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VISCOSITY APPROXIMATION METHODS FOR ZEROS OF ACCRETIVE OPERATORS

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Abstract. Zeros of nonlinear mappings of accretive type are investigated via a viscosity approximation method. Strong convergence theorems of solutions of systems of generalized variational inequalities and the sum problem of two accretive operators are obtained in uniformly convex and q-uniformly smooth Banach spaces with mild conditions on parameters of the approximation method.

Keywords. Accretive operator; Banach space; System of vaiational inequalities; Viscosity method; Forward-Backward method.

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1. Introduction

Let X be a Banach space. Let C be a nonempty convex and closed set in X and let X^* be the dual space of X. Recall that the space X is said to be strictly convex if, for any $x,y \in U$, where $U = \{x \in X : \|x\| = 1\}$, with $x \neq y$, $\|\lambda y + (1-\lambda)x\| < 1$, $\forall \lambda \in (0,1)$. The space X is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x,y \in U$, $\|\frac{x+y}{2}\| > 1-\delta \Rightarrow \|x-y\| < \varepsilon$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that the modulus of convexity of X is the function $\delta_E(\varepsilon) : (0,2] \to [0,1]$ defined by $\delta_X(\varepsilon) = \inf\{\frac{2-\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}$, $0 \leq \varepsilon \leq 2$. Let $p > 1$ be a real number. One says that X is p-uniformly convex iff there exists a constant $c_p > 0$ such that $\delta_X(\varepsilon) \geq c_p \varepsilon^p$ for any $\varepsilon \in (0,2]$. X is said to be uniformly convex if $\delta_X(0) = 0$, and $\delta_E X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. It is clear that the p-uniform convexness implies the uniform convexness.

Recall that the generalized duality mapping, J_q with $q > 1$, is defined by

$$J_q(x) := \{y \in X^* : \langle y,x \rangle = \|x\|^q, \|y\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

If $q = 2$, that is,

$$J_2(x) := \{y \in X^* : \langle y,x \rangle = \|x\|^2, \|y\| = \|x\|\}, \quad \forall x \in X.$$

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then J_2 is called the normalized duality mapping and it is simply denoted by J. It is known that $J_q(x)$ is nonempty, and $\|x\|^{q-2}J(x) = J_q(x)$.

Recall that the modulus of smoothness of Banach space X, ρ_X is defined by

$$\rho_E(t) = \sup\{\frac{\|x+y\| - \|x-y\| - 2}{2} : x \in U, \|y\| \leq t\}.$$

Recall that X is said to be q-uniformly smooth if and only if there exists a fixed constant $c > 0$ such that $\frac{\rho_X(t)}{t^q} \leq c$. For q-uniformly smooth spaces, one has $q \leq 2$. X is said to be uniformly smooth iff $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. One knows that X is p-uniformly convex if and only if X^*, the dual space of X, is q-uniformly smooth, where $p + q = pq$. One remarks that L_p is $\min\{p,2\}$-uniformly smooth and uniformly convex for every $p > 1$. Let $p > 1$ and $r > 0$ be two fixed real numbers.

Recall that The norm of X is said to be Gâteaux differentiable iff the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. In this case, J_q is single-valued and strong-weak* continuous.

Let C be a convex and closed set in X. Let D be a nonempty set in C. Let Π_D^C be a mapping from set C and set D. Recall that Π_D^C is said to be a retraction if and only if $(\Pi_D^C)^2 = \Pi_D^C$; sunny iff, for each $x \in C$ and $t \in (0,1)$, $\Pi_D^C\big((1-t)\Pi_D^C x + tx\big) = \Pi_D^C x$; sunny nonexpansive retractction iff Π_D^C is sunny, nonexpansive and a retraction.

In smooth Banach space X, we have the following useful properties [1, 2, 3]

(1) Π_C^X is sunny and nonexpansive;

(2) $\langle x - \Pi_C^X x, J_q(y - \Pi_C^X x)\rangle \leq 0, \forall x \in X, y \in C$.

(3) $\langle x - y, J_q(\Pi_C^X x - \Pi_C^X y)\rangle \geq \|\Pi_C^X x - \Pi_C^X y\|^2, \forall x, y \in X$.

One also remarks that the sunny and nonexpansive retraction is just the nearest point projections in the setting of Hilbert spaces. Recall that an operator $N : X \to 2^X$ with domain $Dom(N) = \{b \in X : Nb \neq \emptyset\}$ and range $Ran(N) = \cup\{Nb : b \in Dom(N)\}$ is said to be accretive if and only if, for $t > 0$ and $x, y \in Dom(N)$,

$$\|x-y\| \leq \|x-y+t(x'-y')\|, \quad \forall x' \in Nx, y' \in Ny.$$

From Kato [4], one knows that N is accretive if and only if, for $x, y \in Dom(N)$, there exists $J_q(x-y)$ such that

$$\langle x'-y', J_q(x-y)\rangle \geq 0, \quad \forall x' \in Nx, y' \in Ny.$$

An accretive operator N is said to be m-accretive if and only if $Ran(I + bN)$, where b is any positive real number, is fully X. In this paper, we use $N^{-1}(0)$ to denote the zero point set of operator N. Further, for an m-accretive operator N, one can define a single-valued mapping $J_b^N : Ran(I + bN) \to Dom(N)$ associated with N by $J_b^N = (I + bN)^{-1}$ for each $r > 0$. For the single-valued case, $N : C \to X$ is said to be α-inverse strongly accretive if and only if there exist a constant $\alpha > 0$ and some $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Nx - Ny, J_q(x-y)\rangle \geq \alpha\|Nx - Ny\|^q, \quad \forall x, y \in C.$$

Let T be an operator on C. Recall that T is contractive if and only if there exists a constant $\tau \in (0,1)$ such that $\|Tx - Ty\| \leq \tau\|x-y\|$, for all $x, y \in C$. If $\tau = 1$, we say that T is nonexpansive. Indeed, the mapping J_b^N is nonexpnsive.

In 2006, Aoyama, Iiduka and Takahashi [5], based on nonexpansive mappings, proposed an iterative scheme of finding solutions of the following generalized variational inequality associated with an accretive operator A $\langle Ax^*, J(x-x^*)\rangle \geq 0$ $\forall x \in C$. They proved, in both uniformly convex and q-uniformly smooth Banach spaces, that their algorithms are weakly convergent to some zero of the above generalized variational inequality. Recently, many authors investigated the above problems and established various convergence theorems with mild conditions on operators and parameters; see, e.g., [6]-[15] and the references therein.

In this paper, we consider the following problem, associated with a pair of accretive operators, $A_1, A_2 : C \to X$, of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - y^* + \mu_1 A_1 y^*, J(x-x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - x^* + \mu_2 A_2 x^*, J(x-y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.1}$$

where $\mu_1, \mu_2 > 0$ are any two positive real number. This system is called a system of generalized variational inequalities; see [16, 17] and [18] for the version in Hilbert spaces. In addition, if $A_1 = A_2 = A$ and $x^* = y^*$, then it is reduced to the generalized variational inequality studied by Aoyama, Iiduka and Takahashi [5].

In this paper, we are concerned with the above system and the sum of two accretive operators (one is single-valued and the other one is set-valued) via an implicit viscosity forward-backward method. We establish a convergence theorem of solutions in the sense of norm function in q-uniformly smooth, where $1 < q \leq 2$, and uniformly convex Banach spaces with mild conditions on parameters.

2. Preliminaries

Lemma 2.1. [13] *Let X be a real Banach space and let J_q be the generalized duality mapping. Then, for any give $x, y \in X$, one has*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle,$$

for all $j_q(x+y) \in J_q(x+y)$, where $j_q(x+y)$ is the single-valued duality mapping.

Lemma 2.2. [19] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-t_n)a_n + b_n + c_n$, $\forall n \geq 0$, where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence of real numbers. Assume that*

(a) $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, $\sum_{n=0}^{\infty} t_n = \infty$,

(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [20] *Let X a a uniformly smooth Banach space and let C be a nonempty convex closed subset of X. Let $f : C \to C$ be a contractive mapping and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set. For each $t \in (0,1)$, let x_t be the unique solution of the equation $x_t = (1-t)Tx_t + tf(x_t)$. Then $\{x_t\}$ converges strongly to a fixed point $\bar{x} = Proj^C_{\mathrm{Fix}(T)} f(\bar{x})$, where $\Pi^C_{\mathrm{Fix}(T)}$ is the unique sunny nonexpansive retraction from C onto* $\mathrm{Fix}(T)$*, as $t \to 0$.*

Lemma 2.4. [21] *Let X a strictly convex Banach space. Let T and S be a pair of nonexpansive mapping with nonempty common fixed ponts. Let b be a real number in $(0,1)$. Let $W = bT + (1-b)S$. Then W is a nonexpansive with* $\mathrm{Fix}(W)$=$\mathrm{Fix}(T) \cap \mathrm{Fix}(S)$.

Lemma 2.5. [13] *Let X be a real uniformly convex Banach space. Then there exists a convex, strictly increasing and continuous function $g:[0,\infty)\to[0,\infty)$ with $g(0)=0$ such that, for any real $p>1$,*

$$\|ax+by+cz\|^p \le a\|x\|^p + b\|y\|^p + c\|z\|^p - \frac{a^pb+b^pa}{(a+b)^p}g(\|x-y\|),$$

for all $x,y,z\in\{x\in X:\|x\|\le r\}$ and $a,b,c\in[0,1]$ with $a+b+c=1$.

Lemma 2.6. [22] *Let X be a real q-uniformly smooth Banach space. Then the following inequality holds:*

$$\|x+y\|^q \le K_q\|y\|^q + \|x\|^q + q\langle y, J_q(x)\rangle, \quad \forall x,y\in X,$$

where K_q is some fixed positive constant.

Lemma 2.7. [23] *Let $\{x_n\}$ and $\{y_n\}$ be two bounded vector sequences in a Banach space X and let $\{\alpha_n\}$ be a real sequence in $(0,1)$ with $\limsup_{n\to\infty}\alpha_n<1$ and $\liminf_{n\to\infty}\alpha_n>0$. If $x_{n+1}=(1-\alpha_n)y_n+\alpha_n x_n$, and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0$, then $\lim_{n\to\infty}\|x_n-y_n\|=0$.*

Lemma 2.8. [24] *Let M be an m-accretive operator. For the two positive real numbers μ and λ, we have*

$$J_\lambda^M x = J_\mu^M\Big(\frac{\mu}{\lambda}x + \Big(1-\frac{\mu}{\lambda}\Big)J_\lambda^M x\Big).$$

Lemma 2.9. [16] *Let X be q-uniformly smooth Banach space and Π_C be the sunny nonexpansive retraction from X onto its nonempty convex and closed subset C. Let $A_1:C\to X$ be a_1-inverse-strongly accretive of order q and $A_2:C\to X$ be a_1-inverse-strongly accretive of order q. Let the mapping $G:C\to C$ be defined as $Gx:=\Pi_C(I-\mu_1A_1)\Pi_C(I-\mu_2A_2)$, $\forall x\in C$. If $0<\mu_1^{q-1}\kappa_q\le a_1q$ and $0<\mu_2^{q-1}\kappa_q\le a_2q$, then $G:C\to C$ is nonexpansive. Let X be q-uniformly smooth. For given $(x^*,y^*)\in C\times C$, (x^*,y^*) is a solution of system (1.1) if and only if $x^*=\Pi_C(y^*-\mu_1A_1y^*)$, where $y^*=\Pi_C(x^*-\mu_2A_2x^*)$, that is, $x^*=Gx^*$.*

Lemma 2.10. [13] *Let X be a Banach space, C a nonempty convex and closed subset, $B:C\to 2^X$ an m-accretive operator and A an inverse-strongly accretive operator. Let $T_\lambda=J_\lambda^B(I-\lambda A)^{-1}$. Then T_λ is nonexpansive mapping and* $\mathrm{Fix}(T_\lambda)=(A+B)^{-1}0$, $\forall\lambda>0$.

3. MAIN RESULTS

Theorem 3.1. *Let X be a q-uniformly smooth, where $1<q\le 2$, and uniformly convex Banach space. Let C be a nonempty convex and closed set in X. Let $A:C\to X$ be an a-inverse-strongly accretive operator of order q, $A_1:C\to X$ an a_1-inverse-strongly accretive operator of order q, and $A_2:C\to X$ an a_2-inverse-strongly accretive operator of order q. Let $B:C\to 2^X$ be an m-accretive operator and $f:C\to C$ a δ-contractive operator, where $\delta\in(0,1)$. Suppose that $\Omega=(A+B)^{-1}(0)\cap\mathrm{GSVI}(C,A_1,A_2)\neq\emptyset$, where* $\mathrm{GSVI}(C,A_1,A_2)$ *is the fixed point set of $G:=\Pi_C(I-\mu_1A_1)\Pi_C(I-\mu_2A_2)$ with $0<\mu_1^{q-1}\kappa_q<a_1q$ and $0<\mu_2^{q-1}\kappa_q<a_2q$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} u_n=\Pi_C(y_n-\mu_2A_2y_n),\\ y_n=\gamma_nJ_{\lambda_n}^B(I-\lambda_nA)((1-t_n)\Pi_C(u_n-\mu_1A_1u_n)+t_nx_n)+\beta_nx_n+\alpha_nf(x_n),\\ x_{n+1}=\delta_nx_n+(1-\delta_n)J_{\lambda_n}^B(y_n-\lambda_nAy_n) \quad n\ge 0,\end{cases}$$

where Π_C is a sunny nonexpansive retraction from X onto C, $\{\lambda_n\}\subset(0,(\frac{aq}{\kappa_q})^{\frac{1}{q-1}})$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{t_n\}$ are in $(0,1)$ s.t.

(i) $\alpha_n+\beta_n+\gamma_n=1$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\alpha_n=0$;

(ii) $\lim_{n\to\infty}|\beta_n-\beta_{n-1}|=\lim_{n\to\infty}|t_n-t_{n-1}|=\lim_{n\to\infty}|\gamma_n-\gamma_{n-1}|=0$*;*
(iii) $\limsup_{n\to\infty}\gamma_n t_n(1-t_n)<1$ *and* $\liminf_{n\to\infty}\gamma_n(1-t_n)>0$*;*
(iv) $\limsup_{n\to\infty}\delta_n<1$, $\liminf_{n\to\infty}\delta_n<1$ *and* $\liminf_{n\to\infty}\beta_n\gamma_n>0$*;*
(v) $\lim_{n\to\infty}\lambda_n=\lambda<(\frac{aq}{\kappa_q})^{\frac{1}{q-1}}$ *and* $0<\bar{\lambda}\le\lambda_n$*.*

Then $x_n\to x^*\in\Omega$*, which is a unique solution to the variational inequality:* $\langle x^*-f(x^*),J(x^*-p)\rangle\le 0$*,* $\forall p\in\Omega$*.*

Proof. From now on, we set $v_n=\Pi_C(u_n-\mu_1A_1u_n)$ and $T_n=J^B_{\lambda_n}(I-\lambda_nA)$. So, our algorithm can be re-written as

$$\begin{cases} y_n=\gamma_nT_nz_n+\beta_nx_n+\alpha_nf(x_n),\\ x_{n+1}=\delta_nx_n+(1-\delta_n)T_ny_n,\end{cases}$$

where $z_n:=(1-t_n)Gy_n+t_nx_n$. From Lemma 2.9 and Lemma 2.10 and the condition on the parameters, one gets that T_n is a nonexpansive self-mapping on C for each $n\ge 0$. Observe that

$$\alpha_n\delta+\beta_n+\gamma_nt_n+\gamma_n(1-t_n)=1-\alpha_n(1-\delta).$$

One says that sequence $\{x_n\}$ is well defined. Indeed, one can a mapping $\Gamma_n:C\to C$ by

$$\Gamma_n(x)=\alpha_nf(x_n)+\beta_nx_n+\gamma_nT_n((1-t_n)Gx+t_nx_n),\quad \forall x\in C,$$

for each fixed $x_n\in C$. Observe that G is a nonexpansive mapping. Then, for any $x,y\in C$,

$$\begin{aligned}\|\Gamma_n(x)-\Gamma_n(y)\| &=\gamma_n\|T_n((1-t_n)Gx+t_nx_n)-T_n((1-t_n)Gy+t_nx_n)\|\\ &\le(1-t_n)\gamma_n\|Gx-Gy\|\\ &\le(1-t_n)\gamma_n\|x-y\|.\end{aligned}$$

This yields that operator Γ_n is contractive, which further yields from the Banach fixed-point theorem that there exists a unique fixed point $y_n\in C$ s.t.

$$y_n=\alpha_nf(x_n)+\beta_nx_n+\gamma_nT_n((1-t_n)Gy_n+t_nx_n).$$

For any $p\in\Omega=(A+B)^{-1}(0)\cap\mathrm{GSVI}(C,A_1,A_2)$, using Lemma 2.10, one gets $T_np=p$ for each n. These further imply

$$\begin{aligned}\|y_n-p\| &=\|\gamma_n(T_n((1-t_n)Gy_n+t_nx_n)-p)+\beta_n(x_n-p)+\alpha_n(f(x_n)-p)\|\\ &\le\gamma_n\|T_n((1-t_n)Gy_n+t_nx_n)-p\|+\beta_n\|x_n-p\|+\alpha_n(\|f(p)-p\|+\|f(p)-f(x_n)\|)\\ &\le\gamma_n((1-t_n)\|Gy_n-p\|+t_n\|x_n-p\|)+\beta_n\|x_n-p\|+\alpha_n(\delta\|x_n-p\|+\|f(p)-p\|)\\ &\le\gamma_n(1-t_n)\|y_n-p\|+(\alpha_n\delta+\gamma_nt_n+\beta_n)\|x_n-p\|+\alpha_n\|f(p)-p\|,\end{aligned}$$

which hence implies that

$$\begin{aligned}\|y_n-p\| &\le\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\|+\tfrac{\alpha_n\delta+\gamma_nt_n+\beta_n}{1-\gamma_n(1-t_n)}\|x_n-p\|\\ &=\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\|+(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|.\end{aligned}\tag{3.1}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &\le(1-\delta_n)\|T_ny_n-p\|+\delta_n\|x_n-p\|\\ &\le(1-\delta_n)\|y_n-p\|+\delta_n\|x_n-p\|\\ &\le(1-\delta_n)\big(\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\|+(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|\big)+\delta_n\|x_n-p\|\\ &=\alpha_n\tfrac{(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\tfrac{\|f(p)-p\|}{1-\delta}+\big(1-\alpha_n\tfrac{(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\big)\|x_n-p\|\\ &\le\max\{\tfrac{\|f(p)-p\|}{1-\delta},\|x_n-p\|\}.\end{aligned}$$

It is easy to see that

$$\|x_n - p\| \le \max\{\frac{\|f(p)-p\|}{1-\delta}, \|x_0 - p\|\}, \quad \forall n \ge 0.$$

This leads to the boundedness of $\{x_n\}$, which guarantees that $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{v_n\}$ are all bounded. Next, one proves that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Observing

$$\begin{cases} z_{n-1} = (1-t_{n-1})Gy_{n-1} + t_{n-1}x_{n-1}, \\ z_n = (1-t_n)Gy_n + t_n x_n, \end{cases}$$

one gets that

$$z_n - z_{n-1} = t_n(x_n - x_{n-1}) + (t_n - t_{n-1})(x_{n-1} - Gy_{n-1}) + (1-t_n)(Gy_n - Gy_{n-1}),$$

It follows that

$$\begin{aligned} &\|T_{n-1}z_{n-1} - T_n z_n\| \\ &\le |1 - \tfrac{\lambda_{n-1}}{\lambda_n}|\|(I-\lambda_n A)z_{n-1} - J^B_{\lambda_n}(I-\lambda_n A)z_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| + \|z_n - z_{n-1}\| \\ &\le |1 - \tfrac{\lambda_{n-1}}{\lambda_n}|\|T_n z_{n-1} - (I-\lambda_n A)z_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Az_{n-1}\| \\ &\quad + t_n\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\|x_{n-1} - Gy_{n-1}\| + (1-t_n)\|Gy_n - Gy_{n-1}\| \\ &\le |t_n - t_{n-1}|\|x_{n-1} - Gy_{n-1}\| + t_n\|x_n - x_{n-1}\| + (1-t_n)\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1, \end{aligned}$$

where M_1 is an appropriate constant. Observing

$$\begin{cases} y_{n-1} = \gamma_{n-1}T_{n-1}z_{n-1} + \beta_{n-1}x_{n-1} + \alpha_{n-1}f(x_{n-1}), \\ y_n = \gamma_n T_n z_n + \beta_n x_n + \alpha_n f(x_n), \end{cases}$$

one gets that

$$\begin{aligned} y_n - y_{n-1} &= \alpha_n(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) + \beta_n(x_n - x_{n-1}) \\ &\quad + (\beta_n - \beta_{n-1})x_{n-1} + \gamma_n(T_n z_n - T_{n-1}z_{n-1}) + (\gamma_n - \gamma_{n-1})T_{n-1}z_{n-1}. \end{aligned} \tag{3.2}$$

Using Lemma 2.8, the resolvent identity, one deduces that Using (3.2), one gets

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \beta_n\|x_n - x_{n-1}\| + \alpha_n\|f(x_{n-1}) - f(x_n)\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\|T_n z_n - T_{n-1}z_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|T_{n-1}z_{n-1}\| \\ &\le \beta_n\|x_n - x_{n-1}\| + \alpha_n\delta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\big(t_n\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\|x_{n-1} - Gy_{n-1}\| \\ &\quad + (1-t_n)\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1\big) + |\gamma_n - \gamma_{n-1}|\|T_{n-1}z_{n-1}\| \\ &\le (1-t_n)\gamma_n\|y_n - y_{n-1}\| + (\alpha_n\delta + \gamma_n t_n + \beta_n)\|x_n - x_{n-1}\| + M_2(|\alpha_n - \alpha_{n-1}| \\ &\quad + |\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n - \lambda_{n-1}| + |t_n - t_{n-1}|), \end{aligned}$$

where M_2 is an appropriate constant. This implies that

$$\begin{aligned} \|y_n - y_{n-1}\| &\le (1 - \tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n - x_{n-1}\| + \tfrac{M_2}{1-\gamma_n(1-t_n)}(|\alpha_{n-1} - \alpha_n| + |\beta_n - \beta_{n-1}| \\ &\quad + |\lambda_n - \lambda_{n-1}| + |\gamma_n - \gamma_{n-1}| + |t_n - t_{n-1}|) \\ &\le \|x_n - x_{n-1}\| + \tfrac{M_2}{1-\gamma_n(1-t_n)}(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\ &\quad + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|) \end{aligned}$$

and

$$\begin{aligned}\|T_{n-1}y_{n-1}-T_ny_n\| &\le \|T_ny_{n-1}-T_{n-1}y_{n-1}\|+\|T_ny_n-T_ny_{n-1}\|\\ &\le \|J^B_{\lambda_n}(I-\lambda_nA)y_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)y_{n-1}\|+\|y_n-y_{n-1}\|\\ &\le \|J^B_{\lambda_n}(I-\lambda_nA)y_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_nA)y_{n-1}\|\\ &\quad+\|J^B_{\lambda_{n-1}}(I-\lambda_nA)y_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)y_{n-1}\|+\|y_n-y_{n-1}\|\\ &= \|J^B_{\lambda_{n-1}}(\tfrac{\lambda_{n-1}}{\lambda_n}I+(1-\tfrac{\lambda_{n-1}}{\lambda_n})J^B_{\lambda_n})(I-\lambda_nA)y_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_nA)y_{n-1}\|\\ &\quad+\|J^B_{\lambda_{n-1}}(I-\lambda_nA)y_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)y_{n-1}\|+\|y_n-y_{n-1}\|\\ &\le |1-\tfrac{\lambda_{n-1}}{\lambda_n}|\|J^B_{\lambda_n}(I-\lambda_nA)y_{n-1}-(I-\lambda_nA)y_{n-1}\|\\ &\quad+|\lambda_n-\lambda_{n-1}|\|Ay_{n-1}\|+\|y_n-y_{n-1}\|\\ &\le \|y_n-y_{n-1}\|+|\lambda_n-\lambda_{n-1}|M_1.\end{aligned} \tag{3.3}$$

From (3.3), one arrives at

$$\begin{aligned}\|T_{n-1}y_{n-1}-T_ny_n\| &\le \|x_n-x_{n-1}\|+\tfrac{M_2}{1-\gamma_n(1-t_n)}(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|\\ &\quad+|t_n-t_{n-1}|+|\lambda_n-\lambda_{n-1}|)+M_1|\lambda_n-\lambda_{n-1}|.\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_{n\to\infty}(\|T_ny_n-T_{n-1}y_{n-1}\|-\|x_n-x_{n-1}\|)\\ &\le \limsup_{n\to\infty}(\tfrac{M_2}{1-\gamma_n(1-t_n)}(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|\\ &\quad+|t_n-t_{n-1}|+|\lambda_n-\lambda_{n-1}|)+M_1|\lambda_n-\lambda_{n-1}|).\end{aligned}$$

It follows from the conditions on the parameters that

$$\limsup_{n\to\infty}(\|T_ny_n-T_{n-1}y_{n-1}\|-\|x_n-x_{n-1}\|)\le 0.$$

Lemma 2.7 sends us to $\lim_{n\to\infty}\|x_n-T_ny_n\|=0$. Hence

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

Next, one lets $\bar{p}:=\Pi_C(I-\mu_2A_2)p$. From $v_n=\Pi_C(I-\mu_1A_1)u_n$ and $u_n=\Pi_C(I-\mu_2A_2)y_n$, one gets $Gy_n=v_n$. From the sunny nonexpansive retraction and the mapping A_2, one has

$$\begin{aligned}\|u_n-\bar{p}\|^q &= \|\Pi_C(I-\mu_2A_2)p-\Pi_C(I-\mu_2A_2)y_n\|^q\\ &\le \|(I-\mu_2A_2)p-(I-\mu_2A_2)y_n\|^q\\ &\le \|y_n-p\|^q+\mu_2(\kappa_q\mu_2^{q-1}-a_2q)\|A_2y_n-A_2p\|^q.\end{aligned} \tag{3.5}$$

One also has

$$\|v_n-p\|^q\le\|u_n-\bar{p}\|^q+\mu_1(\kappa_q\mu_1^{q-1}-a_1q)\|A_1u_n-A_1\bar{p}\|^q,$$

which together with (3.5) leads to

$$\begin{aligned}\|v_n-p\|^q &\le \|y_n-p\|^q-\mu_1({}_1q-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q\\ &\quad-\mu_2(a_2q-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q.\end{aligned}$$

Using Lemma 2.9 and $Gy_n=v_n$, one has

$$\|z_n-p\|^q\le(1-t_n)\|v_n-p\|^q+t_n\|x_n-p\|^q.$$

This yields that

$$\begin{aligned}\|y_n-p\|^q &= \|\alpha_n(f(p)-p)+\alpha_n(f(x_n)-f(p))+\beta_n(x_n-p)+\gamma_n(T_nz_n-p)\|^q\\ &\le q\alpha_n\langle f(p)-p,J_q(y_n-p)\rangle+\|\alpha_n(f(x_n)-f(p))+\beta_n(x_n-p)+\gamma_n(T_nz_n-p)\|^q\\ &\le q\alpha_n\langle f(p)-p,J_q(y_n-p)\rangle+\alpha_n\|f(x_n)-f(p)\|^q+\beta_n\|x_n-p\|^q+\gamma_n\|T_nz_n-p\|^q\\ &\le \alpha_n\delta^q\|x_n-p\|^q+\beta_n\|x_n-p\|^q+\gamma_n(1-t_n)\|v_n-p\|^q+t_n\gamma_n\|x_n-p\|^q\\ &\quad +q\alpha_n\|f(p)-p\|\|y_n-p\|^{q-1}\\ &\le (\alpha_n\delta+\beta_n+\gamma_nt_n)\|x_n-p\|^q+\gamma_n(1-t_n)\big(\|y_n-p\|^q-\mu_2(a_2q-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q\\ &\quad -\mu_1(a_1q-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q\big)+q\alpha_n\|f(p)-p\|\|y_n-p\|^{q-1}.\end{aligned}$$

One re-write as follows

$$\begin{aligned}&\|y_n-p\|^q\\ &\le (1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^q+\tfrac{\alpha_nq}{1-\gamma_n(1-t_n)}\|f(p)-p\|\|y_n-p\|^{q-1}\\ &\quad -\tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(a_2q-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q+\mu_1(a_1q-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q].\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_{n+1}-p\|^q\\ &\le (1-\delta_n)\|y_n-p\|^q+\delta_n\|x_n-p\|^q\\ &\le \delta_n\|x_n-p\|^q+(1-\delta_n)\Big((1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^q-\tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}\big(\mu_2(a_2q-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q\\ &\quad +\mu_1(a_1q-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q\big)+\tfrac{\alpha_nq}{1-\gamma_n(1-t_n)}\|f(p)-p\|\|y_n-p\|^{q-1}\Big)\\ &= (1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^q-\tfrac{\gamma_n(1-\delta_n)(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(a_2q-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q\\ &\quad +\mu_1(a_1q-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q]+\tfrac{q(1-\delta_n)\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\|\|y_n-p\|^{q-1}\\ &\le \|x_n-p\|^q-\tfrac{(1-\delta_n)\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q\\ &\quad +\mu_1(q\alpha_1-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q]+\alpha_nM_3,\end{aligned}$$

where M_3 is an appropriate constant. Using Lemma 2.6, one has

$$\begin{aligned}&\tfrac{(1-\delta_n)\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2-\kappa_q\mu_2^{q-1})\|A_2y_n-A_2p\|^q+\mu_1(q\alpha_1-\kappa_q\mu_1^{q-1})\|A_1u_n-A_1\bar{p}\|^q]\\ &\le \|x_n-p\|^q-\|x_{n+1}-p\|^q+\alpha_nM_3\\ &\le q\|x_n-x_{n+1}\|\|x_{n+1}-p\|^{q-1}+\kappa_q\|x_n-x_{n+1}\|^q+\alpha_nM_3.\end{aligned}$$

From the conditions $a_1q>\kappa_q\mu_1^{q-1}>0$ and $a_2q>\kappa_q\mu_1^{q-1}>0$, $\lim_{n\to\infty}\alpha_n=0$, $\liminf_{n\to\infty}(1-\delta_n)>0$ and $\liminf_{n\to\infty}\gamma_n(1-t_n)>0$, one has

$$\lim_{n\to\infty}\|A_1u_n-A_1\bar{p}\|=\lim_{n\to\infty}\|A_2y_n-A_2p\|=0. \tag{3.6}$$

Since Π_C is sunny nonexpansive, one has

$$\begin{aligned}\|u_n-\bar{p}\|^2 &= \|\Pi_C(I-\mu_2A_2)y_n-\Pi_C(I-\mu_2A_2)p\|^2\\ &\le \langle (I-\mu_2A_2)y_n-(I-\mu_2A_2)p,J(u_n-\bar{p})\rangle\\ &= \langle y_n-p,J(u_n-\bar{p})\rangle+\mu_2\langle A_2p-A_2y_n,J(u_n-\bar{p})\rangle\\ &\le \mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+\tfrac{1}{2}(\|y_n-p\|^2+\|u_n-\bar{p}\|^2-g_1(\|y_n-u_n-(p-\bar{p})\|)),\end{aligned}$$

which implies that

$$\|u_n-\bar{p}\|^2\le 2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|-g_1(\|y_n-u_n-(p-\bar{p})\|)+\|y_n-p\|^2. \tag{3.7}$$

In the same approach, we derive

$$\|v_n-p\|^2\le 2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|-g_2(\|u_n-v_n+(p-\bar{p})\|)+\|u_n-\bar{p}\|^2. \tag{3.8}$$

Substituting (3.7) into (3.8) yields that

$$\begin{aligned}\|v_n-p\|^2 &\leq \|y_n-p\|^2+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\\ &\quad -g_1(\|y_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|).\end{aligned}$$

Utilizing Lemma 2.5, one has

$$\begin{aligned}\|z_n-p\|^2 &\leq t_n\|x_n-p\|^2+(1-t_n)\|Gy_n-p\|^2-t_n(1-t_n)g_3(\|x_n-Gy_n\|)\\ &\leq t_n\|x_n-p\|^2+(1-t_n)\|v_n-p\|^2-t_n(1-t_n)g_3(\|x_n-Gy_n\|),\end{aligned}$$

and hence

$$\begin{aligned}&\|y_n-p\|^2\\ &\leq 2\alpha_n\langle f(p)-p,J(y_n-p)\rangle+\|\alpha_n(f(x_n)-f(p))+\beta_n(x_n-p)+\gamma_n(T_nz_n-p)\|^2\\ &\leq 2\alpha_n\langle f(p)-p,J(y_n-p)\rangle+\alpha_n\|f(x_n)-f(p)\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|T_nz_n-p\|^2\\ &\quad -\beta_n\gamma_ng_4(\|x_n-T_nz_n\|)\\ &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[(1-t_n)\|v_n-p\|^2+t_n\|x_n-p\|^2\\ &\quad -t_n(1-t_n)g_3(\|x_n-Gy_n\|)]+2\alpha_n\|f(p)-p\|\|y_n-p\|-\beta_n\gamma_ng_4(\|x_n-T_nz_n\|)\\ &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\{t_n\|x_n-p\|^2+(1-t_n)[\|y_n-p\|^2\\ &\quad -g_1(\|y_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|)+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|\\ &\quad +2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|]-t_n(1-t_n)g_3(\|x_n-Gy_n\|)\}+2\alpha_n\|f(p)-p\|\|y_n-p\|\\ &\quad -\beta_n\gamma_ng_4(\|x_n-T_nz_n\|)\\ &\leq (\alpha_n\delta+\beta_n+\gamma_nt_n)\|x_n-p\|^2+\gamma_n(1-t_n)\|y_n-p\|^2-\gamma_n(1-t_n)[g_1(\|y_n-u_n-(p-\bar{p})\|)\\ &\quad +g_2(\|u_n-v_n+(p-\bar{p})\|)]+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\\ &\quad +2\alpha_n\|f(p)-p\|\|y_n-p\|-\gamma_nt_n(1-t_n)g_3(\|x_n-Gy_n\|)-\beta_n\gamma_ng_4(\|x_n-T_nz_n\|).\end{aligned}$$

So, one has

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq (1-\delta_n)\|y_n-p\|^2+\delta_n\|x_n-p\|^2\\ &\leq \delta_n\|x_n-p\|^2+(1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^2-\tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[g_1(\|y_n-u_n-(p-\bar{p})\|)\\ &\quad +g_2(\|u_n-v_n+(p-\bar{p})\|)]+\tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|\\ &\quad +\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\alpha_n\|f(p)-p\|\|y_n-p\|]\\ &\quad -\tfrac{1}{1-\gamma_n(1-t_n)}[\gamma_nt_n(1-t_n)g_3(\|x_n-Gy_n\|)+\beta_n\gamma_ng_4(\|x_n-T_nz_n\|)]\}\\ &\leq (1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^2-\tfrac{1-\delta_n}{1-\gamma_n(1-t_n)}[\gamma_n(1-t_n)(g_1(\|y_n-u_n-(p-\bar{p})\|)\\ &\quad +g_2(\|u_n-v_n+(p-\bar{p})\|))+\gamma_nt_n(1-t_n)g_3(\|x_n-Gy_n\|)+\beta_n\gamma_ng_4(\|x_n-T_nz_n\|)]\\ &\quad +\tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\\ &\quad +\alpha_n\|f(p)-p\|\|y_n-p\|]\\ &\leq \|x_n-p\|^2-\tfrac{1-\delta_n}{1-\gamma_n(1-t_n)}[\gamma_n(1-t_n)(g_1(\|y_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|))\\ &\quad +\gamma_nt_n(1-t_n)g_3(\|x_n-Gy_n\|)+\beta_n\gamma_ng_4(\|x_n-T_nz_n\|)]+\tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|u_n-\bar{p}\|\|A_2p-A_2y_n\|\\ &\quad\quad +\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\alpha_n\|f(p)-p\|\|y_n-p\|].\end{aligned}$$

This further implies that

$$\begin{aligned}
&\tfrac{1-\delta_n}{1-\gamma_n(1-t_n)}[\gamma_n(1-t_n)(g_1(\|y_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|))\\
&\quad+\gamma_n t_n(1-t_n)g_3(\|x_n-Gy_n\|)+\beta_n\gamma_n g_4(\|x_n-T_nz_n\|)]\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|\\
&\quad+\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\alpha_n\|f(p)-p\|\|y_n-p\|]\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|\\
&\quad+\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\alpha_n\|f(p)-p\|\|y_n-p\|].
\end{aligned}$$

Since $\liminf_{n\to\infty}\gamma_n t_n(1-t_n)>0$, $\liminf_{n\to\infty}(1-\delta_n)>0$ and $\liminf_{n\to\infty}\beta_n\gamma_n>0$, one gets that

$$\lim_{n\to\infty} g_1(\|y_n-u_n-(p-\bar{p})\|)=0,$$
$$\lim_{n\to\infty} g_2(\|u_n-v_n+(p-\bar{p})\|)=0,$$
$$\lim_{n\to\infty} g_3(\|x_n-Gy_n\|)=0$$

and

$$\lim_{n\to\infty} g_4(\|x_n-T_nz_n\|)=0.$$

Using the property of g_1, g_2, g_3 and g_3, one has

$$\lim_{n\to\infty}\|y_n-u_n-(p-\bar{p})\|=\lim_{n\to\infty}\|u_n-v_n+(p-\bar{p})\|=0$$

and

$$\lim_{n\to\infty}\|x_n-Gy_n\|=\lim_{n\to\infty}\|x_n-T_nz_n\|=0.$$

These send us to

$$\|y_n-Gy_n\|=\|y_n-v_n\|\le\|u_n-v_n+(p-\bar{p})\|+\|y_n-u_n-(p-\bar{p})\|\to 0\quad(n\to\infty).\tag{3.9}$$

Observe $\gamma_n(T_nz_n-x_n)+\alpha_n(f(x_n)-x_n)=y_n-x_n$ and

$$\|y_n-x_n\|\le\alpha_n\|f(x_n)-x_n\|+\|T_nz_n-x_n\|\to 0\ (n\to\infty).$$

(3.9) leads to

$$\begin{aligned}
\|x_n-Gx_n\| &\le \|x_n-y_n\|+\|y_n-Gy_n\|+\|Gy_n-Gx_n\|\\
&\le 2\|x_n-y_n\|+\|y_n-Gy_n\|\to 0\quad(n\to\infty).
\end{aligned}$$

Next, one shows that $\lim_{n\to\infty}\|x_n-T_\lambda x_n\|=\lim_{n\to\infty}\|x_n-Wx_n\|=0$, where $T_\lambda=J_\lambda^B(I-\lambda A)$ and $Wx=\theta Gx+(1-\theta)T_\lambda x$, $\forall x\in C$, where θ is a number in $(0,1)$. Observe that

$$\begin{aligned}
\|T_ny_n-T_\lambda y_n\| &\le |\lambda_n-\lambda|\|Ay_n\|+|1-\tfrac{\lambda}{\lambda_n}|\|J_{\lambda_n}^B(I-\lambda_nA)y_n-(I-\lambda_nA)y_n\|\\
&=|\lambda_n-\lambda|\|Ay_n\|+|1-\tfrac{\lambda}{\lambda_n}|\|T_ny_n-(I-\lambda_nA)y_n\|.
\end{aligned}$$

Since $\{y_n\},\{T_ny_n\}$, and $\{Ay_n\}$ are bounded and $\lim_{n\to\infty}\lambda_n=\lambda$, we get

$$\lim_{n\to\infty}\|T_ny_n-T_\lambda y_n\|=0.\tag{3.10}$$

Since $0<\bar{\lambda}\le\lambda_n$ and $\lim_{n\to\infty}\lambda_n=\lambda<(\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, one has $0<\bar{\lambda}\le\lambda<(\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$. Using Lemma 2.10 yields that $\mathrm{Fix}(T_\lambda)=(A+B)^{-1}0$ and $T_\lambda:C\to C$ is nonexpansive. It follows from (3.10) and $x_n-y_n\to 0$ that

$$\begin{aligned}
\|T_\lambda x_n-x_n\| &\le \|T_\lambda x_n-T_\lambda y_n\|+\|T_\lambda y_n-T_ny_n\|+\|T_ny_n-x_n\|\\
&\le \|x_n-y_n\|+\|T_\lambda y_n-T_ny_n\|+\|T_ny_n-x_n\|\to 0\quad(n\to\infty).
\end{aligned}\tag{3.11}$$

Using Lemma 2.4, we know that $\mathrm{Fix}(W)=\mathrm{Fix}(G)\cap\mathrm{Fix}(T_\lambda)=\Omega$. Observe that

$$\begin{aligned}\|x_n-Wx_n\| &= \|\theta(x_n-Gx_n)+(1-\theta)(x_n-T_\lambda x_n)\|\\ &\le \theta\|x_n-Gx_n\|+(1-\theta)\|x_n-T_\lambda x_n\|,\end{aligned}$$

which together with (3.11) shows $\lim_{n\to\infty}\|x_n-Wx_n\|=0$.

Next, one proves that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\le 0, \tag{3.12}$$

where $x^*=$s-$\lim_{n\to\infty}x_t$ with x_t is a fixed point of the contractive mapping $x\mapsto tf(x)+(1-t)Wx$ for each $t\in(0,1)$ since f is contractive and W is nonexpansive. So, $\|x_t-x_n\|=\|(1-t)(Wx_t-x_n)+t(f(x_t)-x_n)\|$. By Lemma 2.1, we conclude that

$$\begin{aligned}\|x_t-x_n\|^2 &= \|(1-t)(Wx_t-x_n)+t(f(x_t)-x_n)\|^2\\ &\le 2t\langle f(x_t)-x_n,J(x_t-x_n)\rangle+(1-t)^2\|Wx_t-x_n\|^2\\ &\le 2t\langle f(x_t)-x_n,J(x_t-x_n)\rangle+(1-t)^2(\|Wx_n-x_n\|+\|Wx_t-Wx_n\|)^2\\ &\le 2t\langle f(x_t)-x_n,J(x_t-x_n)\rangle+(1-t)^2(\|x_t-x_n\|+\|Wx_n-x_n\|)^2\\ &= (1-t)^2[\|x_t-x_n\|^2+2\|x_t-x_n\|\|Wx_n-x_n\|+\|Wx_n-x_n\|^2]\\ &\quad +2t\langle f(x_t)-x_t,J(x_t-x_n)\rangle+2t\langle x_t-x_n,J(x_t-x_n)\rangle\\ &= (1-2t+t^2)\|x_t-x_n\|^2+2t\langle f(x_t)-x_t,J(x_t-x_n)\rangle+f_n(t)+2t\|x_t-x_n\|^2,\end{aligned}$$

where

$$f_n(t)=(1-t)^2\|Wx_n-x_n\|(2\|x_t-x_n\|+\|x_n-Wx_n\|)\to 0\quad(n\to\infty).$$

It follows that

$$2t\langle x_t-f(x_t),J(x_t-x_n)\rangle\le t^2\|x_t-x_n\|^2+f_n(t).$$

Letting $n\to\infty$ yields that

$$\limsup_{n\to\infty}\langle x_t-f(x_t),J(x_t-x_n)\rangle\le\frac{t}{2}M_4, \tag{3.13}$$

where M_4 is an appropriate positive real number. Taking $t\to 0$ in (3.13), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t-f(x_t),J(x_t-x_n)\rangle\le 0.$$

On the other hand, we have

$$\begin{aligned}&\langle f(x^*)-x^*,J(x_n-x^*)\rangle\\ &=\langle f(x_t)-x_t,J(x_n-x_t)\rangle+\langle x_t-x^*,J(x_n-x_t)\rangle\\ &\quad+\langle f(x^*)-f(x_t),J(x_n-x_t)\rangle+\langle f(x^*)-x^*,J(x_n-x^*)-J(x_n-x_t)\rangle.\end{aligned}$$

It is not hard to see that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\\ &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)-J(x_n-x_t)\rangle.\end{aligned}$$

Since X is uniformly smooth, the two limits are interchangeable. This proves that (3.12) is true. Note that $x_n-y_n\to 0$ implies $J(y_n-x^*)-J(x_n-x^*)\to 0$. Thus,

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(x^*)-x^*,J(y_n-x^*)\rangle=\limsup_{n\to\infty}\{\langle f(x^*)-x^*,J(x_n-x^*)\rangle\\ &+\langle f(x^*)-x^*,J(y_n-x^*)-J(x_n-x^*)\rangle\}=\limsup_{n\to\infty}\langle f(x^*)-x^*,J(x_n-x^*)\rangle\le 0.\end{aligned} \tag{3.14}$$

Observe from Lemma 2.1 that

$$\begin{aligned}\|y_n-x^*\|^2 &= \|\alpha_n(f(x_n)-f(x^*))+\beta_n(x_n-x^*)+\gamma_n(T_nz_n-x^*)+\alpha_n(f(x^*)-x^*)\|^2\\ &\le 2\alpha_n\langle f(x^*)-x^*,J(y_n-x^*)\rangle+\alpha_n\|f(x_n)-f(x^*)\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|z_n-x^*\|^2\\ &\le 2\alpha_n\langle f(x^*)-x^*,J(y_n-x^*)\rangle+\alpha_n\delta\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2\\ &\quad+\gamma_n(t_n\|x_n-x^*\|^2+(1-t_n)\|y_n-x^*\|^2),\end{aligned}$$

which hence yields

$$\|y_n-x^*\|^2 \le \tfrac{2\alpha_n}{1-\gamma_n(1-t_n)}\langle f(x^*)-x^*,J(y_n-x^*)\rangle+(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-x^*\|^2.$$

By the convexity of $\|\cdot\|^2$, the nonexpansivity of T_n, one has

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\le \delta_n\|x_n-x^*\|^2+(1-\delta_n)\|T_ny_n-x^*\|^2\\ &\le \delta_n\|x_n-x^*\|^2+(1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-x^*\|^2+\tfrac{2\alpha_n}{1-\gamma_n(1-t_n)}\langle f(x^*)-x^*,J(y_n-x^*)\rangle\}\\ &= [1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}]\|x_n-x^*\|^2+\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\cdot\tfrac{2\langle f(x^*)-x^*,J(y_n-x^*)\rangle}{1-\delta}.\end{aligned}$$

In view of (3.14) and Lemma 2.2, we are easy to conclude that $\|x_n-x^*\|\to 0$ as $n\to\infty$. This completes the proof. □

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