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THE APPROXIMATION OF SOLUTIONS FOR GENERALIZED EQUILIBRIUM AND NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce and study a Bregan projection algorithm for treating generalized equilibrium and fixed point problems. Norm convergence of the projection algorithm is established in the framework of reflexive Banach spaces.

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1. Introduction

Let E be a Banach space. Let C be a nonempty convex and closed set in E and let E^* be the dual space of E. Let $M : C \to E^*$ be an operator and let $G : C \times C \to \mathbb{R}$ be a bifunction. Recall that the so called generalized equilibrium problem [1]: find $\tilde{x}$ such that

$$\langle M\tilde{x}, y-\tilde{x}\rangle + G(\tilde{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

If $M = 0$, then the generalized equilibrium problem is reduced to the classical equilibrium problem in the sense of Blum and Oettli [2]

$$G(\tilde{x}, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

If $G = 0$, then the generalized equilibrium problem is reduced to the generalized variational inequality problem

$$\langle M\tilde{x}, y-\tilde{x}\rangle \geq 0, \quad \forall y \in C.$$

In this paper, the set of solutions of the generalized equilibrium problem is denoted by $GEP(M,G)$. The equilibrium problems provides us a a unified approach to investigate problems arising in many optimization problems; see [3, 4, 5, 6, 7, 8, 9] and the references.

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For set C, $CB(C)$ is used to stand for the family of nonempty bounded closed subsets of C. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(C)$ defined by

$$H(B,A) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \quad \forall B,A \in CB(C),$$

where $d(b,A) = \inf\{\|b-a\| : a \in A\}$ is the distance from point b to subset A. Let T be a set-valued mapping from C to $CB(C)$. The set of fixed points of T is denoted by $F(T) := \{p \in C : T(p) = p\}$. Let $f : E \to (-\infty,+\infty]$ be a proper, lower semi-continuous, and convex function. The domain of f is presented by dom$f$, i.e., dom$f := \{x \in E : f(x) < +\infty\}$. Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real numbers and positive integers, respectively. Let any $x \in$ int dom$f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^{\circ}(x,y) = \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}.$$

Recall that the function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int dom$f$; Gâteaux differentiable at x if the limit $f^{\circ}(x,y)$ exists for any y; uniformly Fréchet differentiable on a subset C of E if the limit $f^{\circ}(x,y)$ is attained uniformly for $x \in C$ and $\|y\| = 1$; Fréchet differentiable at x if the limit $f^{\circ}(x,y)$ is attained uniformly in $\|y\| = 1$. It is known if f is Gâteaux differentiable at x, then $f^{\circ}(x,y)$ coincides with $\triangledown f(x)$, the value of the gradient $\triangledown f$ of f at x. Let $x \in$ int dom$f$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) \le f(y) + \langle x^*, x-y\rangle, \quad \forall\, y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty,+\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*,x\rangle - f(x) : x \in E\},\ \forall x^* \in E^*.$$

In the framework of reflexive Banach spaces, we have the following facts: (i) $(\partial f)^{-1} = \partial f^*$ and f is Legendre if and only if f^* is Legendre; (ii) f is essentially smooth if and only if f^* is essentially strictly convex; (iii) If f is Legendre, then ∇f is bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$=dom$\nabla f^*$=int dom$f^*$ and ran∇f^*=dom∇f=int dom$f$.

Recall that a function f is said to be (i) essentially stirctly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom∂f; (ii) essentially smooth if ∂f is both locally bounded and single-valued on its domain; (iii) Legendre, if it is both essentially smooth and essentially strictly convex. Let $f : E \to (-\infty,+\infty]$ be a Gâteaux differentiable function. The Bregman distance with respect to f is the function D_f : dom$f\times$int dom$f \to [0,+\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \triangledown f(x), y-x\rangle.$$

Recall that bifunction $V_f : E \times E^* \to [0,\infty)$ associated with f is defined by

$$V_f(x,x^*) = f^*(x^*) + f(x) - \langle x,x^*\rangle,\ \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and satisfies

$$V_f(x,x^*) = D_f(x,\nabla f^*(x^*)),\ \forall x \in E, x^* \in E^*.$$

$D_f(\cdot,\cdot)$ has the following important property, called the three point identity. For any $x \in$ dom$f$ and $y,\ z \in$ int dom$f$,

$$\langle \triangledown f(z) - \triangledown f(y), x-y\rangle = D_f(x,y) - D_f(x,z) + D_f(y,z).$$

Let $f : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function and let $C \subset \mathrm{dom} f$ be a nonempty, closed, and convex set. The Bregman projection $x \in$ int dom$f$ onto C is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x),x) = \inf\{D_f(y,x) : y \in C\}.$$

Let $B_r := \{z \in E : \|z\| \le r\}$ and $S_E = \{x \in E : \|x\| = 1\}$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E if $\rho_r(t) > 0$ for all $r,t > 0$, where $\rho_r : [0,\infty) \to [0,\infty]$ is defined by

$$\rho_r(t) := \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}.$$

Let $f : E \to (-\infty,+\infty]$ be Gâteaux differentiable. The modulus of total convexity of f at $x \in \mathrm{dom} f$ is the function $\nu_f(x,\cdot) : [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in dom f, \|y-x\| = t\}.$$

The modulus of the total convexity of the function f on the set B is the function ν_f : int dom$f \times [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap dom f\}.$$

Recall that a function f is said to be: (i) totally convex at x if $\nu_f(x,t) > 0$, whenever $t > 0$; (ii) totally convex if it is totally convex at any point $x \in$ int dom$f$; (iii) totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$. A function f is said to be: strongly coercive if $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$; sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded, $\lim_{n\to\infty} D_f(y_n,x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0$. Recall that T is said to be set-valued Bregman quasi-strictly pseudo-contractive with respect to f if $F(T) \ne \emptyset$ and $D_f(p,u) \le D_f(p,x) + kD_f(x,u)$, $\forall u \in Tx, x \in C, p \in F(T)$. Further, T is said to be set-valued Bregman quasi-nonexpansive with respect to f if $F(T) \ne \emptyset$ and $D_f(p,u) \le D_f(p,x)$, $\forall u \in Tx, x \in C, p \in F(T)$.

Normal Mann iterative method is a powerful scheme to dealing with convex optimization problems; see, e.g., [10, 11, 12, 13, 14]. One knows that the normal Mann iterative method is efficient for nonexpansive-type mappings, however, it is only weakly convergent in the framework of infinite dimensional spaces. The research on modified Mann iterative method is now under the spotlight of many researchers; see, e.g., [15, 16, 17, 18, 19] and the references therein. We have to mention that the success achieved in using geometric properties of Hilbert spaces is not easy to carry out to the framework of Banach spaces. The main difficulty is that the normalized duality map appears in most Banach space inequalities. This creates very serious technical difficulties in computation. Recently, attempts with the Bregman distance have been made to overcome these difficulties; see [19, 20, 21] and the references therein.

We in this paper focus on a Bregan projection algorithm for generalized equilibrium and fixed point problems of a family of closed multi-valued Bregman quasi-strict pseudocontractions. To study equilibrium problem (1.1), we need the following restrictions on bifunction G and operator M.

(A1) $G(x,y) \ge \limsup_{t\downarrow 0} G(tz + (1-t)x, y)$, $\forall x,y,z \in C$;
(A2) $G(y,x) + g(x,y) \le 0, \forall x,y \in C$;
(A3) $G(x,x) \equiv 0, \forall a \in C$;
(A4) $y \mapsto G(x,y)$ is convex and weakly lower semi-continuous, $\forall x \in C$;

(A5) M is Bregman inverse-strongly monotone (BISM [22]), that is, dom$(A)\cap$int dom$f \neq \emptyset$ and for any $x,y \in$int dom$f$ and each $x' \in Mx, y' \in My$, $\langle x'-y', \nabla f^*(\nabla f(x)-x') - \nabla f^*(\nabla f(y)-y')\rangle \geq 0$.

2. Preliminaries

For $r>0$, the resolvent operator of bifunction G and operator M, $Res_r^{M,G}: E \to C$ is defined as follows:

$$Res_r^{M,G}(x) = \{z \in C : r\langle Mz, y-z\rangle + \langle y-z, \nabla f(z) - \nabla f(x)\rangle + rG(z,y) \geq 0,\ \forall y \in C\},\ \forall x \in E.$$

From [23], the following lemma is easy to reach.

Lemma 2.1. *Let E be a reflexive Banach space. Let C be a nonempty, closed, and convex subset of E. Let $f: E \to \mathbb{R}$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E and let $M: C \to E^*$ be a Bregman inverse-strongly monotone operator. Let M, G satisfy (R-1)-(R-5) and let $Res_r^{M,G}: E \to C$ be the resolvent defined above. Then the following statements hold:*

(a) $Res_r^{M,G}$ is single-valued;

(b) $F(Res_r^{M,G}) = GEP(M,G)$ is closed and convex;

(c) $D_f(p, Res_r^{M,G}x) + D_f(Res_r^{M,G}x, x) \leq D_f(p,x)$, $\forall\, p \in GEP(M,G)$, $\forall\, x \in E$.

Lemma 2.2. [24] *Suppose $x \in E$ and $y \in$ int dom$f$. If f is essentially strictly convex, then $D_f(x,y) = 0 \Leftrightarrow x = y$. Function f is sequentially consistent if and only if f is totally convex on bounded sets.*

Lemma 2.3. [24] *Let $f: E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 2.4. [24] *Let $f: E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of domf^*=E^* if and only if f is strongly coercive and uniformly convex on bounded subsets of E.*

Lemma 2.5. *Let $f: E \to R$ be a Legendre function which is uniformly Fréchet differentiable and bounded on subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T: C \to CB(C)$ be a multi-valued Bregman quasi-strictly pseudocontractive mapping with respect to f. Then, for any $x \in C$, $u \in Tx$, $p \in F(T)$ and $k \in [0,1)$, $(1-k)D_f(x,u) \leq \langle x-p, \nabla f(x) - \nabla f(u)\rangle$.*

Proof. Let $u \in Tx$, $p \in F(T)$, $x \in C$, and $k \in [0,1)$, one has

$$D_f(p,u) \leq D_f(p,x) + kD_f(x,u).$$

This implies that

$$D_f(p,x) + D_f(x,u) + \langle p-x, \nabla f(x) - \nabla f(u)\rangle \leq D_f(p,x) + kD_f(x,u).$$

It follows that

$$(1-k)D_f(x,u) \leq \langle x-p, \nabla f(x) - \nabla f(u)\rangle.$$

This completes the proof. □

Lemma 2.6. [19] *Let $f: E \to R$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T: C \to CB(C)$ be a multi-valued Bregman quasi-strictly pseudocontractive mapping with respect to f. Then $F(T)$ is a convex and closed set.*

Lemma 2.7. [25] *Suppose that f is Gâteaux differentiable and totally convex on int dom$f$. Let $x \in$ int dom$f$ and let $C \subset$ int dom$f$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent: (i) $\langle \nabla f(x) - \nabla f(\hat{x}), \hat{x} - y \rangle \geq 0$, $\forall\ y \in C$; (i) $D_f(y, \hat{x}) + D_f(\hat{x}, x) \leq D_f(y, x)$, $\forall\ y \in C$; and (iii) $\hat{x} = P_C^f(x)$.*

3. THE STRONG CONVERGENCE THEOREM

In this section, we state and prove our main theorem.

Theorem 3.1. *Let C be a nonempty, convex and closed set in a real reflexive Banach space E. Let M_i be a BISM and let Let G_i be a bifunction with (A1), (A2), (A3), (A4) and (A5) for each $i \in \prod$. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\prod$ be a index set. Let $T_i : C \to CB(C)$ be a closed and multi-valued Bregman quasi-strict pseudocontraction. Assume that $\Omega := \cap_{i\in\prod} F(T_i) \bigcap \cap_{i\in\prod} GEP(M, G_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 \in E,\ C_{1,i} = C,\ C_1 = \cap_{i\in\prod} C_{1,i},\ x_1 = P_{C_1}^f(x_0), \\ y_{n,i} = \nabla f^*[\alpha_{n,i} \nabla f(x_n) + (1-\alpha_{n,i}) \nabla f(z_{n,i})], \quad z_{n,i} \in T_i x_n, \\ r_{n,i} \langle M_i u_{n,i}, y - u_{n,i} \rangle + r_{n,i} G_i(u_{n,i}, y) + \langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i}) \rangle \geq 0, \\ C_{n+1,i} = \{ z \in C_{n,i} : D_f(z, u_{n,i}) \leq D_f(z, y_{n,i}) \leq D_f(z, x_n) \\ \qquad\qquad + \frac{\kappa}{1-\kappa} \langle x_n - z, \nabla f(x_n) - \nabla f(z_{n,i}) \rangle \}, \\ C_{n+1} = \cap_{i\in\prod} C_{n+i,i}, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \end{cases} \tag{3.1}$$

where $\kappa \in [0,1)$, $\liminf_{n\to\infty} (1-\alpha_{n,i})\alpha_{n,i} > 0$ $\liminf_{n\to\infty} r_{n,i} > 0$, for $\forall i \in \prod$. Then $\{x_n\}$ converges in norm to $\widehat{p} = P_\Omega^f(x_1)$, where P_Ω^f is the Bregman projection of E onto Ω.

Proof. Using Lemma 2.1 and Lemma 2.6, one concludes that our solution set is convex and closed. Now, we prove that set C_n is a convex and closed set. One knows that $C_{1,i} = C$ is a convex and closed set. We now let $C_{k,i}$ is a convex and closed subset for every integer $k \geq 1$. Set z_1 and z_2 be two arbitrary points in $C_{m+1,i}$. From the construction of set C_n, one obtains $z_1, z_2 \in C_{k,i}$. Further, one sets

$$z_{1,2} = (1-\lambda) z_2 + \lambda z_1,$$

where λ is a real number in $(0,1)$. It follows that

$$\begin{aligned} D_f(z_{1,2}, u_{k,i}) &\leq D_f(z_{1,2}, y_{k,i}) \\ &\leq D_f(z_{1,2}, x_k) + \frac{\kappa}{1-\kappa} \langle x_k - z_{1,2}, \nabla f(x_k) - \nabla f(z_{k,i}) \rangle. \end{aligned}$$

In view of $z_{1,2} \in C_{n,i}$, we obtain that $C_{n,i} \in C_{k+1,i}$. This proves that $C_{k+1,i}$ is a convex and closed set. Hence, $C_{n,i}$ is also a convex and closed set. One concludes that $P_\Omega^f(x_0)$ is well defined.

Next, one focuses on $\Omega \subset C_n$. Indeed, it is easy to see $\Omega \subset C_1 = C$. Set $\Omega \subset C_{m,i}$. Note that $u_{m,i} = Res_{r_{m,i}}^{M_i,G_i} y_m$. For any $w \in \Omega \subset C_{m,i}$, we reach

$$\begin{aligned}
D_f(w,u_{m,i}) &= f(w) - \langle w, \alpha_{m,i}\nabla f(x_m) + (1-\alpha_{m,i})\nabla f(z_{m,i})\rangle \\
&\quad + f^*(\alpha_{m,i}\nabla f(x_m) + (1-\alpha_{m,i})\nabla f(z_{m,i})) \\
&\leq \alpha_m f(w) - \alpha_{m,i}\langle w, \nabla f(x_m)\rangle + \alpha_{m,i} f^*(x_m) \\
&\quad + (1-\alpha_{m,i})f(w) - (1-\alpha_{m,i})\langle w, \nabla f(z_{m,i})\rangle + (1-\alpha_m) f^*(\nabla f(z_{m,i})) \\
&\leq (1-\alpha_{m,i})[D_f(w,x_m) + kD_f(x_m,z_{m,i})] + \alpha_{m,i} D_f(w,x_m) \\
&\leq \frac{(1-\alpha_{m,i})k}{1-k}\langle x_m - w, \nabla f(x_m) - \nabla f(z_{m,i})\rangle + D_f(w,x_m) \\
&\leq \frac{k\langle x_m - w, \nabla f(x_m) - \nabla f(z_{m,i})\rangle}{1-k} + D_f(w,x_m),
\end{aligned}$$

which yields that $w \in C_{m+1,i}$. Hence, $\Omega \subset C_{n,i}$ and then $\Omega \subset C_n = \cap_{i\in\prod} C_{n,i}$. It follows from Lemma 2.7 that

$$\langle y - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \leq 0, \quad \forall\, y \in C_n,$$

By virtue of $\Omega \subset C_n$, one concludes that

$$\langle w - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \leq 0, \quad \forall\, w \in \Omega. \tag{3.2}$$

By using Lemma 2.7, one has

$$\begin{aligned}
D_f(x_n,x_1) &= D_f(P^f_{C_n}(x_1), x_1) \\
&\leq D_f(w,x_1) - D_f(w, P^f_{C_n}(x_1)) \\
&\leq D_f(w,x_1),
\end{aligned}$$

for each $w \in \Omega$. This proves the sequence $\{D_f(x_n,x_1)\}$ is a bounded sequence. This further proves $\{x_n\}$ is a bounded iterative sequence too. Note that $D_f(x_{n+1},x_1) \geq D_f(x_n,x_1)$. So, $\{D_f(x_n,x_1)\}$ is a nondecreasing sequence. We conclude that $\lim_{n\to\infty} D_f(x_n,x_1)$ exists. Without loss of generality, we may suppose that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup \widehat{p}$, which lies in C_n. On the other hand, one has

$$D_f(x_{n_j},x_1) \leq D_f(\widehat{p},x_1) \tag{3.3}$$

and

$$\begin{aligned}
D_f(\widehat{p},x_1) &= f(\widehat{p}) - f(x_1) - \langle \nabla f(x_1), \widehat{p} - x_1\rangle \\
&\leq \liminf_{j\to\infty}\{f(x_{n_j}) - f(x_1) - \langle \nabla f(x_1), x_{n_j} - x_1\rangle\} \\
&\leq \liminf_{j\to\infty} D_f(x_{n_j},x_1).
\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), one arrives at

$$D_f(\widehat{p},x_1) \geq \limsup_{j\to\infty} D_f(x_{n_j},x_1) \geq \liminf_{j\to\infty} D_f(x_{n_j},x_1) \geq D_f(\widehat{p},x_1).$$

This yields that $D_f(\widehat{p},x_1) = \lim_{j\to\infty} D_f(x_{n_j},x_1)$. Employing Lemma 2.7, one obtains that

$$D_f(\widehat{p},x_{n_j}) + D_f(x_{n_j},x_1) \leq D_f(\widehat{p},x_1.$$

Hence, $\lim_{j\to\infty} D_f(\widehat{p}, x_{n_j}) = 0$. Using Lemma 2.2 that $\lim_{j\to\infty} x_{n_j} = \widehat{p}$. Since $\{D_f(x_n, x_0)\}$ is a convergent sequence, one obtains that

$$\lim_{n\to\infty} D_f(x_n, x_1) = D_f(\widehat{p}, x_1). \tag{3.5}$$

Using Lemma 2.7, one has

$$D_f(\widehat{p}, x_n) + D_f(x_n, x_1) \le D_f(\widehat{p}, x_1). \tag{3.6}$$

Further, Lemma 2.2 implies that

$$\lim_{n\to\infty} x_n = \widehat{p}. \tag{3.7}$$

Note that

$$\begin{aligned} D_f(x_{n+1}, u_{n,i}) &\le D_f(x_{n+1}, y_{n,i}) \\ &\le D_f(x_{n+1}, x_n) + \frac{\kappa}{1-\kappa}\langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(z_{n,i})\rangle. \end{aligned}$$

From the situation that $x_n \to \widehat{p}$ as $n \to \infty$, one asserts that

$$\lim_{n\to\infty} D_f(x_{n+1}, u_{n,i}) = \lim_{n\to\infty} D_f(x_{n+1}, y_{n,i}) = 0.$$

It follows that

$$\lim_{n\to\infty} \|x_{n+1} - u_{n,i}\| = \lim_{n\to\infty} \|x_{n+1} - y_{n,i}\| = 0. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\lim_{n\to\infty} \|x_n - u_{n,i}\| = \lim_{n\to\infty} \|x_n - y_{n,i}\| = 0. \tag{3.9}$$

It follows that

$$\lim_{n\to\infty} \|\nabla f(y_{n,i}) - \nabla f(x_n)\| = 0. \tag{3.10}$$

It follows that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_{n,i})\| = \lim_{n\to\infty} \frac{1}{1-\alpha_n}\|\nabla f(x_n) - \nabla f(y_{n,i})\| = 0. \tag{3.11}$$

Using Lemma 2.5 yields that $\lim_{n\to\infty} \|z_{n,i} - x_n\| = 0$. Therefore $\lim_{n\to\infty} z_{n,i} = \lim_{n\to\infty} x_n = \widehat{p}$. From the closedness of each T_i and $z_{n,i} \in T_i x_n$, one sees $\widehat{p} \in F(T_i)$. Hence, $\widehat{p} \in \cap_{i\in\prod} F(T_i)$.

Next, we prove $\widehat{p} \in \cap_{i\in\prod} GEP(M_i, G_i)$. Note that

$$\|y_{n,i} - u_{n,i}\| \le \|x_n - u_{n,i}\| + \|x_n - y_{n,i}\|.$$

By using (3.9), we have $\lim_{n\to\infty} \|y_{n,i} - u_{n,i}\| = 0$. Since ∇f is uniformly norm-to-norm continuous on bounded subsets of E, one has

$$\lim_{n\to\infty} \frac{\|\nabla f(y_{n,i}) - \nabla f(u_{n,i})\|}{r_{n,i}} = 0. \tag{3.12}$$

From the definition of the resolvent of the generalized equilibrium problem, we have $u_{n,i} = Res_{r_{n,i}}^{M_i,G_i} y_{n,i}$. Hence,

$$r_{n,i}\langle M u_{n,i}, y - u_{n,i}\rangle + r_{n,i} g_i(u_{n,i}, y) + \langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle \ge 0, \quad \forall\, y \in C.$$

Hence, one has

$$\begin{aligned} \|y - u_{n,i}\| \frac{\|\nabla f(y_{n,i}) - \nabla f(u_{n,i})\|}{r_{n,i}} &\ge \frac{\langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle}{r_{n,i}} \\ &\ge W_i(y, u_{n,i}), \quad \forall y \in C, \end{aligned}$$

where

$$W_i(y, u_{n,i}) = \langle M u_{n,i}, y - u_{n,i}\rangle + G(u_{n,i}, y).$$

It follows from (3.12) that $W_i(y,\widehat{p}) \le 0$, $\forall y \in C$. Let

$$y_{t_i} = t_i y + (1-t_i)\widehat{p},$$

where $y \in C$, and $t_i \in (0,1)$. It follows that $W_i(y_{t_i},\widehat{p}) \le 0$. Hence

$$t_i W_i(y_{t_i},y) \ge t_i W_i(y_{t_i},y) + (1-t_i)W_i(y_{t_i},p) \ge W_i(y_{t_i},y_{t_i}) = 0.$$

This implies $W_i(y_t,y) \ge 0$, $\forall y \in C$ and then $W_i(\widehat{p},y) \ge 0$, $\forall y \in C$. Hence

$$\widehat{p} \in \cap_{i\in\prod} EP(W_i) = GEP(M_i,G_i).$$

This proves that $\widehat{p} \in \Omega$. Finally, we take $n \to \infty$ in (3.2) to obtain that

$$\langle w - \widehat{p}, \nabla f(x_1) - \nabla f(x_n)\rangle \le 0, \quad \forall\, w \in \Omega.$$

Using Lemma 2.7, one has $\widehat{p} = P^f_\Omega(x_1)$. This completes the proof. □

If $f(x) = \|x\|^2$, $\forall x \in E$, then the class of multi-valued Bregman quasi-strict pseudo-contractions is reduced to the class of multi-valued quasi-strict pseudo-contractions [26]. We have the following result.

Corollary 3.2. *Let C be a nonempty, convex and closed set in a real reflexive Banach space E. Let M_i be a BISM and let Let G_i be a bifunction with (A1), (A2), (A3), (A4) and (A5) for each $i \in \prod$. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\prod$ be a index set. Let $T_i : C \to CB(C)$ be a closed and multi-valued Bregman quasi-strict pseudocontraction. Assume that $\Omega := \cap_{i\in\prod} F(T_i) \bigcap \cap_{i\in\prod} GEP(M,G_i) \ne \emptyset$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 \in E,\ C_{1,i} = C,\ C_1 = \cap_{i\in\prod} C_{1,i},\ x_1 = P^f_{C_1}(x_0), \\ y_{n,i} = J^{-1}[\alpha_{n,i} J(x_n) + (1-\alpha_{n,i})J(z_{n,i})], \quad z_{n,i} \in T_i x_n, \\ r_{n,i}\langle M_i u_{n,i}, y - u_{n,i}\rangle + r_{n,i} G_i(u_{n,i},y) + \langle y - u_{n,i}, J(u_{n,i}) - J(y_{n,i})\rangle \ge 0, \\ C_{n+1,i} = \{z \in C_{n,i} : D_f(z,u_{n,i}) \le D_f(z,y_{n,i}) \le D_f(z,x_n) \\ \qquad\qquad + \frac{\kappa}{1-\kappa}\langle x_n - z, J(x_n) - J(z_{n,i})\rangle\}, \\ C_{n+1} = \cap_{i\in\prod} C_{n+i,i}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \end{cases}$$

where $\kappa \in [0,1)$, $\liminf_{n\to\infty}(1-\alpha_{n,i})\alpha_{n,i} > 0$ $\liminf_{n\to\infty} r_{n,i} > 0$, for $\forall i \in \prod$. Then $\{x_n\}$ converges strongly to $\widehat{p} = P^f_\Omega(x_1)$, where P^f_Ω is the generalized projection of E onto Ω.

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