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EXISTENCE OF SOLUTIONS FOR A CLASS OF NONLINEAR TYPE PROBLEMS INVOLVING THE $p(x)$-LAPLACIAN OPERATOR

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Abstract. The aim of this paper is to establish the existence of at least one nontrivial weak solution of the following problem

$$\begin{cases} -\Delta_{p(x)}u+\alpha(x)|u|^{p(x)-2}u=\lambda a(x)|u|^{q(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}+\beta(x)|u|^{p(x)-2}u=\lambda b(x)|u|^{r(x)-2}u \quad \text{on } \partial\Omega. \end{cases}$$

Our technical approach is based on the direct variational method combined with the mountain pass theorem and the theory of Lebesgue and Sobolev spaces.

Keywords. Variational method; Mountain Pass Theorem; $p(x)$-Laplacian operator; Generalized Lebesgue-Sobolev spaces.

1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} -\Delta_{p(x)}u+\alpha(x)|u|^{p(x)-2}u=\lambda a(x)|u|^{q(x)-2}u \quad \text{in } \Omega, \\ \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}+\beta(x)|u|^{p(x)-2}u=\lambda b(x)|u|^{r(x)-2}u \quad \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega\subset\mathbb{R}^N$, $(N\geq 2)$, is a smooth bounded domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, λ is a positive parameter and p,q,r are continuous functions on $\overline{\Omega}$ with $q^-:=\inf_{x\in\overline{\Omega}}q(x)>1$, $r^-:=\inf_{x\in\overline{\Omega}}r(x)>1$, $(\alpha,\beta)\in L^\infty(\Omega)\times L^\infty(\partial\Omega)$ with $\alpha^-:=\inf_{x\in\Omega}\alpha(x)>0$ and $\beta^-:=\inf_{x\in\partial\Omega}\beta(x)>0$, and a,b are measurable functions satisfying $a(x)>0$ for all $x\in\Omega$ and $b(x)>0$ for all $x\in\partial\Omega$ such that $a(x)\in L^{\gamma(x)}(\Omega)$ and $b(x)\in L^{\delta(x)}(\partial\Omega)$ with $\gamma\in C(\overline{\Omega})$ and $\delta\in C(\partial\Omega)$.

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Throughout this paper, we assume the following conditions

$$1 < r^- \leq r^+ < p^- \leq p^+ < q^- \leq q^+ < (p^-)^\partial \quad \text{and } p^+ < N, \tag{1.1}$$

$$q(x) < \frac{\gamma(x)-1}{\gamma(x)} p^*(x) \quad \forall x \in \Omega \quad \text{and} \quad r(x) < \frac{\delta(x)-1}{\delta(x)} p^\partial(x) \quad \forall x \in \partial\Omega, \tag{1.2}$$

where p^*, p^∂ are the critical Sobolev exponent of p defined by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } N < p(x), \\ \infty, & \text{if } N \geq p(x), \end{cases} \quad \text{and} \quad p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } N < p(x), \\ \infty, & \text{if } N \geq p(x). \end{cases}$$

It is well known that operator $-\Delta_{p(x)} u := -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The study of various mathematical problems with variable exponent has been received considerable attention in recent years. These problems are interesting in applications (see, e.g., [1, 2]) and raise many difficult mathematical problems. We refer to [3, 4] for the overview and [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems under Dirichlet, Neumann, Steklov and Robin boundary conditions. These problems are very interesting from a purely mathematical point of view as well.

In [15], Mavinga considered the problem (P_λ) in the particular case $p(x) = q(x) = r(x) \equiv 2$, i.e.,

$$\begin{cases} -\Delta u + \alpha(x)u = \lambda a(x)u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta(x)u = \lambda b(x)u & \text{on } \partial\Omega, \end{cases}$$

and proved the existence of an unbounded sequence of eigenvalues $(\lambda_k)_{k\geq 1}$. Moreover, the first eigenvalue λ_1 is simple, and its associated eigenfunction does not change sign in Ω. In [16], Li, Liu and Cheng considered the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \sigma, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \Gamma, \end{cases}$$

with $\partial\Omega = \sigma \cup \Gamma$ and $\sigma \cap \Gamma = \emptyset$. Using Ljusternik-Schnirelman principle, they proved the existence of nonnegative eigenvalues. Also the simplicity and isolation of the first eigenvalue were obtained. The particularity of those two problems is the presence of the spectral parameter λ both in the differential equation and on the boundary. In [17], Benouhiba and Bounouala proved the existence of weak solutions of the following problem

$$\begin{cases} -\Delta_{p(x)} u + h(x)|u|^{p(x)-2}u = \lambda g(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = k(x)|u|^{r(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

under suitable assumptions on functions h, g and k.

Motivated by the above papers, we consider those problems in a more general form. The main result of the present paper is the following theorem

Theorem 1.1. *If $p, q, r \in C_+(\overline{\Omega})$ satisfy (1.1) and (1.2), then there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, problem (P_λ) has at last one nontrivial weak solution.*

Notice that our result is new even in the constant case $p(x) = p$. This paper is organized as follows. In Section 2, we recall some basic facts about variable exponent Lebesgue and Sobolev spaces. Moreover, we introduce a new norm on the space $W^{1,p(x)}(\Omega)$ which be used later. In Section 3, our main result is established.

2. Preliminaries

In this section, we introduce the Lebesgue and Sobolev spaces with variable exponent and recall their main properties. For more details, we refer to the book by Musielak [18], Diening, Harjulehto, Hästö and Ruzicka [19], and the papers by Edmunds et al. [20, 21], Kovacik and Rakosnik [22].

Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^- := \inf_{x\in\overline{\Omega}} h(x) \quad \text{and} \;\; h^+ := \sup_{x\in\overline{\Omega}} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\}.$$

On this space, we define a norm, the so-called Luxemburg norm, by the formula

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \tau > 0; \int_\Omega \left| \frac{u}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For a constant function p ,the variable exponent Lebesgue space coincides with the standard Lebesgue space.

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Variable exponent Lebesgue spaces do not have the mean continuity property: if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm (see Kovacik and Rakosnik [22]).

Most of the problems in the development of the theory of $L^{p(x)}(\Omega)$ spaces arise from the fact that these spaces are virtually never translation invariant. The use of convolution is also limited: the Young inequality

$$\|f * g\|_{L^{p(x)}(\Omega)} \leq C \|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^1(\Omega)},$$

holds if and only if p is constant.

Next, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

On $W^{1,p(x)}(\Omega)$, we may consider one of the following equivalent norms

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)},$$

or

$$\|u\| = \inf\left\{\tau > 0; \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \left|\frac{u}{\tau}\right|^{p(x)}\right) dx \le 1\right\}.$$

It is well known (see [23]) that both $L^{p(x)}(\Omega)$ and $W^{p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces. Moreover Hölder's inequality holds, namely

$$\int_\Omega |uv|dx \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)}, \quad \forall u \in L^{p(x)}(\Omega), \quad \forall v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Proposition 2.1. *(See [22, 23]). Assume that Ω is bounded and has a Lipchitz boundary with the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$.*
(i) If $q \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$, $\forall x \in \overline{\Omega}$, then there is a compact and continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.
(ii) If $q \in C(\overline{\Omega})$ and $1 \le q(x) < p^\partial(x)$, $\forall x \in \partial\Omega$, then there is a compact and continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.

Proposition 2.2. *(See [21]). Let p and q be measurable functions and $1 < p(x)q(x) \le \infty$ for a. e. $x \in \Omega$. If $f \in L^{q(x)}(\Omega)$, then*

$$\begin{aligned} &\|f\|^{p^-}_{L^{p(x)q(x)}(\Omega)} \le \| \, |f|^{p(x)}\|_{L^{q(x)}(\Omega)} \le \|f\|^{p^+}_{L^{p(x)q(x)}(\Omega)} \quad if \quad \|f\|_{L^{p(x)q(x)}(\Omega)} > 1 \\ &\|f\|^{p^+}_{L^{p(x)q(x)}(\Omega)} \le \| \, |f|^{p(x)}\|_{L^{q(x)}(\Omega)} \le \|f\|^{p^-}_{L^{p(x)q(x)}(\Omega)} \quad if \quad \|f\|_{L^{p(x)q(x)}(\Omega)} \le 1. \end{aligned} \tag{2.1}$$

In particular, if $p(x) \equiv p$ is a constant, then $\| \, |f|^p|\|_{L^{q(x)}(\Omega)} = \|f\|^p_{L^{q(x)}(\Omega)}$.

Arguing as in the proof of Proposition 2.2, we can easily obtain the following.

Proposition 2.3. *Let p and q be measurable functions and $1 < p(x)q(x) \le \infty$ for a. e. $x \in \partial\Omega$. Let $f \in L^{q(x)}(\partial\Omega)$ then*

$$\begin{aligned} &\|f\|^{p^-}_{L^{p(x)q(x)}(\partial\Omega)} \le \| \, |f|^{p(x)}\|_{L^{q(x)}(\partial\Omega)} \le \|f\|^{p^+}_{L^{p(x)q(x)}(\partial\Omega)} \quad if \quad \|f\|_{L^{p(x)q(x)}(\partial\Omega)} > 1 \\ &\|f\|^{p^+}_{L^{p(x)q(x)}(\partial\Omega)} \le \| \, |f|^{p(x)}\|_{L^{q(x)}(\partial\Omega)} \le \|f\|^{p^-}_{L^{p(x)q(x)}(\partial\Omega)} \quad if \quad \|f\|_{L^{p(x)q(x)}(\partial\Omega)} \le 1. \end{aligned} \tag{2.2}$$

In particular, if $p(x) \equiv p$ is constant then $\| \, |f|^p|\|_{L^{q(x)}(\partial\Omega)} = \|f\|^p_{L^{q(x)}(\partial\Omega)}$.

Let $a : \Omega \to \mathbb{R}$ and $b : \partial\Omega \to \mathbb{R}$ be two measurable functions such that $a(x) > 0$ for $x \in \Omega$ and $b(x) > 0$ for $x \in \partial\Omega$. Define the weighted variable exponent Lebesgue spaces $L^{p(x)}_{a(x)}(\Omega)$, $L^{p(x)}_{b(x)}(\partial\Omega)$ by

$$L^{p(x)}_{a(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \quad \text{such that } u \text{ is measurable and } \int_\Omega a(x)|u|^{p(x)}dx < \infty\right\},$$

$$L^{p(x)}_{b(x)}(\partial\Omega) = \left\{u : \partial\Omega \to \mathbb{R} \quad \text{such that } u \text{ is measurable and } \int_{\partial\Omega} b(x)|u|^{p(x)}dx < \infty\right\},$$

with the norms, respectively,

$$\|u\|_{L^{p(x)}_{a(x)}(\Omega)} = \inf\left\{\tau > 0; \int_\Omega a(x)\left|\frac{u}{\tau}\right|^{p(x)} dx \le 1\right\},$$
$$\|u\|_{L^{p(x)}_{b(x)}(\partial\Omega)} = \inf\left\{\tau > 0; \int_{\partial\Omega} b(x)\left|\frac{u}{\tau}\right|^{p(x)} dx \le 1\right\}.$$

Then $L^{p(x)}_{a(x)}(\Omega)$, $L^{p(x)}_{b(x)}(\partial\Omega)$ are Banach spaces [9, 10]. In particular, when $a(x) \equiv 1$ on Ω and $b(x) \equiv 1$ on $\partial\Omega$, $L^{p(x)}_{a(x)}(\Omega) = L^{p(x)}(\Omega)$ and $L^{p(x)}_{b(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

Proposition 2.4. *(See* [10]*). Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. Suppose that $a \in L^{\gamma(x)}(\Omega)$, $a(x) > 0$ for $x \in \Omega$, $\gamma \in C(\overline{\Omega})$ and $\gamma^- > 1$. If $s \in C(\overline{\Omega})$ and*

$$1 \le s(x) < \frac{\gamma(x)-1}{\gamma(x)} p^*(x), \quad \forall x \in \overline{\Omega},$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}_{a(x)}(\Omega)$.

Proposition 2.5. *(See* [9]*). Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. Suppose that $b \in L^{\delta(x)}(\partial\Omega)$, $b(x) > 0$ for $x \in \partial\Omega$, $\delta \in C(\partial\Omega)$ with $\delta(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. If $\ell \in C(\partial\Omega)$ and*

$$1 \le \ell(x) < \frac{\delta(x)-1}{\delta(x)} p^\partial(x), \quad \forall x \in \partial\Omega.$$

Then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{\ell(x)}_{b(x)}(\partial\Omega)$.

Theorem 2.6. *For any $u \in W^{1,p(x)}(\Omega)$, let*

$$\|u\|_W := \|\nabla u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\partial\Omega)} \text{ and } \|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\Omega)}.$$

Then $\|.\|_W$ is a norm on $W^{1,p(x)}(\Omega)$, which is equivalent to $\|.\|$.

Proof. By Proposition 2.1, there exists a constant $C > 1$ such that

$$\|u\|_{L^{p(x)}(\partial\Omega)} \le (C-1)\|u\|.$$

It follows that

$$\|u\|_W \le C\|u\|.$$

Obviously, we have $\|u\| \le \|u\|_W$. Then

$$\|u\| \le \|u\|_W \le C\|u\|,$$

which mean that the two norms are equivalent. □

Now, taking into account the particularity of problem (P_λ), we introduce a norm, which will be used later. For any $u \in W^{1,p(x)}(\Omega)$, we define

$$\|u\|_{\alpha,\beta} = \inf\left\{\tau > 0: \int_\Omega \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx + \int_\Omega \alpha(x)\left|\frac{u}{\tau}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x)\left|\frac{u}{\tau}\right|^{p(x)} d\sigma \le 1\right\}.$$

Since $(\alpha,\beta) \in L^\infty(\Omega) \times L^\infty(\partial\Omega)$ with $\alpha^- > 0$ and $\beta^- > 0$. From Theorem 2.6, $\|u\|_{\alpha,\beta}$ is also a norm on $W^{1,p(x)}(\Omega)$, which is equivalent to $\|.\|$ and $\|.\|_W$. The modular inequalities that were appropriate for the norm of the Lebesgue space, can be extended to this situation, by proceeding similarly to (Theorems 1.2-1.3, [23]). More precisely, by the definition of $\|.\|_{\alpha,\beta}$, we have the following proposition.

Proposition 2.7. *Let $I_{\alpha,\beta}(u)=\int_\Omega|\nabla u|^{p(x)}dx+\int_\Omega\alpha(x)|u|^{p(x)}dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)}d\sigma$, with $\alpha^->0$ and $\beta^->0$. For any $u,u_k\in W^{1,p(x)}(\Omega)\quad(k=1,2,...)$, we have*

(1) $\|u\|_{\alpha,\beta}\geq 1\Rightarrow\|u\|^{p^-}_{\alpha,\beta}\leq I_{\alpha,\beta}(u)\leq\|u\|^{p^+}_{\alpha,\beta}$;
(2) $\|u\|_{\alpha,\beta}\leq 1\Rightarrow\|u\|^{p^+}_{\alpha,\beta}\leq I_{\alpha,\beta}(u)\leq\|u\|^{p^-}_{\alpha,\beta}$;
(3) $\|u\|_{\alpha,\beta}\to 0\Leftrightarrow I_{\alpha,\beta}(u)\to 0$ *(as $k\to\infty$)*;
(4) $\|u\|_{\alpha,\beta}\to\infty\Leftrightarrow I_{\alpha,\beta}(u)\to\infty$ *(as $k\to\infty$)*.

Finally, we introduce the Mountain Pass Theorem, which is the main tool of our paper.

Definition 2.8. (See [24]). Let E be a Banach space and $I\in C^1(E,\mathbb{R})$. We say that I satisfies Palais-Smale condition in X if any sequence $(u_n)\in X$ such that $(I(u_n))$ is bounded and $I'(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence.

Theorem 2.9. *(See [24]). Let E be a Banach space, and let $I\in C^1(E,\mathbb{R})$ satisfy the Palais-Smale condition. Assume that $I(0)=0$, and there exists a positive real number ρ and $u,v\in E$ such that*

(i) $\|u\|>\rho$, $I(v)\leq I(0)$;
(ii) $\tau=\inf\{I(u):u\in E,\|u\|=\rho\}>0$.

Put $G=\inf\{g\in C([0,1],E):g(0)=0,g(1)=v\}\neq\emptyset$ and $\zeta=\inf\limits_{g\in\Gamma}\sup\limits_{t\in[0,1]}I(g(t))$. Then, $\zeta\geq\tau$ and ζ is a critical value of I.

Since we have established some basic properties of the space $W^{1,p(x)}(\Omega)$, we are ready to introduce the definition of a weak solution to our problem. To this end, we consider a smooth function u that verifies (P_λ). By applying Green's formula, we get

$$\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx+\int_\Omega\alpha(x)|u|^{p(x)-2}uv dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv d\sigma$$

$$=\lambda\int_\Omega a(x)|u|^{q(x)-2}uv dx+\lambda\int_{\partial\Omega}b(x)|u|^{r(x)-2}uv d\sigma,\quad\forall v\in C^\infty(\Omega).$$

Taking into consideration the fact that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$ ([25, Theorem 3.7]) together with the boundary condition, we have the following.

Definition 2.10. We say that $u\in W^{1,p(x)}(\Omega)$ is a weak solution of the boundary value problem (P_λ) if and only if

$$\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx+\int_\Omega\alpha(x)|u|^{p(x)-2}uv dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv d\sigma$$

$$-\lambda\int_\Omega a(x)|u|^{q(x)-2}uv dx-\lambda\int_{\partial\Omega}b(x)|u|^{r(x)-2}uv d\sigma=0,\quad\forall v\in W^{1,p(x)}(\Omega).$$

3. Proof of the main result

Let $X := W^{1,p(x)}(\Omega)$ and define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) := \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{\alpha(x)}{p(x)}|u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma. \tag{3.1}$$

The energy functional $J_\lambda : X \to \mathbb{R}$ corresponding to problem (P_λ) is defined as

$$J_\lambda(u) := \Phi(u) - \lambda \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}dx - \lambda \int_{\partial\Omega} \frac{b(x)}{r(x)}|u|^{r(x)}d\sigma. \tag{3.2}$$

Standard arguments imply that J_λ is well-defined; $J_\lambda \in C^1(X,\mathbb{R})$ and its Gâteaux derivative $J'_\lambda(u)$ at $u \in X$ is given by

$$\begin{aligned} \langle J'_\lambda(u), v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega \alpha(x)|u|^{p(x)-2}uv dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv d\sigma \\ &- \lambda \int_\Omega a(x)|u|^{q(x)-2}uv dx - \lambda \int_{\partial\Omega} b(x)|u|^{r(x)-2}uv d\sigma, \end{aligned}$$

for all $u, v \in X$. Thus the weak solutions of the problem are exactly the critical points of J_λ. If u is a solution of problem (P_λ) and $u \neq 0$, then λ is called the eigenvalue corresponding to the eigenfunction u of (P_λ). Moreover, we have

$$\lambda := \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}d\sigma}{\int_\Omega a(x)|u|^{q(x)} + \int_{\partial\Omega} b(x)|u|^{r(x)}d\sigma}.$$

Hence $\lambda > 0$. Theorem 1.1 ensures that problem (P_λ) has a continuous family of positive eigenvalues that lie in a neighborhood of the origin. Furthermore, we obtain

$$\inf_{u \in X\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}d\sigma}{\int_\Omega a(x)|u|^{q(x)} + \int_{\partial\Omega} b(x)|u|^{r(x)}d\sigma} = 0.$$

To prove our main theorem, we will use the well-known Mountain Pass Theorem. In addition, we also need some auxiliary results.

Proposition 3.1. *The mapping $\Phi' : X \to X^*$ defined by*

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega \alpha(x)|u|^{p(x)-2}uv dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv d\sigma,$$

satisfies condition (S^+), namely, $u_n \rightharpoonup u$ in X and $\limsup\langle \Phi'(u_n), u_n - u\rangle \le 0$, imply $u_n \to u$ in X, where $\rightharpoonup$ and $\to$ denote the weak and strong convergence respectively.

Proof. Define the functional $H, K : X \to \mathbb{R}$ such that

$$H(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{\alpha(x)}{p(x)}|u|^{p(x)}dx \text{ and } K(u) = \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}d\sigma.$$

Then $\Phi = H + K$. It is well known that the mapping $H' : X \to X^*$ is of type (S^+) (see [11, Proposition 2.5]). Under the assumption (1.1) and by Proposition 2.1, it is easy to see that the mapping $K' : X \to X^*$ is weakly-strongly continuous. Then $\Phi' = H' + K'$ is of type (S^+) since a sum of a mapping of type (S^+) and a weakly-strongly continuous mapping is still a mapping of type (S^+). ☐

Lemma 3.2. *There exists $\lambda^* > 0$ such that, for any $\lambda \in (0,\lambda^*)$, there exist $\rho,\tau > 0$ such that $J_\lambda(u) \geq \tau > 0$ for any $u \in X$ with $\|u\|_{\alpha,\beta} = \rho$.*

Proof. From (1.2), we have $\frac{\gamma(x)q(x)}{\gamma(x)-1} < p^*(x), \forall x \in \Omega$ and $\frac{\delta(x)r(x)}{\delta(x)-1} < p^\partial(x), \forall x \in \partial\Omega$. Using Proposition 2.1, one finds that there exists $C_1 > 0$ such that

$$\|u\|_{L^{\frac{\gamma(x)q(x)}{\gamma(x)-1}}(\Omega)} \leq C_1\|u\|_{\alpha,\beta} \quad \text{and } \|u\|_{L^{\frac{\delta(x)r(x)}{\delta(x)-1}}(\partial\Omega)} \leq C_1\|u\|_{\alpha,\beta} \quad \text{for all } u \in X. \tag{3.3}$$

Fixing $\rho \in (0,1)$ with $\rho < \frac{1}{C_1}$, we have from (3.3) that $\|u\|_{L^{\frac{\gamma(x)q(x)}{\gamma(x)-1}}(\Omega)} < 1$ and $\|u\|_{L^{\frac{\delta(x)r(x)}{\delta(x)-1}}(\partial\Omega)} < 1$ for all $u \in X$ with $\|u\|_{\alpha,\beta} = \rho$. By using Hölder's inequality, Propositions 2.2 and 2.3, we have

$$\int_\Omega a(x)|u|^{q(x)}dx \leq 2\|a\|_{L^{\gamma(x)}(\Omega)}\left\||u|^{q(x)}\right\|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)} \leq 2\|a\|_{L^{\gamma(x)}(\Omega)}\|u\|^{q^-}_{L^{\frac{\gamma(x)q(x)}{\gamma(x)-1}}(\Omega)}, \tag{3.4}$$

$$\int_{\partial\Omega} b(x)|u|^{r(x)}dx \leq 2\|b\|_{L^{\delta(x)}(\partial\Omega)}\left\||u|^{r(x)}\right\|_{L^{\frac{\delta(x)}{\delta(x)-1}}(\partial\Omega)} \leq 2\|b\|_{L^{\delta(x)}(\partial\Omega)}\|u\|^{r^-}_{L^{\frac{\delta(x)r(x)}{\delta(x)-1}}(\partial\Omega)}, \tag{3.5}$$

for all $u \in X$ with $\|u\|_{\alpha,\beta} = \rho$. Combining (3.3), (3.4), (3.5), we obtain, for all $u \in X$ with $\|u\|_{\alpha,\beta} = \rho$,

$$\int_\Omega a(x)|u|^{q(x)}dx \leq 2C_1^{q^-}\|a\|_{L^{\gamma(x)}(\Omega)}\|u\|^{q^-}_{\alpha,\beta} \tag{3.6}$$

and

$$\int_\Omega b(x)|u|^{r(x)}d\sigma \leq 2C_1^{r^-}\|b\|_{L^{\delta(x)}(\partial\Omega)}\|u\|^{r^-}_{\alpha,\beta}. \tag{3.7}$$

Hence, we deduce that, for any $u \in X$ with $\|u\|_{\alpha,\beta} = \rho$,

$$\begin{aligned} J_\lambda(u) \;&\geq\; \frac{1}{p^+}\Big(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|u|^{p(x)}d\sigma\Big) \\ &-\; \frac{\lambda}{q^-}\int_\Omega a(x)|u|^{q(x)}dx - \frac{\lambda}{r^-}\int_{\partial\Omega} b(x)|u|^{r(x)}d\sigma \\ &\geq\; \frac{1}{p^+}\|u\|^{p^+}_{\alpha,\beta} - \frac{2\lambda}{q^-}C_1^{q^-}\|a\|_{L^{\gamma(x)}(\Omega)}\|u\|^{q^-}_{\alpha,\beta} - \frac{2\lambda}{r^-}C_1^{r^-}\|b\|_{L^{\delta(x)}(\partial\Omega)}\|u\|^{r^-}_{\alpha,\beta}. \end{aligned}$$

If

$$\lambda^* = \min\left\{\frac{q^-\rho^{p^+-q^-}}{8C_1^{q^-}p^+\|a\|_{L^{\gamma(x)}(\Omega)}}, \frac{r^-\rho^{p^+-r^-}}{8C_1^{r^-}p^+\|b\|_{L^{\delta(x)}(\partial\Omega)}}\right\} \quad \text{and } \tau = \frac{\rho^{p^+}}{2p^+},$$

then

$$J_\lambda(u) \geq \tau > 0, \quad \text{for all } u \in X \text{ with } \|u\|_{\alpha,\beta} = \rho.$$

This completes the proof. □

Lemma 3.3. *There exists $\varphi \in X$ such that $\varphi \geq 0, \varphi \neq 0$ and $J_\lambda(t\varphi) < 0$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 0$, $\varphi \not\equiv 0$ and $t \in (0,1)$. We have

$$\begin{aligned} J_\lambda(t\varphi) &\leq \frac{t^{p^-}}{p^-}\left(\int_\Omega |\nabla\varphi|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|\varphi|^{p(x)}d\sigma\right) \\ &- \frac{\lambda t^{q^+}}{q^+}\int_\Omega a(x)|\varphi|^{q(x)}dx - \frac{\lambda t^{r^+}}{r^+}\int_{\partial\Omega} b(x)|\varphi|^{r(x)}d\sigma \\ &\leq \frac{t^{p^-}}{p^-}\left(\int_\Omega |\nabla\varphi|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|\varphi|^{p(x)}d\sigma\right) \\ &- \frac{\lambda t^{r^+}}{r^+}\int_{\partial\Omega} b(x)|\varphi|^{r(x)}d\sigma. \end{aligned}$$

For $t < \eta^{\frac{1}{p^- - r^+}}$ with

$$0 < \eta < \min\left\{1, \frac{\lambda p^- \int_{\partial\Omega} b(x)|\varphi|^{r(x)}d\sigma}{r^+\left(\int_\Omega |\nabla\varphi|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|\varphi|^{p(x)}d\sigma\right)}\right\},$$

we deduce that $J_\lambda(t\varphi) < 0$. This completes the proof. □

Lemma 3.4. *The functional J_λ satisfies the Palais-Smale condition (PS).*

Proof. Let $(u_n) \in X$ be a (PS) sequence, i.e.,

$$|J_\lambda(u_n)| \leq M \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

We show that (u_n) is bounded in X. Passing eventually to a subsequence, still denote by (u_n), we have $\|u_n\|_{\alpha,\beta} \to \infty$ as $n \to \infty$. We assume $\|u_n\|_{\alpha,\beta} > 1$. Since $J_\lambda(u_n)$ is bounded, we have, for n large enough,

$$
\begin{aligned}
&M+1\\
&\geq J_\lambda(u_n)-\frac{1}{q^-}\langle J'_\lambda(u_n),u_n\rangle+\frac{1}{q^-}\langle J'_\lambda(u_n),u_n\rangle\\
&=\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}dx+\int_\Omega\frac{\alpha(x)}{p(x)}|u_n|^{p(x)}dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma\\
&\quad-\lambda\int_\Omega\frac{a(x)}{q(x)}|u_n|^{q(x)}dx-\lambda\int_{\partial\Omega}\frac{b(x)}{r(x)}|u_n|^{r(x)}d\sigma\\
&\quad-\frac{1}{q^-}\left(\int_\Omega|\nabla u_n|^{p(x)}dx+\int_\Omega\alpha(x)|u_n|^{p(x)}dx+\int_{\partial\Omega}\beta(x)|u_n|^{p(x)}d\sigma\right)\\
&\quad+\frac{\lambda}{q^-}\int_\Omega a(x)|u_n|^{q(x)}dx+\frac{\lambda}{q^-}\int_{\partial\Omega}b(x)|u_n|^{r(x)}d\sigma+\frac{1}{q^-}\langle J'_\lambda(u_n),u_n\rangle\\
&\geq\frac{1}{p^+}I_{\alpha,\beta}(u_n)-\frac{\lambda}{q^-}\int_\Omega a(x)|u_n|^{q(x)}dx-\frac{\lambda}{r^-}\int_\Omega b(x)|u_n|^{r(x)}d\sigma-\frac{1}{q^-}I_{\alpha,\beta}(u_n)\\
&\quad+\frac{\lambda}{q^-}\int_\Omega a(x)|u_n|^{q(x)}dx+\frac{\lambda}{q^-}\int_\Omega b(x)|u_n|^{r(x)}d\sigma+\frac{1}{q^-}\langle J'_\lambda(u_n),u_n\rangle\\
&\geq\left(\frac{1}{p^+}-\frac{1}{q^-}\right)I_{\alpha,\beta}(u_n)-\lambda\left(\frac{1}{r^-}-\frac{1}{q^-}\right)C_2\|b\|_{L^{\delta(x)}(\partial\Omega)}\|u_n\|_{\alpha,\beta}^{r^+}-\frac{1}{q^-}\|J'_\lambda(u_n)\|_{X'}\|u_n\|_{\alpha,\beta}\\
&\geq\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\|u_n\|_{\alpha,\beta}^{p^-}-\lambda\left(\frac{1}{r^-}-\frac{1}{q^-}\right)C_2\|b\|_{L^{\delta(x)}(\partial\Omega)}\|u_n\|_{\alpha,\beta}^{r^+}-\frac{C_3}{q^-}\|u_n\|_{\alpha,\beta},
\end{aligned}
$$

where C_2, C_3 are two positive constants. Since $r^-\leq r^+<p^-\leq p^+<q^-$, we obtain that (u_n) is bounded in X. Therefore, there exists a subsequence, again denoted by (u_n) and $u\in X$ such that $u_n\rightharpoonup u$ in X. By (3.8), we have

$$
\begin{aligned}
\langle J'_\lambda(u_n),u_n-u\rangle &= \langle\Phi'(u_n),u_n-u\rangle-\lambda\int_\Omega a(x)|u_n|^{q(x)-2}u_n(u_n-u)dx\\
&\quad-\lambda\int_{\partial\Omega}b(x)|u_n|^{r(x)-2}u_n(u_n-u)d\sigma\to 0.
\end{aligned}\tag{3.9}
$$

On the other hand, using Hölder's inequality, Proposition 2.2 and the compact embedding $X\hookrightarrow L^{q(x)}_{a(x)}(\Omega)$ (Proposition 2.4), we have

$$\int_\Omega a(x)|u_n|^{q(x)-2}u_n(u_n-u)dx\to 0.\tag{3.10}$$

Similarly, using Hölder's inequality, Proposition 2.3 and the compact embedding $X\hookrightarrow L^{r(x)}_{b(x)}(\partial\Omega)$ (Proposition 2.5), we also have

$$\int_{\partial\Omega}b(x)|u_n|^{r(x)-2}u_n(u_n-u)d\sigma\to 0.\tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we deduce that $\langle\Phi'(u_n),u_n-u\rangle\to 0$. Thus by Proposition 3.1, it follows that $u_n\to u$ in X. This completes the proof. $\square$

Now we are in position to prove our main result. To apply the Mountain Pass theorem, we need to prove that

$$J_\lambda(tu) \to -\infty \text{ as } t \to +\infty,$$

for a certain $u \in X$. Let $\omega \in C_0^\infty(\Omega)$ such that $\omega \geq 0$, $\omega \not\equiv 0$. We have, for $t > 1$,

$$\begin{aligned} J_\lambda(t\omega) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\omega|^{p(x)}dx + \int_\Omega \frac{t^{p(x)}}{p(x)}\alpha(x)|\omega|^{p(x)}dx + \int_{\partial\Omega} \frac{t^{p(x)}}{p(x)}\beta(x)|\omega|^{p(x)}d\sigma \\ &- \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}a(x)|\omega|^{q(x)}dx - \lambda\int_{\partial\Omega} \frac{t^{r(x)}}{r(x)}b(x)|\omega|^{r(x)}d\sigma \\ &\leq \frac{t^{p^+}}{p^-}\Big(\int_\Omega |\nabla\omega|^{p(x)}dx + \int_\Omega \alpha(x)|u|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|\omega|^{p(x)}d\sigma\Big) \\ &- \frac{\lambda t^{q^-}}{q^+}\int_\Omega a(x)|\omega|^{p(x)}dx - \frac{\lambda t^{r^-}}{r^+}\int_{\partial\Omega} b(x)|\omega|^{r(x)}d\sigma. \end{aligned}$$

Since $r^-, p^+ < q^-$, we have $J_\lambda(t\omega) \to -\infty$ as $t \to +\infty$. It follows that there exists $e \in X$ such that $\|e\|_{\alpha,\beta} > \rho$ and $J_\lambda(e) < 0$. According to the Mountain Pass Theorem, J_λ admits a critical value $\zeta \geq \tau$, which is characterized by

$$\zeta := \inf_{g\in\Gamma}\sup_{t\in[0,1]} J_\lambda(g(t)),$$

where

$$\Gamma = \Big\{g \in C([0,1],X) : g(0) = 0, g(1) = e\Big\}.$$

This completes the proof.

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