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WEAK SOLUTIONS FOR SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL INTEGRAL BOUNDARY CONDITIONS IN BANACH SPACES

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Dedicated to Professor Sotiris K. Ntouyas on the occasion of his 69 birthday

Abstract. In this paper, we study the existence of weak solutions for some nonlinear fractional differential equations with fractional integral boundary conditions involving the fractional Caputo derivative of order $1 < \alpha \leq 2$ in Banach spaces. Our main results are proved by applying the Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. In addition, an example is given to demonstrate the applications of our results.

Keywords. Fractional differential equations; Weak solution; Measure of weak noncompactness; Pettis integrals; Banach space.

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1. Introduction

The topic of fractional differential equations has recently emerged as a popular field of research due to its extensive development and applications in several disciplines such as, physics, mechanics, chemistry and engineering. For more details, we refer the reader to [1, 2, 3, 4]. Recent developments of fractional differential and integral equations are given in [5, 6, 7, 8, 9].

Many authors have studied the existence of solution of the fractional boundary value problems under various boundary conditions and by different approaches. We refer the readers to the papers [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and references therein.

Our investigation relies upon Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. This technique was introduced by De Blasi [28]. The strong measure of noncompactness was considered first by Banaŝ and Goebel [29] and subsequently developed and used

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in many papers; see, for example, Akhmerov *et al.* [30], Alvàrez [31], Belmekki and Mekhalfi [32], Benchohra, Henderson and Seba [33], Guo, Lakshmikantham and Liu [34], and the references therein. Recently, there are also many results on weak solutions of nonlinear fractional differential equations; see [33, 35, 36, 37, 38, 39] and the references therein.

In this paper, we discuss the existence of weak solutions for the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = f\big(t, x(t)\big), \quad t \in J = [0, T], \\ a_1x(0) + b_1x(T) = \lambda_1 I^{\sigma_1}x(\eta), \\ a_2\, {}^cD^{\sigma_2}_{0^+}x(\xi) + b_2\, {}^cD^{\sigma_3}_{0^+}x(\eta) = \lambda_2, \end{cases} \tag{1.1}$$

where ${}^cD^{\mu}_{0^+}$ is the Caputo fractional derivative of order $\mu \in \{\alpha, \sigma_2, \sigma_3\}$ such that $1 < \alpha \leq 2, 0 < \sigma_2, \sigma_3 \leq 1, I^{\sigma_1}$ is the Riemann–Liouville fractional integral of order $\sigma_1 > 0$ and $f : [0,1] \times E \longrightarrow E$ is a given function satisfying some assumptions that will be specified later, E is a reflexive Banach space with norm $\|\cdot\|, a_i, b_i, \lambda_i, i = 1, 2$ are real constants, and $\xi,\ \eta \in (0, T)$.

We remark that when $b_1 = T = \sigma_2 = 1, a_1 = \lambda_2 = 0$, problem (1.1) reduces to the case considered in [19] in the scalar case using the Banach contraction principal, the Schaefer's fixed point theorem and the Krasnoselskii's fixed point theorem. Here we extend the results of [19] to cover the abstract case.

The organization of this work is as follows. In Section 2, we introduce some notations, definitions and lemmas that will be used later. Section 3 treats the existence of weak solutions in Banach spaces by using the Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. Finally, we illustrate the obtained results by an example in Section 3.

2. Preliminaries

In this section, we state definitions and notations that are used in the remainder of the paper. Denote by $L^1(J)$ the Banach space of real-valued Lebesgue integrable functions, on the interval J. E denotes the real Banach space with norm $\|\cdot\|$ and dual E^* also $(E, w) = (E, \sigma(E, E^*))$ denotes the space E with its weak topology.

Let $L^\infty(J)$ be the Banach space of real-valued essentially bounded and measurable functions defined over J equipped with the norm $\|\cdot\|_{L^\infty}$. $C(J, E)$ is the Banach space of continuous functions $x : J \to E$, with the usual supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\|,\ t \in J\}.$$

Definition 2.1. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E (i.e. for any $(x_n)_n$ in E with $x_n \to x$ in (E, w), $h(x_n) \to h(x)$ in (E, w)).

Definition 2.2 ([40])**.** The function $x : J \to E$ is said to be Pettis integrable on J if and only if there is an element $x_I \in E$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in E^*$, where the integral on the right is supposed to exist in the sense of Lebesgue.

We have $x_I = \int_I x(s)ds$. Let $P(J, E)$ be the space of all E-valued Pettis integrable functions in the interval J.

Proposition 2.3 ([40])**.** *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$is Pettis integrable.*

Definition 2.4 ([28]). Let E be a Banach space. Let Ω_E be the bounded subsets of E and let B_1 be the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0,\infty)$ defined by

$$\beta(X) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : X \subset \varepsilon B_1 + \Omega\}.$$

Property 2.5. *The De Blasi measure of noncompactness satisfies some properties for more details (see [28])*

(a) $A \subset B \Leftrightarrow \beta(A) \leq \beta(B)$,
(b) $\beta(A) = 0 \Rightarrow A$ *is relatively weakly compact,*
(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,
(d) $\beta(\overline{A}^w) = \beta(A)$, *where* $\overline{A}^w$ *denotes the weak closure of* A,
(e) $\beta(A+B) \leq \beta(A) + \beta(B)$,
(f) $\beta(\lambda A) = |\lambda|\beta(A)$,
(g) $\beta(conv(A)) = \beta(A)$,
(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The following result directly follows from the Hahn-Banach theorem.

Proposition 2.6. *Let E be a normed space with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

Let us now recall the definitions of the Pettis integral and Caputo derivative of fractional order.

Definition 2.7. [41] Let $h : J \to E$ be a function. The fractional Pettis integral of the function h of order $\alpha > 0$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

where the sign $\int$ denotes the Pettis integral and Γ is the Gamma function.

Definition 2.8. [1] For a function $h : J \to E$, the Caputo fractional-order derivative of h is defined by

$${}^cD^\alpha_{0^+} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

Lemma 2.9 ([1]). *If $\alpha > 0$, then the differential equation*

$$\left({}^cD^\alpha_{0^+} f\right)(t) = 0,$$

has solutions

$$f(t) = \sum_{j=0}^{m-1} c_j t^j, \quad c_j \in \mathbb{R}, j = 0 \dots m-1,$$

where $m = [\alpha] + 1$.

Lemma 2.10 ([1]). *If $\alpha > 0$, then*

$$I^\alpha_{0^+}\, {}^cD^\alpha_{0^+} f(t) = f(t) + \sum_{j=0}^{m-1} c_j t^j,$$

for some $c_j \in \mathbb{R}$, $j = 0, 1, 2, \dots, m-1$, where $m = [\alpha] + 1$.

Theorem 2.11 ([42]). *Let D be a closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(J,E)$ such that $0 \in D$. Assume that $N : D \to D$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup N(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{2.1}$$

holds for every subset $V \subset D$, then N has a fixed point.

Lemma 2.12 ([34]). *Let $H \subset C(J,E)$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on J, and*

$$\beta_C(H) = \max_{t\in J} \beta(H(t)),$$

and

$$\beta\Big(\int_J u(s)ds\Big) \leq \int_J \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H, s \in J\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

3. Main Results

Definition 3.1. By a weak solution of (1.1), we mean a function $x : I \to E$ such that the weak fractional derivative ${}^cD^{\alpha}_{0^+}x(t)$ exists and is weakly continuous and satisfies problem (1.1).

Lemma 3.2. *Let $1 < \alpha \leq 2$ and h be continuous function on $J := [0,T]$. Then the linear problem*

$${}^cD^{\alpha}_{0^+}x(t) = h(t), \tag{3.1}$$

with boundary conditions

$$a_1x(0) + b_1x(T) = \lambda_1 I^{\sigma_1}x(\eta), \;\; a_2\, {}^cD^{\sigma_2}_{0^+}x(\xi) + b_2\, {}^cD^{\sigma_3}_{0^+}x(\eta) = \lambda_2. \tag{3.2}$$

has a unique solution given by

$$\begin{aligned} x(t) &= I^{\alpha}h(t) + \frac{1}{v_0}\Big\{\lambda_1 I^{\alpha+\sigma_1}h(\eta) - b_1 I^{\alpha}h(T)\Big\} \\ &+ \frac{v_1}{v_0v_2}\Big\{a_2I^{\alpha-\sigma_2}h(\xi) + b_2I^{\alpha-\sigma_3}h(\eta) - \lambda_2\Big\} \\ &+ \frac{t}{v_2}\Big\{\lambda_2 - \big(a_2I^{\alpha-\sigma_2}h(\xi) + b_2I^{\alpha-\sigma_3}h(\eta)\big)\Big\}, \end{aligned} \tag{3.3}$$

where

$$v_0 = a_1 + b_1 - \frac{\lambda_1\eta^{\sigma_1}}{\Gamma(\sigma_1+1)},\; v_1 = b_1T - \frac{\lambda_1\eta^{\sigma_1+1}}{\Gamma(\sigma_1+2)},\; v_2 = \frac{a_2\xi^{1-\sigma_2}}{\Gamma(2-\sigma_2)} + \frac{b_2\eta^{1-\sigma_3}}{\Gamma(2-\sigma_3)}, \tag{3.4}$$

and $v_0v_2 \neq 0$.

Proof. By applying Lemma 2.10, we may reduce (3.1) to an equivalent integral equation

$$x(t) = I^{\alpha}h(t) - c_0 - c_1t, \quad c_0, c_1 \in \mathbb{R}. \tag{3.5}$$

Applying the boundary conditions (3.2) in (3.5), we obtain

$$I^{\sigma_1}x(\eta) = I^{\sigma_1+\alpha}h(\eta) - c_0\frac{\eta^{\sigma_1}}{\Gamma(\sigma_1+1)} - c_1\frac{\eta^{\sigma_1+1}}{\Gamma(\sigma_1+2)},$$

$${}^cD^{\sigma_i}_{0^+}x(t) = I^{\alpha-\sigma_i}h(t) - c_1\frac{\Gamma(2)}{\Gamma(2-\sigma_i)}t^{1-\sigma_i},\; i = 2,3.$$

After collecting the similar terms in one part, we have the following equations:

$$\Big(a_1+b_1-\frac{\lambda_1\eta^{\sigma_1}}{\Gamma(\sigma_1+1)}\Big)c_0+\Big(b_1T-\frac{\lambda_1\eta^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\Big)c_1=b_1I^{\alpha}h(T)-\lambda_1I^{\sigma_1+\alpha}h(\eta) \tag{3.6}$$

and

$$-\Big(\frac{a_2\xi^{1-\sigma_2}}{\Gamma(2-\sigma_2)}+\frac{b_2\eta^{1-\sigma_3}}{\Gamma(2-\sigma_3)}\Big)c_1=\lambda_2-\big(a_2I^{\alpha-\sigma_2}h(\xi)+b_2I^{\alpha-\sigma_3}h(\eta)\big). \tag{3.7}$$

Rewriting equations (3.6) and (3.7) by using (3.4), we obtain

$$\begin{aligned} v_0c_0+v_1c_1&=b_1I^{\alpha}h(T)-\lambda_1I^{\sigma_1+\alpha}h(\eta)\\ -v_2c_1&=\lambda_2-\big(a_2I^{\alpha-\sigma_2}h(\xi)+b_2I^{\alpha-\sigma_3}h(\eta)\big). \end{aligned} \tag{3.8}$$

Solving (3.8), we find that

$$c_1=\frac{-\lambda_2+\big(a_2I^{\alpha-\sigma_2}h(\xi)+b_2I^{\alpha-\sigma_3}h(\eta)\big)}{v_2}$$

and

$$c_0=\frac{b_1I^{\alpha}h(T)-\lambda_1I^{\sigma_1+\alpha}h(\eta)}{v_0}+\frac{v_1}{v_0v_2}\Big\{\lambda_2-a_2I^{\alpha-\sigma_2}h(\xi)+b_2I^{\alpha-\sigma_3}h(\eta)\Big\}.$$

Substituting the value of c_0,c_1 in (3.5), we get (3.3). □

In order to present and prove our main results, we consider the following hypotheses:

(H1) For each $t\in J$, the function $f(t,\cdot)$ is weakly sequentially continuous;

(H2) For each $x\in C(J,E)$, the function $f(\cdot,x(\cdot))$ is Pettis integrable on J;

(H3) There exist $p\in L^{\infty}(J)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\longrightarrow(0,+\infty)$ such that

$$\|f(t,x(t))\|\le p(t)\psi(\|x\|);$$

(H4) There exists a constant $R>0$ such that

$$\|p\|_{L^{\infty}}\psi(R)M+\frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0v_2|}\le R, \tag{3.9}$$

where

$$\begin{aligned} M=&\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|b_1|T^{\alpha}}{|v_0|\Gamma(\alpha+1)}+\frac{|\lambda_1|}{|v_0|\Gamma(\alpha+\sigma_1+1)}\eta^{\alpha+\sigma_1}\\ &+\frac{|a_2|(|v_1|+|v_0|T)\xi^{\alpha-\sigma_2}}{|v_0v_2|\Gamma(\alpha-\sigma_2+1)}+\frac{|b_2|(|v_1|+|v_0|T)\eta^{\alpha-\sigma_3}}{|v_0v_2|\Gamma(\alpha-\sigma_3+1)}; \end{aligned}$$

(H5) For each bounded set $D\subset E$, and each $t\in J$, the following inequality holds

$$\beta(f(t,D))\le p(t)\beta(D).$$

Now we are able to establish the main result.

Theorem 3.3. *Assume that assumptions (H1)-(H5) hold. If*

$$\|p\|_{L^{\infty}}M<1, \tag{3.10}$$

then boundary value problem (1.1) has at least one solution.

Proof. Transform boundary value problem (1.1) into a fixed point equation and consider the operator $\mathscr{N} : C(J,E) \longrightarrow C(J,E)$ defined by

$$\begin{aligned}\mathscr{N}x(t) &= I^{\alpha} f\big(s,x(s)\big)(t) + \frac{1}{v_0}\Big\{\lambda_1 I^{\alpha+\sigma_1} f\big(s,x(s)\big)(\eta) - b_1 I^{\alpha} f\big(s,x(s)\big)(T)\Big\}\\ &+ \frac{v_1}{v_0 v_2}\Big\{\big(a_2 I^{\alpha-\sigma_2} f\big(s,x(s)\big)(\xi) + b_2 I^{\alpha-\sigma_3} f\big(s,x(s)\big)(\eta)\big) - \lambda_2\Big\}\\ &+ \frac{t}{v_2}\Big\{\lambda_2 - a_2 I^{\alpha-\sigma_2} f\big(s,x(s)\big)(\xi) + b_2 I^{\alpha-\sigma_3} f\big(s,x(s)\big)(\eta)\Big\}. \end{aligned} \tag{3.11}$$

For $x \in C(J,E)$, we have $f(\cdot,x(\cdot)) \in P(J,E)$ (assumption (H2)). Since $\frac{(t-\cdot)^{\alpha-1}}{\Gamma(\alpha)}$, $\frac{(T-\cdot)^{\alpha-1}}{\Gamma(\alpha)}$, $\frac{(\eta-\cdot)^{\alpha+\sigma_1-1}}{\Gamma(\alpha+\sigma_1)}$, $\frac{(\xi-\cdot)^{\alpha-\sigma_2-1}}{\Gamma(\alpha-\sigma_2)}$, $\frac{(\eta-\cdot)^{\alpha-\sigma_3-1}}{\Gamma(\alpha-\sigma_3)}$ are $\in L^{\infty}(J)$, one has $\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(\cdot,x(\cdot))$, $\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(\cdot,x(\cdot))$, $\frac{(\eta-s)^{\alpha+\sigma_1-1}}{\Gamma(\alpha+\sigma_1)} f(\cdot,x(\cdot))$, $\frac{(\xi-s)^{\alpha-\sigma_2-1}}{\Gamma(\alpha-\sigma_2)} f(\cdot,x(\cdot))$, $\frac{(\eta-s)^{\alpha-\sigma_3-1}}{\Gamma(\alpha-\sigma_3)} f(\cdot,x(\cdot))$ for all $t \in J$ are Pettis integrable (Proposition 2.3). Thus $\mathscr{N}$ is well defined. Let $R > 0$, and consider the set

$$\begin{aligned} D = \Big\{x \in C(J,E) : \|x\|_{\infty} \le R, \|x(t_2) - x(t_1)\| \le \|p\|_{L^{\infty}} \psi(R)\Big(\frac{t_2^{\alpha} - t_1^{\alpha}}{\Gamma(\alpha+1)}\\ + \frac{t_2 - t_1}{|v_2|}\Big[\frac{|a_2|\xi^{\alpha-\sigma_2}}{\Gamma(\alpha-\sigma_2+1)} + \frac{|b_2|\eta^{\alpha-\sigma_3}}{\Gamma(\alpha-\sigma_3+1)}\Big]\Big)\\ + \Big|\frac{\lambda_2}{v_2}\Big|(t_2 - t_1), \text{ for } t_1, t_2 \in J\Big\}, \end{aligned}$$

where R satisfies inequality (3.9). Notice that D is a closed, convex, bounded, and equicontinuous subset of $C(J,E)$. We shall show that $\mathscr{N}$ satisfies all the assumptions of Theorem 2.11. The proof will be given in several steps.

Step 1. We show that $\mathscr{N}$ maps D into D.

Take $x \in D, t \in J$ and assume that $\mathscr{N}x(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|\mathscr{N}x(t)\| = \varphi(\mathscr{N}x(t))$. Thus

$$\begin{aligned}\|\mathscr{N}x(t)\| &= \varphi\Big(I^{\alpha} f\big(s,x(s)\big)(t) + \frac{1}{v_0}\Big\{\lambda_1 I^{\alpha+\sigma_1} f\big(s,x(s)\big)(\eta) - b_1 I^{\alpha} f\big(s,x(s)\big)(T)\Big\}\\ &+ \frac{v_1}{v_0 v_2}\Big\{\big(a_2 I^{\alpha-\sigma_2} f\big(s,x(s)\big)(\xi) + b_2 I^{\alpha-\sigma_3} f\big(s,x(s)\big)(\eta)\big) - \lambda_2\Big\}\\ &+ \frac{t}{v_2}\Big\{\lambda_2 - a_2 I^{\alpha-\sigma_2} f\big(s,x(s)\big)(\xi) + b_2 I^{\alpha-\sigma_3} f\big(s,x(s)\big)(\eta)\Big\}\Big)\\ &\le I^{\alpha}\varphi\big(f(s,x(s))\big)(t) + \frac{|b_1|}{|v_0|} I^{\alpha}\varphi\big(f(s,x(s))\big)(T) + \frac{|\lambda_1|}{|v_0|} I^{\alpha+\sigma_1}\varphi\big(f(s,x(s))\big)(\eta)\\ &+ \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} + \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_2}\varphi\big(f(s,x(s))\big)(\xi)\\ &+ \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_3}\varphi\big(f(s,x(s))\big)(\eta). \end{aligned}$$

Using hypothesis (H3), we get

$$\begin{aligned}
\|\mathscr{N}x(t)\| &\le \psi(\|x\|)\Big\{I^{\alpha}p(s)(T)+\frac{|b_1|}{|v_0|}\,I^{\alpha}p(s)(T)+\frac{|\lambda_1|}{|v_0|}I^{\alpha+\sigma_1}p(s)(\eta)\\
&\quad+\frac{|a_2|(|v_1|+|v_0|T)}{|v_0v_2|}I^{\alpha-\sigma_2}p(s)(\xi)+\frac{|b_2|(|v_1|+|v_0|T)}{|v_0v_2|}I^{\alpha-\sigma_3}p(s)(\eta)\Big\}\\
&\quad+\frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0v_2|}\\
&\le \|p\|_{L^\infty}\psi(R)\Big\{I^{\alpha}(1)(T)+\frac{|b_1|}{|v_0|}I^{\alpha}(1)(T)+\frac{|\lambda_1|}{|v_0|}I^{\alpha+\sigma_1}(1)(\eta)\\
&\quad+\frac{|a_2|(|v_1|+|v_0|T)}{|v_0v_2|}I^{\alpha-\sigma_2}(1)(\xi)+\frac{|b_2|(|v_1|+|v_0|T)}{|v_0v_2|}I^{\alpha-\sigma_3}(1)(\eta)\Big\}\\
&\quad+\frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0v_2|}\\
&\le \|p\|_{L^\infty}\psi(R)\Big\{\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|b_1|T^{\alpha}}{|v_0|\Gamma(\alpha+1)}+\frac{|\lambda_1|}{|v_0|\Gamma(\alpha+\sigma_1+1)}\eta^{\alpha+\sigma_1}\\
&\quad+\frac{a_2(|v_1|+|v_0|T)\xi^{\alpha-\sigma_2}}{|v_0v_2|\Gamma(\alpha-\sigma_2+1)}+\frac{|b_2|(|v_1|+|v_0|T)\eta^{\alpha-\sigma_3}}{|v_0v_2|\Gamma(\alpha-\sigma_3+1)}\Big\}+\frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0v_2|}\\
&= \|p\|_{L^\infty}\psi(R)M+\frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0v_2|}\\
&\le R.
\end{aligned}$$

Next, let $t_1,t_2\in J, t_1<t_2, x\in D$, so $\mathscr{N}x(t_2)-\mathscr{N}x(t_1)\neq 0$. Then there exists $\varphi\in E^*$ such that

$$\|\mathscr{N}(x)(t_2)-\mathscr{N}(x)(t_1)\|=\varphi(\mathscr{N}(x)(t_2)-\mathscr{N}(x)(t_1)).$$

Hence,

$$\begin{aligned}
&\|\mathscr{N}(x)(t_2)-\mathscr{N}(x)(t_1)\|\\
&\le I^{\alpha}\varphi\big(f(s,x(s))(t_2)-f(s,x(s))(t_1)\big)\\
&+\frac{t_2-t_1}{|v_2|}\Big\{|a_2|I^{\alpha-\sigma_2}\varphi\big(f(s,x(s))\big)(\xi)+|b_2|I^{\alpha-\sigma_3}\varphi\big(f(s,x(s))\big)(\eta)+|\lambda_2|\Big\},\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big]\varphi\big(f(s,x(s))\big)ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi\big(f(s,x(s))\big)ds\\
&+\frac{t_2-t_1}{|v_2|}\Big\{|a_2|I^{\alpha-\sigma_2}\varphi\big(f(s,x(s))\big)(\xi)+|b_2|I^{\alpha-\sigma_3}\varphi\big(f(s,x(s))\big)(\eta)+|\lambda_2|\Big\},\\
&\le \frac{\|p\|_{L^\infty}\psi(R)}{\Gamma(\alpha)}\Big\{\int_0^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\Big\}\\
&+\frac{(t_2-t_1)\|p\|_{L^\infty}\psi(R)}{|v_2|}\Big\{|a_2|I^{\alpha-\sigma_2}(1)(\xi)+|b_2|I^{\alpha-\sigma_3}(1)(\eta)\Big\}+\left|\frac{\lambda_2}{v_2}\right|(t_2-t_1),\\
&\le \|p\|_{L^\infty}\psi(R)\Big(\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}+\frac{t_2-t_1}{|v_2|}\Big[\frac{|a_2|\xi^{\alpha-\sigma_2}}{\Gamma(\alpha-\sigma_2+1)}+\frac{|b_2|\eta^{\alpha-\sigma_3}}{\Gamma(\alpha-\sigma_3+1)}\Big]\Big)+\left|\frac{\lambda_2}{v_2}\right|(t_2-t_1).
\end{aligned}$$

It follows that $\mathscr{N}(D)\subset D$.

Step 2. We show that $\mathscr{N}$ is weakly sequentially continuous.

Let (x_n) be a sequence in D and let $x_n(t) \to x(t)$ in (E,w) for each $t \in J$. Fix $t \in J$. Since f satisfies assumption (H1), we have that $f(t,x_n(t))$ converges weakly to $f(t,x(t))$. Hence the Lebesgue dominated convergence theorem for the Pettis integral implies that $\mathcal{N}x_n(t)$ converges weakly to $\mathcal{N}x(t)$ in (E,w). We do it for each $t \in J$, so $\mathcal{N}x_n \to Nx$. Then $\mathcal{N} : D \longrightarrow D$ is weakly sequentially continuous.

Step 3. The implication (2.1) holds.

Now let V be a bounded and equicontinuous subset of D. Hence $t \mapsto v(t) = \beta(V(t))$ is continuous on J such that $V \subset \overline{conv}(\{0\} \cup \mathcal{N}(V))$. Clearly, $V(t) \subset \overline{conv}(\{0\} \cup \mathcal{N}(V))$ for al $t \in J$. Hence $\mathcal{N}V(t) \subset \mathcal{N}D(t)$, $t \in J$ is bounded in E. By assumption (H5), and the properties of measure β, we have, for each $t \in J$,

$$
\begin{aligned}
v(t) &\le \beta\left(\overline{conv}\left(\mathcal{N}(V)(t) \cup \{0\}\right)\right) \le \beta\left(\mathcal{N}(V)(t)\right) \\
&\le \beta\Big(I^{\alpha} f\big(s,V(s)\big)(t) + \frac{1}{v_0}\Big\{\lambda_1 I^{\alpha+\sigma_1} f\big(s,V(s)\big)(\eta) - b_1 I^{\alpha} f\big(s,V(s)\big)(T)\Big\} \\
&+ \frac{|v_1|}{|v_0 v_2|}\Big\{\big(a_2 I^{\alpha-\sigma_2} f\big(s,V(s)\big)(\xi) + b_2 I^{\alpha-\sigma_3} f\big(s,V(s)\big)(\eta)\big) - \lambda_2\Big\} \\
&+ \frac{t}{|v_2|}\Big\{\lambda_2 - a_2 I^{\alpha-\sigma_2} f\big(s,V(s)\big)(\xi) + b_2 I^{\alpha-\sigma_3} f\big(s,V(s)\big)(\eta)\Big\}\Big), \\
&\le I^{\alpha}\beta\Big(f\big(s,V(s)\big)\Big)(t) + \frac{|b_1|}{|v_0|}\, I^{\alpha}\beta\Big(f\big(s,V(s)\big)\Big)(T) + \frac{|\lambda_1|}{|v_0|}\, I^{\alpha+\sigma_1}\beta\Big(f\big(s,V(s)\big)\Big)(\eta) \\
&+ \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_2}\beta\Big(f\big(s,V(s)\big)\Big)(\xi) + \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_3}\beta\Big(f\big(s,V(s)\big)\Big)(\eta), \\
&\le I^{\alpha}\big(p(s)v(s)\big)(t) + \frac{|b_1|}{|v_0|}\, I^{\alpha}\big(p(s)v(s)\big)(T) + \frac{|\lambda_1|}{|v_0|} I^{\alpha+\sigma_1}\big(p(s)v(s)\big)(\eta) \\
&+ \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_2}\big(p(s)v(s)\big)(\xi) + \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_3}\big(p(s)v(s)\big)(\eta)\Big\}, \\
&\le \|p\|_{L^\infty}\|v\|_\infty\Big\{I^{\alpha}(1)(T) + \frac{|b_1|}{|v_0|} I^{\alpha}(1)(T) + \frac{|\lambda_1|}{|v_0|} I^{\alpha+\sigma_1}(1)(\eta) \\
&+ \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_2}(1)(\xi) + \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I^{\alpha-\sigma_3}(1)(\eta)\Big\}, \\
&\le \|p\|_{L^\infty}\|v\|_\infty\Big\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha}}{|v_0|\Gamma(\alpha+1)} + \frac{\lambda_1}{|v_0|\Gamma(\alpha+\sigma_1+1)}\eta^{\alpha+\sigma_1} \\
&+ \frac{|a_2|(|v_1| + |v_0|T)\xi^{\alpha-\sigma_2}}{|v_0 v_2|\Gamma(\alpha-\sigma_2+1)} + \frac{|b_2|(|v_1| + |v_0|T)\eta^{\alpha-\sigma_3}}{|v_0 v_2|\Gamma(\alpha-\sigma_3+1)}\Big\} \\
&= \|p\|_{L^\infty}\|v\|_\infty M,
\end{aligned}
$$

which gives

$$\|v\|_\infty \le \|p\|_{L^\infty}\|v\|_\infty M.$$

This means that

$$\|v\|_\infty\Big(1 - \|p\|_{L^\infty}M\Big) \le 0.$$

By (3.10), it follows that $\|v\|_\infty = 0$, that is, $v(t) = 0$ for each $t \in J$. Then $V(t)$ is relatively weakly compact in E. Applying Theorem 2.11, we conclude that $\mathcal{N}$ has a fixed point which is a solution of problem (1.1). □

4. An Example

In this section, we give an example to illustrate the usefulness of our main result. Let

$$E = l^1 = \left\{ x = (x_1, x_2, \dots, x_n, \dots), \sum_{n=1}^{\infty} |x_n| < \infty \right\}$$

be the Banach space with the norm $\|x\|_E = \sum_{n=1}^{\infty} |x_n|$. Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{\frac{7}{4}} x_n(t) = \frac{1}{e^{t+4}}\big(1 + |x_n(t)|\big) \quad \forall t \in J = [0,1] \\ x_n(0) - x_n(1) = 2I^{\frac{3}{4}} x_n(\frac{1}{3}), \\ 3\,{}^cD_{0^+}^{\frac{1}{7}} x_n(\frac{2}{3}) - {}^cD_{0^+}^{\frac{1}{2}} x_n(\frac{1}{3}) = 1. \end{cases} \tag{4.1}$$

Here

$$T = 1,\ \alpha = \frac{7}{4},\ a_1 = 1,\ b_1 = -1,\ a_2 = 3,\ b_2 = -1,\ \sigma_1 = \frac{3}{4},$$

$$\sigma_2 = \frac{1}{7},\ \sigma_3 = \frac{1}{2},\ \xi = \frac{2}{3},\ \eta = \frac{1}{3},\ \lambda_1 = 2,\ \lambda_2 = 1.$$

Set

$$x = (x_1, x_2, \dots, x_n, \dots),\ f = (f_1, f_2, \dots, f_n, \dots),$$

$$f(t, x_n) = \frac{1}{e^{t+4}}\big(1 + |x_n(t)|\big),\ t \in J.$$

For each $x_n \in \mathbb{R}$, $t \in J$, we have

$$|f(t, x_n)| \leq \frac{1}{e^{t+4}}\big(1 + |x_n(t)|\big).$$

Hence conditions (H1), (H2) and (H3) hold with

$$p(t) = \frac{1}{e^{t+4}},\ t \in J,$$

and

$$\psi(u) = 1 + u,\ u \in [0, \infty).$$

For any bounded set $D \subset l^1$, we have

$$\beta(f(t, D)) \leq \frac{1}{e^{t+4}} \beta(D),\ \text{for each}\ t \in J.$$

Hence (H5) is satisfied. Using the Matlab program, we can find that

$$\|p\|_{L^\infty} M = \frac{10}{173} < 1,$$

and the inequality

$$\|p\|_{L^\infty}(1 + R)M + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} \leq R,$$

is satisfied for

$$R > \frac{1072}{687}.$$

Consequently, Theorem 2.11 implies that problem (4.1) has a solution defined on J.

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