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THREE KINDS OF HYBRID ALGORITHMS AND THEIR NUMERICAL REALIZATIONS FOR A FINITE FAMILY OF QUASI-NONEXPANSIVE MAPPINGS

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Abstract. The purpose of this paper is to propose three new hybrid projection methods for a finite family of quasi-nonexpansive mappings. The strong convergence of the algorithms is proved in real Hilbert spaces. Some numerical experiments are also provided to compare and illustrate the effectiveness of the proposed algorithm.

Keywords. Quasi-nonexpansive mapping; Hybrid algorithms; Strong convergence; Numerical experiments.

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1. Introduction

Suppose that H is a real Hilbert space and C is a closed convex nonempty subset of H. We use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote the inner product and the norm, respectively. Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the fixed-point set of T, i.e., $F(T) = \{x \in C : x = Tx\}$. Recall that $T : C \to C$ is called a nonexpansive mappings if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Recall that $T : C \to C$ is called a quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad x \in C, p \in F(T)$$

Obviously, a nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-nonexpansive mapping, but the converse may be not true. Since common fixed-point problems have a lot of real world applications, such as image recovery, and signal processing [1, 2, 3], the construction of common fixed points for a finite family of nonlinear mappings is of practical importance. In particular, iterative algorithms for finding common fixed points of a finite family of nonexpansive mappings have been extensively studied; see [4, 5, 6, 7] and the references therein

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In 2003, Nakajo and Takahashi [8] first introduced a hybrid algorithm for fixed points of nonexpansive mappings. Thereafter, some hybrid algorithms have been studied extensively since they have strong convergence without any compact assumptions; see, for example, [9, 10, 11, 12, 13] and the references therein.

Recently, Anh and Chung [14] proposed a parallel hybrid algorithm for a finite family of nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$ in a Hilbert space H as following:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ z_k = P_C(x_k), \\ y_k^i = \alpha_k z_k + (1-\alpha_k) T_i(z_k), i = 1, 2, \cdots, N, \\ i_k = \arg\max\{\|y_k^i - x_k\| : i = 1, 2, \cdots, N\}, \\ C_k = \{v \in H : \|v - y_k^{i_k}\| \leq \|v - x_k\|\}, \\ Q_k = \{u \in H : \langle x_0 - x_k, x_k - u \rangle \geq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0). \end{cases} \tag{1.1}$$

They proved that $\{x_n\}$ strongly converges to a particular common fixed points of $T_1, T_2, \cdots, T_N$, $P_{\bigcap_{i=1}^N F(T_i)} x_0$ in the framework of Hilbert spaces.

Very recently, Dong, Lu and Yang [15] proposed a cyclic algorithm for a finite family of nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$ in a Hilbert space H as following:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_k^1 = \alpha_k x_k + (1-\alpha_k) T_1(x_k), \\ y_k^{i+1} = \alpha_k y_k^i + (1-\alpha_k) T_{i+1}(y_k^i), i = 1, 2, \cdots, N-1, \\ i_k = \arg\max\{\|y_k^i - x_k\| : i = 1, 2, \cdots, N\}, \\ C_k = \{u \in C : \|u - y_k^{i_k}\| \leq \|u - x_k\|\}, \\ Q_k = \{v \in C : \langle x_0 - x_k, x_k - v \rangle \geq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0). \end{cases} \tag{1.2}$$

They proved that $\{x_n\}$ strongly converges to a particular common fixed points of $T_1, T_2, \cdots, T_N$, $P_{\bigcap_{i=1}^N F(T_i)} x_0$ in the framework of Hilbert spaces.

In this paper, motivated by Anh and Dong's results, we design three simple hybrid methods for a finite family of quasi-nonexpansive mappings. Some strong convergence theorems are obtained by using new methods. We also give some numerical experiments to compare and illustrate the effectiveness of the proposed algorithms. The results obtained in this paper improve and extend the related ones obtained by many authors recently.

2. Preliminaries

In this section, we mainly give some necessary lemmas.

The following two lemmas are trivial.

Lemma 2.1. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation $\langle x - z, y - z \rangle \leq 0$, $y \in C$.*

Lemma 2.2. *Let C be a closed convex nonempty subset of H and let P_C be the projection from H onto C. Then*

$$\|y-P_Cx\|^2+\|x-P_Cx\|^2 \leq \|x-y\|^2,\ x\in H,\ y\in C.$$

Lemma 2.3. [16] *Let H be a Hilbert space. Then the following equality holds*

$$\|\sum_{i=1}^{n}\alpha_i x_i\|^2=\sum_{i=1}^{n}\alpha_i\|x_i\|^2-\sum_{1\leq i<j\leq n}\alpha_i\alpha_j\|x_i-x_j\|^2,$$

where $x_i\in H$ and $\alpha_i\in[0,1](i=1,2,\cdots,n)$ such that $\sum_{i=1}^{n}\alpha_i=1$.

Lemma 2.4. [17] *Let C be a closed convex nonempty subset of H, $T:C\to C$ be a quasi-nonexpansive mapping. Then F(T) is a convex closed subset of C.*

3. Main results

Theorem 3.1. *Let C is a closed convex nonempty subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}:C\to C$ be a finite family of closed quasi-nonexpansive mappings such that $F=\bigcap_{i=1}^{N}F(T_i)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process:*

$$\begin{cases} x_0\in H \ \ chosen\ arbitrarily,\\ C_1=C,\\ y_n^1=\alpha_n x_n+(1-\alpha_n)T_1(x_n),\\ y_n^{i+1}=\alpha_n y_n^i+(1-\alpha_n)T_{i+1}(y_n^i), i=1,2,\cdots,N-1,\\ i_n=\arg\max\{\|y_n^i-x_n\|: i=1,2,\cdots,N\},\\ C_{n+1}=\{w\in C_n:\|w-y_n^{i_n}\|\leq\|w-x_n\|\},\\ x_{n+1}=P_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0,\alpha]$, where α is a real number such that $\alpha<1$. Then $\{x_n\}$ strongly converges to P_Fx_0.

Proof. By Lemma 2.4 and assumpiton $F\neq\emptyset$, we know that P_Fx_0 is well defined for every $x_0\in C$. From the constructions of C_n, one easily see that C_n is closed and convex, $\forall n\geq 1$.

Next, we split our proof into five steps.

Step 1. Show that $F\subset C_n$ for all $n\geq 1$.

It is obvious that $F \subset C_1 = C$. Assume that $F \subset C_n$ for some $n \geq 1$. For any $q \in F \subset C_n$, from (3.1), we have

$$\begin{aligned}\|q - y_n^{i_n}\| &= \|q - \{\alpha_n y_n^{i_n - 1} + (1-\alpha_n) T_i y_n^{i_n-1}\}\| \\ &\leq \alpha_n \|q - y_n^{i_n-1}\| + (1-\alpha_n)\|q - T_i y_n^{i_n-1}\| \\ &\leq \alpha_n \|q - y_n^{i_n-1}\| + (1-\alpha_n)\|q - y_n^{i_n-1}\| \\ &= \|q - y_n^{i_n-1}\| \\ &\leq \cdots \\ &\leq \|q - y_n^1\| \\ &\leq \|q - x_n\|.\end{aligned}$$

Therefore $q \in C_{n+1}$ and hence $F \subset C_n$ for all $n \geq 1$.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

In view of (3.1), one has $x_n = P_{C_n}(x_0)$. Since $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$, for all $n \geq 1$, one has

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \forall n \geq 1. \tag{3.2}$$

On the other hand, as $F \subset C_n$ by step 1, one concludes that

$$\|x_n - x_0\| \leq \|z - x_0\|, \forall z \in F, \forall n \geq 1. \tag{3.3}$$

Combining (3.2) and (3.3), one sees that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Step 3. Show that $x_n \to v$ as $n \to \infty$, $v \in C$.

For $m > n \geq 1$, one has $x_m = P_{C_m}(x_0) \in C_m \subset C_n$. By Lemma 2.2, one has

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.4}$$

Taking $m,\ n \to \infty$ in (3.4), one gets $x_m - x_n \to 0$ as $m,\ n \to \infty$, which proves that $\{x_n\}$ is a Cauchy sequence in C. By completeness of space H and closedness of set C, one can assume that $x_n \to v$ as $n \to \infty$.

Step 4. Show that $x_n - T_i x_n \to 0 (n \to \infty),\ i = 1,\ 2,\ \cdots,\ N$.

Since $x_n \to v(n \to \infty)$, one has

$$\|x_{n+1} - x_n\| \to 0 (n \to \infty). \tag{3.5}$$

Since $x_{n+1} \in C_{n+1}$, one has

$$\|x_{n+1} - y_n^{i_n}\| \leq \|x_{n+1} - x_n\|,$$

which implies from (3.5) that

$$\begin{aligned}\|x_n - y_n^{i_n}\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n^{i_n}\| \\ &\leq 2\|x_{n+1} - x_n\| \to 0 (n \to \infty).\end{aligned}$$

By the definition of i_n, one has

$$\|x_n - y_n^i\| \to 0 (n \to \infty),\ i = 1,\ 2,\ \cdots,\ N, \tag{3.6}$$

which yields that

$$\|y_n^{i+1}-y_n^i\| \le \|y_n^{i+1}-x_n\| + \|y_n^i - x_n\| \to 0 (n\to\infty),\ i=1,\ 2,\ \cdots,\ N-1.$$

Using (3.1), one has

$$\|x_n - T_1x_n\| = \frac{1}{1-\alpha_n}\|y_n^1 - x_n\| \to 0(n\to\infty), \tag{3.7}$$

and

$$\|y_n^i - T_{i+1}y_n^i\| = \frac{1}{1-\alpha_n}\|y_n^i - y_n^{i+1}\| \to 0(n\to\infty),\ i=1,\ 2,\ \cdots,\ N-1. \tag{3.8}$$

Note that T_i is quasi-nonexpansive. By (3.6) and (3.8), one gets that

$$\begin{aligned}\|x_n - T_{i+1}x_n\| &\le \|x_n - y_n^i\| + \|y_n^i - T_{i+1}y_n^i\| + \|T_{i+1}x_n - T_{i+1}y_n^i\| \\ &\le 2\|x_n - y_n^i\| + \|y_n^i - T_{i+1}y_n^i\|.\end{aligned}$$

This implies from (3.6) and (3.8) that

$$\|x_n - T_{i+1}x_n\| \to 0,\ i=1,\ 2,\ \cdots,\ N-1. \tag{3.9}$$

Step 5. Show that $v = P_F x_0$.

From step 3, one knows that $x_n \to v(n\to\infty)$. By (3.7), (3.9) and closedness of T_i, one has $v = T_i v,\ i = 1,\ 2,\ \cdots,\ N$. Therefore $v\in F$. Note that $F\subset C_n$ and $x_n = P_{C_n}x_0$. By Lemma 2.1, one concludes that

$$\langle z - x_n, x_0 - x_n\rangle \le 0, \forall z\in F.$$

It follows that

$$\langle z - v, x_0 - v\rangle \le 0, \forall z\in F.$$

By Lemma 2.1, one concludes that $v = P_F x_0$. □

Remark 3.2. Theorem 3.1 generalizes the main result obtained in [15] as follows:

(i) Nonexpansive mappings are extended to quasi-nonexpansive mappings.

(ii) Iterative algorithm (3.1) is more simple than iterative algorithm (1.2), that is, there is no set "Q_n" in iterative algorithm (3.1).

Theorem 3.3. *Let C be a closed convex nonempty subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N : C\to C$ be a finite family of closed quasi-nonexpansive mappings such that $F=\bigcap_{i=1}^N F(T_i)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process:*

$$\begin{cases} x_0\in H \ \ chosen\ arbitrarily, \\ C_1 = C, \\ y_n^i = \alpha_n x_n + (1-\alpha_n)T_i x_n, i = 1,2,\cdots,N, \\ i_n = \arg\max\{\|y_n^i - x_n\| : i=1,2,\cdots,N\}, \\ C_{n+1} = \{w\in C_n : \|w - y_n^{i_n}\| \le \|w - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases} \tag{3.10}$$

where $\{\alpha_n\}$ is a real sequence in $(0,\alpha]$, where α is a real number such that $\alpha<1$. Then $\{x_n\}$ strongly converges to $P_F x_0$.

Proof. The proof is also split into five steps. Steps 1, 2 and 3 are similar to the proof in Theorem 3.1. So, we omit them here. Next we prove that $x_n - T_i x_n \to 0 (n \to \infty),\ i = 1,\ 2,\ \cdots,\ N$. Since $x_n \to v (n \to \infty)$, one has

$$\|x_{n+1} - x_n\| \to 0 (n \to \infty).$$

Since $x_{n+1} \in C_{n+1}$, one has

$$\|x_{n+1} - y_n^{i_n}\| \leq \|x_{n+1} - x_n\|.$$

It follows that

$$\begin{aligned} \|x_n - y_n^{i_n}\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n^{i_n}\| \\ &\leq 2\|x_{n+1} - x_n\| \to 0 (n \to \infty). \end{aligned}$$

By the definition of i_n, one has

$$\|x_n - y_n^i\| \to 0 (n \to \infty), i = 1, 2, \cdots, N.$$

Using (3.10), one has

$$\|x_n - T_i x_n\| = \frac{1}{1 - \alpha_n} \|y_n^i - x_n\| \to 0 (n \to \infty),$$

The remainder of the proof follows from Step 5 in Theorem 3.1. This completes the proof. □

Remark 3.4. Theorem 3.3 generalizes the related results obtained in [14] as follows:

(i) Nonexpansive mappings are extended to quasi-nonexpansive mappings.

(ii) Iterative algorithm (3.10) is more simple than iterative algorithm (1.1), i.e., there is no set "Q_n" in iterative algorithm (3.10).

Finally, we give another kind of a parallel iterative algorithm for a finite family of quasi-nonexpansive mappings.

Theorem 3.5. *Let C be a closed convex nonempty subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of closed quasi-nonexpansive mappings such that $F = \bigcap_{i=1}^N F(T_i)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process:*

$$\begin{cases} x_0 \in H \ \ chosen\ arbitrarily, \\ C_1 = C, \\ y_n = \alpha_{n_0} x_n + \alpha_{n_1} T_1 x_n + \cdots + \alpha_{n_N} T_N x_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases} \tag{3.11}$$

where α_{n_i} is a real sequence in $[0,1]$, $i = 0,\ 1,\ 2,\ \cdots,\ N$ such that $\alpha_{n_0} + \alpha_{n_1} + \cdots + \alpha_{n_N} = 1$ and $\liminf_{n\to\infty} \alpha_{n_0} \alpha_{n_i} > 0$. *Then $\{x_n\}$ strongly converges to $P_F x_0$.*

Proof. As shown in Theorem 3.1, one has that $P_F x_0$ is well defined for every $x_0 \in H$, and C_n is closed and convex.

Next, one proves that $F \subset C_n, \forall n \geq 1$. For $q \in F$, one has

$$\begin{aligned}
\|y_n - q\| &= \|\alpha_{n_0} x_n + \alpha_{n_1} T_1 x_n + \cdots + \alpha_{n_N} T_N x_n - q\| \\
&\leq \alpha_{n_0} \|x_n - q\| + \alpha_{n_1} \|T_1 x_n - q\| + \cdots + \alpha_{n_N} \|T_N x_n - q\| \\
&\leq \|x_n - q\|,
\end{aligned}$$

which implies that $q \in C_{n+1}$. Therefore $\{x_n\}$ is well defined, $\forall n \geq 1$. Furthermore, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Hence, $x_n \to v(n \to \infty)$, $v \in C$.

Next, we prove that $x_n - T_i x_n \to 0(n \to \infty)$, $i = 1, 2, \cdots, N$. Since $x_n \to v(n \to \infty)$, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_{n+1}$, one has

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|.$$

It follows that

$$\|x_n - y_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| \leq 2\|x_{n+1} - x_n\| \to 0 \quad (as\ n \to \infty). \tag{3.12}$$

Note that T_i is quasi-nonexpansive. For $w \in F$, we obtain from (3.11) and Lemma 2.3 that

$$\begin{aligned}
\|y_n - w\|^2 &= \|\alpha_{n_0}(x_n - w) + \alpha_{n_1}(T_1 x_n - w) + \cdots + \alpha_{n_N}(T_N x_n - w)\|^2 \\
&= \alpha_{n_0}\|x_n - w\|^2 + \sum_{i=1}^{N} \alpha_{n_i}\|T_i x_n - w\|^2 - \alpha_{n_0}\alpha_{n_i}\|T_i x_n - x_n\|^2 \\
&\leq \|x_n - w\|^2 - \alpha_{n_0}\alpha_{n_i}\|T_i x_n - x_n\|^2.
\end{aligned}$$

Hence

$$\alpha_{n_0}\alpha_{n_i}\|T_i x_n - x_n\|^2 \leq \|x_n - w\|^2 - \|y_n - w\|^2.$$

By (3.12), one has

$$\begin{aligned}
\|x_n - w\|^2 - \|y_n - w\|^2 &= \langle x_n - w, x_n - w\rangle - \langle y_n - w, y_n - w\rangle \\
&= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, w\rangle \\
&\leq (\|x_n\| + \|y_n\|)(\|x_n - y_n\|) + 2\|x_n - y_n\| \cdot \|w\| \to 0(n \to \infty).
\end{aligned}$$

Since $\liminf_{n\to\infty} \alpha_{n_0}\alpha_{n_i} > 0$, one has

$$\|T_i x_n - x_n\| \to 0(n \to \infty), i = 1, 2, \cdots, N.$$

The remainder of the proof follows from Step 5 in Theorem 3.1. This completes the proof. □

4. Rate of Convergence and Numerical Experiments

In this section, we provide some numerical examples to show that our algorithms are effective. We also compare the rate of convergence of these algorithms. In order to compare two fixed point iteration, Rhoades [18] introduced the following concept in 1976.

Definition 4.1. [18] Let $\{x_n\}$, $\{z_n\}$ be two iteration schemes which converge to the same fixed point p. We say that $\{x_n\}$ is better than $\{z_n\}$ if $\|x_n - p\| \leq \|z_n - p\|$ for all n.

Berinde [19] introduced the following definition, which is slightly different from definition 4.1.

Definition 4.2. [19] Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

(a) If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.
(b) If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Phuengrattana and Suantai [20] presented numerical examples to compare the convergence speed of Mann, Ishikawa, Noor and SP-iterations based on the Definition 4.1.

Motivated by the above results, we present numerical examples to compare the convergence speed of algorithms (3.1), (3.10), and (3.11) by using the Definition 4.1, the Definition 4.2 and other methods. In the numerical experiments, we take the mapping $T_1, T_2 : C \to \mathbb{R}$ by

$$T_1 x = \frac{1}{2} x \cos x, T_2 x = \frac{2}{3} x \sin x, x \in C,$$

where $C = [0, 2\pi]$, $\mathbb{R}$ denotes the set of real numbers. One has that $F(T_1) = F(T_2) = \{0\}$ and T_1 and T_2 are two quasi-nonexpansive mappings. Obviously, $F(T_1) \cap F(T_2) = \{0\}$. Algorithms (3.1), (3.10), and (3.11) are iterated to step 70 respectively.

For algorithms (3.1), and (3.10), we take $\alpha_n = 0.1$. For algorithm (3.11), we take $\alpha_{n_0} = 0.1$, $\alpha_{n_1} = 0.2$. We choose $x_0 \in [0, 2\pi]$ arbitrarily. Then, for 50 different initial values, one can see all the results are convergent in Figure 1, Figure 2 and Figure 3.

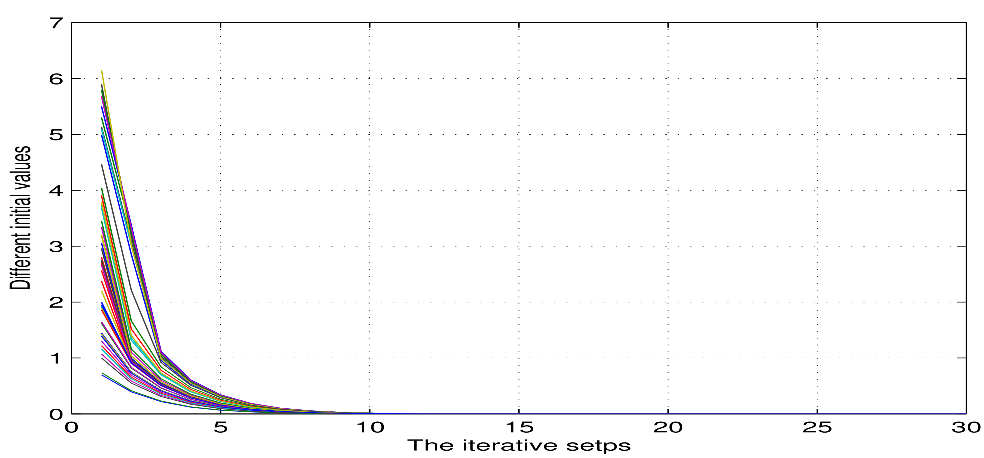

Figure 1. The iterative curves of algorithm (3.1) under different initials.

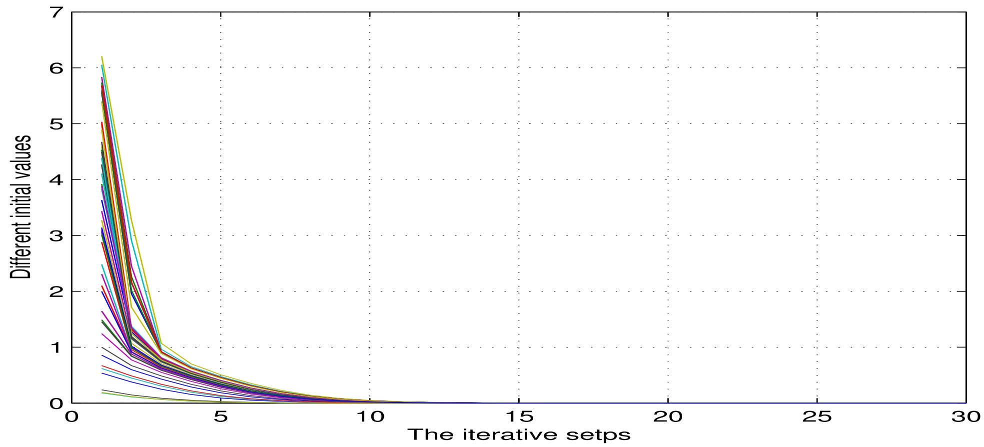

Figure 2. The iterative curves of algorithm (3.10) under different initials.

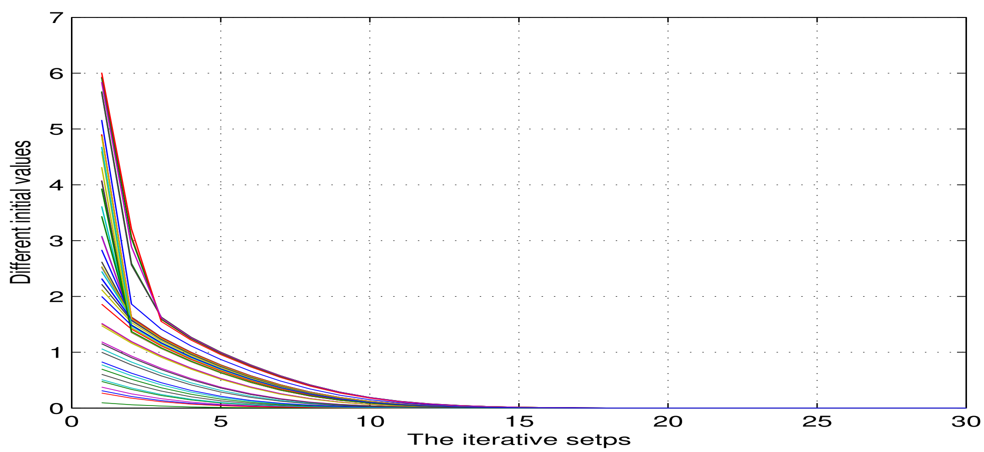

Figure 3. The iterative curves of algorithm (3.11) under different initials.

Next, we compare algorithms (3.1), (3.10), and (3.11) for the above given examples. In the following Table 1, *Iter.* and *Sec*. denote the number of iterations and the running time in seconds, respectively. We chose $x_0 \in [0, 2\pi]$ as initial points. We take $\|x_n - 0\| \leq \varepsilon_1$ as the stopping criterion and $\varepsilon_1 = 10^{-12}$ in Table 1. The algorithms are coded in Matlab 2013 and run on a personal laptop. For different initial points 1, 4 and 5, the numerical results of the above algorithms (3.1), (3.10), and (3.11) are shown in Table 1. From Table 1, we observe that algorithm (3.1) is the best, from the points of view of number of iterations and running time.

TABLE 1. The numerical results of the three algorithms with $\varepsilon_1 = 10^{-12}$

x_0	Algorithm(3.1)		Algorithm(3.10)		Algorithm(3.11)	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
1	45	0.018679	50	0.02251	58	0.027706
4	47	0.018463	52	0.022052	60	0.023901
5	47	0.018915	51	0.022997	61	0.026087

In Table 2 and Table 3, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence generated by (3.1), $\{y_n\}_{n=0}^{\infty}$ be the iterative sequence generated by (3.10) and $\{z_n\}_{n=0}^{\infty}$ be the iterative sequence generated by (3.11). The comparison of the convergence of algorithms (3.1), (3.10), and (3.11) to the common fixed point $p = 0$ are given in Table 2 and Table 3, with the initial $x_0 = y_0 = z_0 = 5$.

From Table 2, we see that algorithm (3.1) converges faster than algorithms (3.10) and (3.11) by using the Definition 4.1. From Table 3, we see that algorithm (3.1) converges faster than algorithm (3.10) and algorithm (3.10) converges faster than algorithm (3.11) by using the Definition 4.2.

TABLE 2. Comparison of rate of convergence of the three algorithms by the def. 4.1

n	Algorithm (3.1)	Algorithm (3.10)	Algorithm (3.11)
	x_n	y_n	z_n
10	0.01342	0.033201	0.13211
⋮	⋮	⋮	⋮
45	2.5566e-12	2.828e-11	2.59e-09
46	1.3486e-12	1.5554e-11	1.554e-09
47	7.114e-13	8.5547e-12	9.3239e-10

TABLE 3. Comparison of rate of convergence of the three algorithms by the def. 4.2

n	$\frac{\lvert x_n - 0\rvert}{\lvert y_n - 0\rvert}$	$\frac{\lvert x_n - 0\rvert}{\lvert z_n - 0\rvert}$	$\frac{\lvert y_n - 0\rvert}{\lvert z_n - 0\rvert}$
10	0.4042	0.10158	0.25131
20	0.25688	0.024715	0.096213
30	0.16915	0.0068125	0.040273
46	0.086704	0.00086782	0.010009

5. CONCLUSIONS

Three kinds of hybrid algorithms for a finite family of quasi-nonexpansive mappings were proposed in this paper. Their strong convergence was obtained in the framework of Hilbert spaces. Numerical examples were provided to compare and illustrate the effectiveness of the three algorithms. The results obtained in this paper extend some known results in the literature.

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