

□

The normal cone N_C to a set C at a point $x \in C$ is defined by

$$N_C(x) := \{x^* \in E^* : \langle x - y, x^*\rangle \geq 0, \ \forall y \in C\}.$$

Let $g : C \to \mathbb{R}$ be a function. The subdifferential of g at x is defined by

$$\partial g(x) = \{w \in E^* : g(y) - g(x) \geq \langle w, y - x\rangle, \ \forall y \in C\}.$$

Lemma 2.6. *Let C be a nonempty convex subset of a Banach space E and $g : C \to \mathbb{R} \cup \{+\infty\}$ be a convex, subdifferentiable and lower semicontinuous function. Furthermore, the function g satisfies the following regularity condition*

$$\textit{Either } int(C) \neq \emptyset \textit{ or } g \textit{ is continuous at a point in } C.$$

Then, x^* is a solution to the following convex optimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone to C at x^*.

3. Proposed Method

Let E be a 2-uniformly convex Banach space which is uniformly smooth and let C be a nonempty closed and convex subset of E. A bifunction $f : E \times E \to \mathbb{R}$ is said to be:

i. strongly monotone on C if there exists a constant $\gamma > 0$ such that
$$f(x,y) + f(y,x) \leq -\gamma ||x-y||^2, \ \forall x,y \in C;$$

ii. monotone on C if
$$f(x,y) + f(y,x) \leq 0, \ \forall x,y \in C;$$

iii. pseudomonotone on C if
$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

It is easy to see that i $\Rightarrow$ ii $\Rightarrow$ iii. A bifunction $f : E \times E \to \mathbb{R}$ is said to satisfy Lipschitz-type condition if there exists two constants L_1 and L_2 such that
$$f(x,y) + f(y,z) \geq f(x,z) - L_1||x-y||^2 - L_2||y-z||^2, \ \forall x,y,z \in E.$$

We now give the following modified subgradient extragradient iterative algorithm for the approximation of solutions of pseudomonotone equilibrium problems in the framework of 2-uniformly convex Banach spaces which are uniformly smooth. First, let us state the following conditions that will be required in the analysis of our convergence result for solving EP (1.1).

Condition B:
(B1) f is pseudomonotone on C and $f(x,x) = 0$, for all $x \in C$;
(B2) f satisfies Lipschitz-type condition on E with Lipschitz-type constants L_1 and L_2;
(B3) $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C for each fixed point $y \in C$, i.e., if $\{x_n\} \subset C$ is a sequence converging weakly to $x \in C$, then $\limsup_{n\to\infty} f(x_n, y) \leq f(x,y)$;
(B4) $f(x, \cdot)$ is convex and subdifferentiable on E for every fixed $x \in C$;
(B5) $0 < \lambda < \frac{c}{2L_2+4L_1}$, where c is the 2-uniformly convexity constant of E and L_1 and L_2 are the two Lipschitz-type constants of f;
(B6) The solution set $EP(f,C)$ of EP (1.1) is nonempty.

It has been proved that under the conditions (B1)-(B4), the solution set $EP(f,C)$ of EP (1.1) is closed and convex [14].

Algorithm 3.1. Initialization: Choose $x_0 \in E$, $y_0 \in C$, a control parameter $\lambda > 0$, and compute
$$x_1 = \arg\min_{y\in C}\{\lambda f(y_0,y) + \frac{1}{2}\phi(y,x_0)\},$$
$$y_1 = \arg\min_{y\in C}\{\lambda f(y_0,y) + \frac{1}{2}\phi(y,x_1)\}$$

Iterative step for $n \geq 1$.
Step 1. Select $w_n \in \partial_2 f(y_{n-1}, y_n) = \partial f(y_{n-1}, \cdot)(y_n)$ and construct a half space
$$T_n = \{z \in E : \langle Jx_n - \lambda w_n - Jy_n, z - y_n\rangle \leq 0\}.$$

Step 2. Solve two strongly convex optimization programs
$$\begin{cases} x_{n+1} = \arg\min_{y\in T_n}\{\lambda f(y_n,y) + \frac{1}{2}\phi(y,x_n)\}, \\ y_{n+1} = \arg\min_{y\in C}\{\lambda f(y_n,y) + \frac{1}{2}\phi(y,x_{n+1})\}. \end{cases} \tag{3.1}$$

Stopping criterion: If $y_{n+1} = y_n = x_{n+1}$, stop. Otherwise, Set $n := n+1$ and go to Step 1.

4. Main results

Now, we are ready to give our main results.

Lemma 4.1. *Let $\{x_n\}$ and $\{y_n\}$ be the two sequences generated by Algorithm 3.1.*

i. $\lambda(f(y_n,y) - f(y_n,y_{n+1})) \geq \langle Jx_{n+1} - Jy_{n+1}, y - y_{n+1}\rangle$ for all $y \in C$ and $n \geq 0$.

ii. If $y_{n+1} = y_n = x_{n+1}$, then $x_{n+1} \in EP(f,C)$.

iii. For all $x^* \in EP(f,C)$, the following estimate holds

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - (1 - \frac{2\lambda L_2}{c})\phi(x_{n+1}, y_n)\\ &\quad -(1 - \frac{4\lambda L_1}{c})\phi(y_n, x_n) + \frac{4\lambda L_1}{c}\phi(x_n, y_{n-1}). \end{aligned} \tag{4.1}$$

Proof. i. From the definition of y_{n+1} and Lemma 2.6, we have that

$$0 \in \partial_2 \lambda f(y_n, y_{n+1}) + \frac{1}{2}\partial_1 \phi(y_{n+1}, x_n) + N_C(y_{n+1}).$$

Therefore, there exists $w \in \partial_2 f(y_n, y_{n+1})$ and $\bar{w} \in N_C(y_{n+1})$ such that

$$\lambda w + Jy_{n+1} - Jx_{n+1} + \bar{w} = 0.$$

Thus, we have from the definition of N_C that

$$\begin{aligned}\langle Jx_{n+1} - Jy_{n+1}, y - y_{n+1}\rangle &= \lambda\langle w, y - y_{n+1}\rangle + \langle \hat{w}, y - y_{n+1}\rangle\\ &\leq \lambda\langle w, y - y_{n+1}\rangle,\ \forall y \in C. \end{aligned} \tag{4.2}$$

Again, since $w \in \partial_2 f(y_n, y_{n+1})$, we have

$$\langle w, y - y_{n+1}\rangle \leq f(y_n, y) - f(y_n, y_{n+1}),\ \forall y \in C. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\lambda(f(y_n,y) - f(y_n,y_{n+1})) \geq \langle Jx_{n+1} - Jy_{n+1}, y - y_{n+1}\rangle,\ \forall y \in C. \tag{4.4}$$

ii. If $y_{n+1} = y_n = x_{n+1}$, then from (4.4), Condition (A1) and $\lambda > 0$, we obtain $f(y_n, y) \geq 0$, for all $y \in C$. Thus $x_{n+1} = y_n \in EP(f,C)$.

iii. From $x_{n+1} \in T_n$ and the definition of T_n, we have

$$\langle Jx_n - \lambda w_n - Jy_n, x_{n+1} - y_n\rangle \leq 0.$$

Hence

$$\langle Jx_n - Jy_n, x_{n+1} - y_n\rangle \leq \lambda\langle w_n, x_{n+1} - y_n\rangle. \tag{4.5}$$

Moreover, since $w_n \in \partial_2 f(y_{n-1}, y_n)$, we have

$$\langle w_n, y - y_n\rangle \leq f(y_{n-1}, y) - f(y_{n-1}, y_n),\ \forall y \in E \tag{4.6}$$

Put $y = x_{n+1}$. From (4.5) and (4.6), we obtain

$$\langle Jx_n - Jy_n, x_{n+1} - y_n\rangle \leq \lambda(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)). \tag{4.7}$$

From the definition of x_{n+1} and by similar arguments as in the proof of (i), we have

$$\lambda(f(y_n,y)-f(y_n,x_{n+1}))\geq\langle Jx_n-Jx_{n+1},y-x_{n+1}\rangle,\ \forall y\in T_n.$$

Now, letting $y=x^*$, we have

$$\lambda(f(y_n,x^*)-f(y_n,x_{n+1}))\geq\langle Jx_n-Jx_{n+1},x^*-x_{n+1}\rangle. \tag{4.8}$$

Since $x^*\in EP(f,C)$, $f(x^*,y_n)\geq 0$. Therefore, from the pseudomonotonicity of f, we have $f(y_n,x^*)\leq 0$. Hence, it follows from relation (4.8) that

$$-\lambda f(y_n,x_{n+1})\geq\langle Jx_n-Jx_{n+1},x^*-x_{n+1}\rangle.$$

Thus

$$\langle Jx_n-Jx_{n+1},x_{n+1}-x^*\rangle\geq\lambda f(y_n,x_{n+1}). \tag{4.9}$$

Since f is Lipschitz-type continuous, we have

$$\begin{aligned} f(y_n,x_{n+1}) &\geq (f(y_{n-1},x_{n+1})-f(y_{n-1},y_n))\\ &\quad -L_1||y_{n-1}-y_n||^2-L_2||y_n-x_{n+1}||^2. \end{aligned} \tag{4.10}$$

But from the triangular inequality, we have

$$\begin{aligned} ||y_{n-1}-y_n||^2 &\leq (||y_{n-1}-x_n||+||x_n-y_n||)^2\\ &\leq 2(||y_{n-1}-x_n||^2+||x_n-y_n||^2). \end{aligned} \tag{4.11}$$

We now have from (4.10), (4.11) and Lemma 2.4 that

$$\begin{aligned} f(y_n,x_{n+1}) &\geq (f(y_{n-1},x_{n+1})-f(y_{n-1},y_n))-2L_1||y_{n-1}-x_n||^2\\ &\quad -2L_1||x_n-y_n||-L_2||y_n-x_{n+1}||^2\\ &\geq (f(y_{n-1},x_{n+1})-f(y_{n-1},y_n)-\frac{2L_1}{c}\phi(x_n,y_{n-1})\\ &\quad -\frac{2L_1}{c}\phi(y_n,x_n)-\frac{L_2}{c}\phi(x_{n+1},y_n). \end{aligned} \tag{4.12}$$

Therefore, from (4.7), (4.8), (4.9) and (4.12), we have

$$\begin{aligned} \langle Jx_n-Jx_{n+1},x_{n+1}-x^*\rangle &\geq \langle Jx_n-Jy_n,x_{n+1}-y_n\rangle-\frac{2\lambda L_1}{c}\phi(x_n,y_{n-1})\\ &\quad -\frac{2\lambda L_1}{c}\phi(y_n,x_n)-\frac{\lambda L_2}{c}\phi(x_{n+1},y_n). \end{aligned} \tag{4.13}$$

Observe that

$$\langle Jx_n-Jx_{n+1},x_{n+1}-x^*\rangle=\frac{1}{2}[\phi(x^*,x_n)-\phi(x^*,x_{n+1})-\phi(x_{n+1},x_n)] \tag{4.14}$$

and

$$\begin{aligned} \langle Jx_n-Jy_n,x_{n+1}-y_n\rangle &= -\langle Jx_n-Jy_n,y_n-x_{n+1}\rangle\\ &= -\frac{1}{2}[\phi(x_{n+1},x_n)-\phi(x_{n+1},y_n)-\phi(y_n,x_n)]\\ &= \frac{1}{2}[\phi(x_{n+1},y_n)+\phi(y_n,x_n)-\phi(x_{n+1},x_n)]. \end{aligned} \tag{4.15}$$

Thus, from (4.13), (4.14) and (4.15), we get

$$\begin{aligned}\phi(x^*,x_n)-\phi(x^*,x_{n+1}) \quad &\geq \quad \phi(x_{n+1},y_n)+\phi(y_n,x_n)-\frac{2\lambda L_1}{c}\phi(x_n,y_{n-1})\\ &\quad -\frac{2\lambda L_1}{c}\phi(y_n,x_n)-\frac{\lambda L_2}{c}\phi(x_{n+1},y_n),\end{aligned} \tag{4.16}$$

which implies

$$\begin{aligned}\phi(x^*,x_{n+1}) \quad &\leq \quad \phi(x^*,x_n)-(1-\frac{2\lambda L_2}{c})\phi(x_{n+1},y_n)\\ &\quad -(1-\frac{4\lambda L_1}{c})\phi(y_n,x_n)+\frac{4\lambda L_1}{c}\phi(x_n,y_{n-1}).\end{aligned} \tag{4.17}$$

□

Theorem 4.2. *Let C be a nonempty closed and convex subset of E and $f : E \times E \to \mathbb{R}$ be a bifunction such that Conditions (A1)-(A6) are satisfied. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 converge weakly to some $p \in EP(f,C)$. Moreover, $p = \lim_{n\to\infty} \Pi_{EP(f,C)}(x_n)$.*

Proof. First, we show that $\{x_n\}$ is bounded.
Summing up inequality (4.17) for every $N \geq 1$, we obtain

$$\begin{aligned}\phi(x^*,x_{N+1}) \quad &\leq \quad \phi(x^*,x_0)+\frac{4\lambda L_1}{c}\phi(x_1,y_0)\\ &\quad -\sum_{n=1}^{N}(1-\frac{2\lambda L_2}{c}-\frac{4\lambda L_1}{c})\phi(x_{n+1},y_n)\\ &\quad -\sum_{n=1}^{N}(1-\frac{4\lambda L_1}{c})\phi(y_n,x_n).\end{aligned} \tag{4.18}$$

From (B5), we have that

$$1-\frac{2\lambda L_2}{c}-\frac{4\lambda L_1}{c}>0$$

and consequently $1-\frac{4\lambda L_1}{c}>0$. This together with inequality (4.18) gives that $\{\phi(x^*,x_{N+1})\}$ is bounded for all $N \geq 1$. Thus, $\{x_n\}$ is also bounded. Moreover, it follows from inequality (4.18) that

$$\sum_{n=1}^{N}(1-\frac{2\lambda L_2}{c}-\frac{4\lambda L_1}{c})\phi(x_{n+1},y_n)<\infty \tag{4.19}$$

and

$$\sum_{n=1}^{N}(1-\frac{4\lambda L_1}{c})\phi(y_n,x_n)<\infty. \tag{4.20}$$

Therefore, we have from (B5) that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=\lim_{n\to\infty}\phi(y_n,x_n)=0,$$

which implies that

$$\lim_{n\to\infty}||x_{n+1}-y_n||=\lim_{n\to\infty}||y_n-x_n||=0. \tag{4.21}$$

Now from the triangular inequality, we have

$$||x_{n+1}-x_n|| \leq ||x_{n+1}-y_n||+||y_n-x_n|| \to 0, n\to\infty. \tag{4.22}$$

Also,

$$||y_{n-1}-y_n|| \leq ||y_{n-1}-x_n|| + ||x_n-y_n|| \to 0, n\to\infty. \tag{4.23}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Also, since $||x_{n_i} - y_{n_i}|| \to 0$, we have $y_{n_i} \rightharpoonup p$. Since C is a closed and convex subset of E, we have that C is weakly closed. Hence, it follows from $\{y_{n_i}\} \subset C$ that $p \in C$. Also, from (4.4), with $n = n_i$ and the Lipschitz-type continuity of f, we have

$$\begin{aligned}\lambda f(y_{n_i},y) &\geq \lambda f(y_{n_i},y_{n_i+1}) + \langle Jx_{n_i+1} - Jy_{n_i+1}, y-y_{n_i+1}\rangle \\ &\geq \lambda(f(y_{n_i-1},y_{n_i+1}) - f(y_{n_i-1},y_{n_i})) - \lambda L_1||y_{n_i-1}-y_{n_i}||^2 \\ &\quad -\lambda L_2||y_{n_i}-y_{n_i+1}||^2 + \langle Jx_{n_i+1} - Jy_{n_i+1}, y-y_{n_i+1}\rangle. \end{aligned} \tag{4.24}$$

Again from (4.4) with $n = n_i - 1$, we obtain

$$\lambda(f(y_{n_i-1},y) - f(y_{n_i-1},y_{n_i}) \geq \langle Jx_{n_i} - Jy_{n_i}, y-y_{n_i}\rangle, \ \forall y \in C.$$

This with $y = y_{n_i+1}$ leads to

$$\lambda(f(y_{n_i-1},y) - f(y_{n_i-1},y_{n_i}) \geq \langle Jx_{n_i} - Jy_{n_i}, y_{n_i+1}-y_{n_i}\rangle. \tag{4.25}$$

Combining relations (4.24) and (4.25), we have

$$\begin{aligned}\lambda f(y_{n_i},y) &\geq \langle Jx_{n_i} - Jy_{n_i}, y_{n_i+1}-y_{n_i}\rangle \\ &\quad -\lambda L_1||y_{n_i-1}-y_{n_i}||^2 - \lambda L_2||y_{n_i}-y_{n_i+1}||^2 \\ &\quad +\langle Jx_{n_i+1} - Jy_{n_i+1}, y-y_{n_i+1}\rangle, \ \forall y \in C. \end{aligned} \tag{4.26}$$

Passing to the limit in (4.26) as $n_i \to \infty$ and using hypothesis (B3), (4.21) and the uniformly norm-to-norm continuity of the normalised duality mapping J, we have $f(p,y) \geq 0$ for all $y \in C$ or $p \in EP(f,C)$.

Next, we show that $x_n \rightharpoonup p$. Suppose for contradiction that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ and $q \neq p$. From (4.1) and $1 - \frac{2\lambda L_2}{c} - \frac{4\lambda L_1}{c} > 0$, we have

$$\phi(x^*,x_{n+1}) + \frac{4\lambda L_1}{c}\phi(x_{n+1},y_n) \leq \phi(x^*,x_n) + \frac{4\lambda L_1}{c}\phi(x_n,y_{n-1}). \tag{4.27}$$

Therefore, for all $x^* \in EP(f,C)$, there is

$$\lim_{n\to\infty}(\phi(x^*,x_n) + \frac{4\lambda L_1}{c}\phi(x_n,y_{n-1})) \in \mathbb{R}.$$

Applying Lemma 2.5 twice, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\Big(\phi(p,x_n)+\frac{4\lambda L_1}{c}\phi(x_n,y_{n-1})\Big) &= \lim_{i\to\infty}\Big(\phi(p,x_{n_i})+\frac{4\lambda L_1}{c}\phi(x_{n_i},y_{n_i-1})\Big)\\
&= \liminf_{i\to\infty}\phi(p,x_{n_i}) < \liminf_{i\to\infty}\phi(q,x_{n_i})\\
&= \lim_{i\to\infty}\Big(\phi(q,x_{n_i})+\frac{4\lambda L_1}{c}\phi(x_{n_i},y_{n_i-1})\Big)\\
&= \lim_{n\to\infty}\Big(\phi(q,x_n)+\frac{4\lambda L_1}{c}\phi(x_n,y_{n-1})\Big)\\
&= \lim_{j\to\infty}\Big(\phi(q,x_{n_j})+\frac{4\lambda L_1}{c}\phi(x_{n_j},y_{n_j-1})\Big)\\
&= \liminf_{j\to\infty}(\phi(q,x_{n_j}) < \liminf_{j\to\infty}\phi(p,x_{n_j})\\
&= \lim_{j\to\infty}\Big(\phi(p,x_{n_j})+\frac{4\lambda L_1}{c}\phi(x_{n_j},y_{n_j-1})\Big)\\
&= \lim_{n\to\infty}\Big(\phi(p,x_n)+\frac{4\lambda L_1}{c}\phi(x_n,y_{n-1})\Big).
\end{aligned}$$

This is an absurdity. Thus, we conclude that $p=q$.

We proceed to show that

$$p=\lim_{n\to\infty}\Pi_{EP(f,C)}x_n. \tag{4.28}$$

Clearly,

$$\begin{aligned}
\phi(\Pi_{EP(f,C)}x_{n+1},x_{n+1}) &\leq \phi(\Pi_{EP(f,C)}x_n,x_{n+1})\\
&\leq \phi(\Pi_{EP(f,C)}x_n,x_n)+\frac{4\lambda L_1}{c}\phi(x_n,y_{n-1}).
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\phi(x_n,y_{n-1})<\infty$, we have

$$\lim_{n\to\infty}\phi(\Pi_{EP(f,C)}x_n,x_n)\in\mathbb{R}.$$

Thus it follows from Lemma 2.3 that

$$\begin{aligned}
\phi(\Pi_{EP(f,C)}x_n,\Pi_{EP(f,C)}x_m) &\leq \phi(\Pi_{EP(f,C)}x_n,x_m)-\phi(\Pi_{EP(f,C)}x_m,x_m)\\
&\leq \phi(\Pi_{EP(f,C)}x_n,x_{m-1})-\phi(\Pi_{EP(f,C)}x_m,x_m)\\
&\quad +\frac{4\lambda L_1}{c}\phi(x_{m-1},y_{m-2})\\
&\quad \vdots\\
&\leq \phi(\Pi_{EP(f,C)}x_n,x_n)-\phi(\Pi_{EP(f,C)}x_m,x_m)\\
&\quad +\frac{4\lambda L_1}{c}\sum_{k=n}^{m}\phi(x_{k-1},y_{k-2}),\ m>n.
\end{aligned} \tag{4.29}$$

Passing to the limit in (4.29) as $m,n\to\infty$, we obtain

$$\lim_{n,m\to\infty}\phi(\Pi_{EP(f,C)}x_n,\Pi_{EP(f,C)}x_m)=0,$$

which implies that

$$\lim_{n,m\to\infty}||\Pi_{EP(f,C)}x_n-\Pi_{EP(f,C)}x_m||=0.$$

Thus $\{\Pi_{EP(f,C)}x_n\}$ is a Cauchy sequence. Hence there exists the limit

$$\lim_{n\to\infty}\Pi_{EP(f,C)}x_n = z \in EP(f,C).$$

From Lemma 2.2, we have

$$\langle J\Pi_{EP(f,C)}x_n - Jx_n, p - \Pi_{EP(f,C)}x_n\rangle \geq 0.$$

Passing to the limit as $n\to\infty$, we obtain $\langle Jz - Jp, p - z\rangle \geq 0$, which implies

$$\phi(z,p) + \phi(p,z) = 2\langle Jz - Jp, z - p\rangle \leq 0.$$

Hence $z = p$. □

5. Applications

Let E be a real Banach space and let C be a nonempty, closed and convex subset of E. Let $A : E \to E^*$ be an operator. The variational inequality problem, (for short, $VI(A,C)$) associated with A and C is defined as: find $x \in C$ such that

$$\langle Ax; z - x\rangle \geq 0;\ \forall z \in C. \tag{5.1}$$

We denote by Γ the solution set of variational inequality (5.1).

Remark 5.1. In the special case where $f(x,y) = \langle Ax, y - x\rangle$, the EP (1.1) becomes the $VI(A,C)$ (5.1). Moreover Algorithm 3.1 is reduced to the following.

Algorithm 5.2. Initialization: Choose $x_0 \in E$, $y_0 \in C$, a control parameter $\lambda > 0$, and compute

$$x_1 = \Pi_C J^{-1}(Jx_0 - \lambda Ay_0),$$
$$y_1 = \Pi_C J^{-1}(Jx_1 - \lambda Ay_0).$$

Iterative step for $n \geq 1$.
Step 1. construct a half space

$$T_n = \{z \in E : \langle Jx_n - \lambda Ay_{n-1} - Jy_n, z - y_n\rangle \leq 0\}.$$

Step 2.

$$\begin{cases} x_{n+1} = \Pi_{T_n} J^{-1}(Jx_n - \lambda y_n), \\ y_{n+1} = \Pi_C J^{-1}(Jx_{n+1} - \lambda Ay_n). \end{cases} \tag{5.2}$$

Stopping criterion: If $y_{n+1} = y_n = x_{n+1}$, stop. Otherwise, Set $n := n+1$ and go to Step 1.

Therefore, if we let $f(x,y) = \langle Ax, y - x\rangle$, and assume that $A : E \to E^*$ satisfies the following conditions:
(1) A is a pseudo-monotone operator on C, that is, for all $x,y \in C$, $\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, x - y\rangle \leq 0$;
(2) A is L-Lipschitz-continuous on E, that is, there exists a constant $L > 0$ such that $||Ax - Ay|| \leq L||x - y||$ for all $x,y \in E$ (It was shown [11] that if A is a Lipschitz continuous operator, $f(x,y) = \langle Ax, y - x\rangle\rangle$ satisfies the Lipschitz-type condition with $L_1 = L_2 = \frac{L}{2}$.
(3) A is sequentially weakly continuous on C, that is, for each sequence $\{x_n\} \subset C$, we have that $\{x_n\}$ converges weakly to $x \in E$ implies $\{Ax_n\}$ converges weakly to Ax.

(4) The solution set of $VI(A,C)$ (5.1) is nonempty.

Then from Lemma 4.1 and Theorem 4.2 with $f(x,y) = \langle Ax, y-x\rangle$, and A satisfying conditions (1)-(4), we obtain the desired convergence results.

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