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OSCILLATION CRITERIA FOR A CLASS OF NONLINEAR CONFORMABLE FRACTIONAL DAMPED DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we consider a class of nonlinear damped dynamic equations on time scales within conformable fractional derivative. We accommodate the newly defined conformable time-scale fractional calculus to establish new oscillation criteria for the solutions of canonical and noncanonical form of the proposed equation. Our approach is based on the implementation of generalized Riccati transformation, some properties of conformable time-scale fractional calculus and certain mathematical inequalities. Applications of the established results demonstrate that one can obtain oscillation criteria for both fractional differential equations and fractional difference equations simultaneously. The validity of the main results are also illustrated by some examples.

Keywords. Oscillation of solutions; Conformable damped dynamic equations; Conformable time-scale fractional calculus.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order. Recently, it has been realized that the fractional calculus has numerous applications in engineering, economics and finance, signal processing, dynamics of earthquakes, geology, probability and statistics, chemical engineering, physics, splines, thermodynamics and neural networks; see, e.g., [1, 2, 3, 4, 5, 6] and the references cited therein. Due to their widespread applications in the field engineering, the investigations of dynamic equations on time scale have attracted many researchers during the last decades. This considerable interest is owed to the fact that the dynamic equations on time scale, which was introduced by Hilger [7], have the remarkable feature of unifying the continuous and discrete calculus. In the literatures, one

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can easily find that there are tremendous number of important results discussing the properties of dynamic equations [8, 9]. Particularly, the investigation of the oscillation of solutions of dynamic equations has gained extensive attention [10, 11, 12, 13, 14, 15]. Comparably, it is not the case for fractional differential/difference equations where only few results were recently reported; see [16, 17, 18, 19, 20, 21].

In recent years, there have appeared different types of fractional derivatives. Despite their flexible applicabilities, all these definitions of fractional derivatives have undeviating disadvantages. Fortunately, Khalil *et al.* [22] defined a new well-behaved fractional derivative, which is called the "conformable fractional derivative" or simply "conformable derivative" depending entirely on the basic limit definition of the derivative. Thereafter, researchers developed the conformable derivative and obtained different results exposing its features [23, 24]. In [25, 26], the conformable time-scale fractional calculus was introduced. Applications of the established results demonstrate that the newly defined calculus can be used to research oscillation for both fractional differential equations and fractional difference equations at the same time. Therefore, the determination of oscillation of solutions of conformable fractional dynamic equations has become promising topic for researchers. To the best of our observation, the oscillation of conformable fractional dynamic equations was only studied in [27].

2. Preliminaries

We consider a class of nonlinear conformable fractional damped dynamic equations on time scales of the form

$$(r(t)(x^{(\alpha)}(t))^{\gamma})^{(\alpha)}+p(t)(x^{(\alpha)}(t))^{\gamma}+q(t)f(x(t))=0,\ \ t\in[t_0,\infty)_{\mathbb{T}}, \tag{2.1}$$

where $\mathbb{T}$ denotes an arbitrary time scale, $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$, $(\cdot)^{(\alpha)}$ is the conformable fractional derivative of order α, $r(t),p(t),q(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$, $f\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$ and $\gamma\geq 1$ is a quotient of two odd positive integers. We establish new oscillation and oscillatory behavior criteria for the solutions of equation (2.1) when the nonlinear function f is increasing and nonincreasing. Besides, the results are carried out in light of the following two cases:

(C_1) $\displaystyle\int_{t_0}^{\infty}\frac{\Delta^{\alpha}s}{r(s)}=\infty$,

(C_2) $\displaystyle\int_{t_0}^{\infty}\frac{\Delta^{\alpha}s}{r(s)}<\infty$.

Equation (2.1) is said to be in canonical form if (C_1) holds whereas it is called noncanonical form if (C_2) holds. Our approach is based on the implementation of generalized Riccati transformation, some properties of conformable time scale fractional calculus and certain mathematical inequalities. It is worthy mentioning that our results have merits and differ from the ones obtained [27].

By a solution of (2.1), we mean a nontrivial real valued function $x(t)$ satisfying (2.1) for $t\geq t_0$. A solution of (2.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (2.1) is said to be oscillatory if its all solutions are oscillatory.

Before we proceed to the main results, we assemble essential preliminaries on conformable time-scale fractional calculus that will be used to justify further discussion. Terms and definitions are adopted from [8, 25].

Definition 2.1. [8, p. 1] On any time scale $\mathbb{T}$, define the forward and backward jump operators by $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$, respectively. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, right-scattered if $\sigma(t) > t$. The graininess function $\mu(t)$ of the time scale is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^k = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}) & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$

Definition 2.2. [8, p. 5] A function $f : \mathbb{T} \to \mathbb{R}$ is said to be Δ-differentiable if $\lim_{s \to t} \dfrac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$ exists for $s \in \mathbb{T}/\{\sigma(t)\}$. The limit is denoted by $f^{\Delta}(t)$, the Δ-derivative of $f(t)$.

Definition 2.3. [8, p. 2] A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point and if there exists a finite left limit of f at all left dense points.

Definition 2.4. [8, p. 58] A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for $t \in \mathbb{T}^k$. Let $\mathscr{R}$ be the set of all $f : \mathbb{T} \to \mathbb{R}$, which are rd-continuous and regressive. We define $\mathscr{R}^+ := \{f \in \mathscr{R} : 1 + \mu(t)f(t) > 0, t \in \mathbb{T}\}$.

Definition 2.5. [8, p. 59] If $p \in \mathscr{R}$, then the exponential function is defined by

$$e_p(t,s) = \exp\left(\int_s^t \zeta_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for } t \in \mathbb{T},\ s \in \mathbb{T}^k,$$

where $\zeta_h(z)$ is the cylindrical transformation given by $\zeta_h(z) = \begin{cases} \frac{log(1+hz)}{h}, h \neq 0, \\ z, h = 0. \end{cases}$

Also $e_p(t,s)$ is a nonzero real valued function and $e_p(\sigma(t);t_0) = [1 + \mu(t)p(t)]e_p(t,t_0)$.

Definition 2.6. [25] For $t \in \mathbb{T}^k$, $\alpha \in (0,1]$ and a function $f : \mathbb{T} \to \mathbb{R}$, the conformable fractional derivative of order α for f at t is denoted by $f^{(\alpha)}(t)$ (provided that it exists) with property such that, for every $\varepsilon > 0$, there exists a neighborhood $\mathfrak{N}$ of t satisfying

$$|[f(\sigma(t)) - f(s)]t^{1-\alpha} - f^{(\alpha)}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in \mathfrak{N}.$$

Definition 2.7. [25] If $F^{(\alpha)}(t) = f(t)$, $t \in \mathbb{T}^k$, then F is called an α-order anti-derivative of f and the Cauchy α-fractional integral of f is defined by

$$\int_a^b f(t)\Delta^{\alpha}t = \int_a^b f(t)t^{\alpha-1}\Delta t = F(b) - F(a), \text{ where } a, b \in \mathbb{T}.$$

Theorem 2.8. [25] *For $t \in \mathbb{T}^k$, $\alpha \in (0,1]$ and a function $f : \mathbb{T} \to \mathbb{R}$, the following statements hold:*

(i) *If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.*
(ii) *If f is continuous at t and t is right scattered, then f is conformable fractional differentiable of orderα at t with $f^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}$.*
(iii) *If t is right dense, then f is conformal fractional differentiable of order α at t if and only if $f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} t^{1-\alpha}$ exists as a finite number.*

(iv) *If f is conformal fractional differentiable of order α at t then*

$$f(\sigma(t)) = f(t) + \mu(t)t^{1-\alpha} f^{(\alpha)}(t).$$

Theorem 2.9. [25] *According to the definition of the conformable fractional differentiable of order α, it holds that $f^{(\alpha)}(t) = t^{1-\alpha} f^{\Delta}(t)$. Also, if $f^{(\alpha)}(t) > 0 (< 0)$ for $t > 0$, then f is increasing (decreasing) for $t > 0$.*

Theorem 2.10. [27] *Let $\tilde{p}(t) = t^{\alpha-1} p(t), \alpha \in (0,1]$. If $\tilde{p} \in \mathscr{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_{\tilde{p}}(t,t_0)$ is the unique solution of the initial value problem*

$$y^{(\alpha)}(t) = p(t)y(t), \;\; y(t_0) = 1 \text{ on } \mathbb{T}.$$

Theorem 2.11. [25] *Let $f, g : \mathbb{T} \to \mathbb{R}$ be conformable fractional differentiable of order α at $t \in \mathbb{T}^k$. Then*

(i) $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t) = f(t)g^{(\alpha)}(t) + f^{(\alpha)}(t)g(\sigma(t))$,

(ii) $\left(\dfrac{f}{g}\right)^{(\alpha)}(t) = \dfrac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}$, *provided* $g(t)g(\sigma(t)) \neq 0$,

(iii) $f(\sigma(t)) = f(t) + \mu(t)t^{1-\alpha} f^{(\alpha)}(t)$.

Theorem 2.12. [25] *Let $\alpha \in (0,1]$; $f, g : \mathbb{T} \to \mathbb{R}$ be rd-continuous functions and $a, b \in \mathbb{T}$. Then*

$$\int_a^b f(t)g^{(\alpha)}(t)\Delta^{\alpha}t = [f(t)g(t)]_a^b - \int_a^b f^{(\alpha)}(t)g(\sigma(t))\Delta^{\alpha}t.$$

Definition 2.13. Let $\mathbb{D} = \{(t,s) | t \geq s \geq t_0\}$. Then the class $\mathscr{H}$ is a collection of functions $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ such that $H(t,t) = 0$ for $t \geq t_0$; $H(t,s) > 0$ for $t > s \geq t_0$ and H has a non positive continuous α-partial fractional derivative $H_s^{(\alpha)}(t,s)$ with respect to the second variable.

Lemma 2.14. [28] *Assume that U and V are nonnegative real numbers and $\gamma \geq 1$ is a quotient of two odd positive integers. Then*

$$(U-V)^{1-\frac{1}{\gamma}} \geq U^{1-\frac{1}{\gamma}} + \frac{1}{\gamma} V^{1-\frac{1}{\gamma}} - \left(1 - \frac{1}{\gamma}\right) V^{\frac{1}{\gamma}} U.$$

Lemma 2.15. [28] *Assume that X and Y are nonnegative real numbers and $\lambda > 1$. Then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda}.$$

In the sequel, we will use the following notations for convenience:

$$R(t) = -\frac{t^{\alpha-1}p(t)}{r(t)}, \quad \mathfrak{U}(t,t_0) = \left(\frac{e_{R(t)}(t,t_0)}{r(t)}\right)^{\frac{1}{\gamma}}, \;\; \mathfrak{U}_1(t,t_0) = \frac{e_{R(t)}(t,t_0)}{r(t)}$$

3. Results and Discussion

This section is devoted to the main results of the paper. We will present the results in two folds based on the monotonicity of f.

3.1. **Oscillation criteria for equation (2.1) when f is not necessarily increasing.** To establish oscillation criteria, we make use of the following assumption:

(H_1) $\dfrac{f(x)}{x^\gamma} \geq K$ for some $K > 0$ and for all $x \neq 0$.

Lemma 3.1. *Let $R(t) \in \mathscr{R}^+$. If (C_1) and (H_1) hold and (2.1) has an eventually positive solution x. Then there exists a sufficiently large $t^* \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$\left(\frac{r(t)(x^{(\alpha)}(t))^\gamma}{e_{R(t)}(t,t_0)}\right)^{(\alpha)} < 0 \quad \textit{and} \quad x^{(\alpha)}(t) > 0 \textit{ on } [t^*, \infty)_{\mathbb{T}}.$$

Proof. Since x is eventually a positive solution of (2.1), there exists a sufficiently large t_1 such that $x(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. Now, we have

$$\begin{aligned}\left(\frac{r(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)}\right)^{(\alpha)} &= \frac{e_{R(t)}(t,t_0)(r(t)x^{(\alpha)}(t))^{(\alpha)} - r(t)x^{(\alpha)}(t)(e_{R(t)}(t,t_0))^{(\alpha)}}{e_{R(t)}(t,t_0)e_{R(t)}(\sigma(t),t_0)} \\ &= \frac{e_{R(t)}(t,t_0)(r(t)x^{(\alpha)}(t))^{(\alpha)} + p(t)x^{(\alpha)}(t)e_{R(t)}(t,t_0)}{e_{R(t)}(t,t_0)e_{R(t)}(\sigma(t),t_0)} \\ &= \frac{-q(t)f(x(t))}{e_{R(t)}(\sigma(t),t_0)} < 0. \end{aligned} \tag{3.1}$$

Since $R(t) \in \mathscr{R}^+$, we get $e_{R(t)}(\sigma(t),t_0) > 0$ and $\dfrac{r(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)}$ is decreasing on $[t_1, \infty)_{\mathbb{T}}$. Therefore, by the assumption of $r(t) > 0$ and $R(t) \in \mathscr{R}^+$, we obtain that $x^{(\alpha)}(t)$ is eventually of one sign. Suppose that $x^{(\alpha)}(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$ for sufficiently large $t_2 > t_1$. Then

$$x(t) - x(t_2) = \int_{t_2}^{t} \frac{r(s)x^{(\alpha)}(s)}{r(s)} \Delta^\alpha s \leq r(t_2)x^{(\alpha)}(t_2) \int_{t_2}^{t} \frac{\Delta^\alpha s}{r(s)}.$$

Letting $t \to \infty$ and using (C_1), we get $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction to $x(t) > 0$. Therefore $x^{(\alpha)}(t) > 0$. □

Theorem 3.2. *Let $R(t) \in \mathscr{R}^+$. If $(C_1), (H_1)$ hold and there exist two function $z(t) > 0$ and $\psi(t) \geq 0$ on $[T, \infty)_{\mathbb{T}}$ such that*

$$\begin{aligned}\limsup_{t\to\infty} \quad \int_T^t \Bigg\{ &\frac{Kq(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + z(s)\mathfrak{U}(s,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(s)) \\ &- \left[\frac{z^{(\alpha)}(s) + (\gamma+1)z(s)\mathfrak{U}(s,t_0)\psi^{\frac{1}{\gamma}}(\sigma(s))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,t_0)}\right]^{\gamma+1}\Bigg\} \Delta^\alpha s = \infty, \end{aligned} \tag{3.2}$$

then equation (2.1) is oscillatory.

Proof. Assume on the contrary that x is a nonoscillatory solution of (2.1). Without loss of generality, assume that $x(t) > 0$ in $[t_1, \infty)_{\mathbb{T}}$ for $t_1 > t_0$. Then by Lemma 3.1, there exists $t_2 > t_1$

such that $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Define a generalized Riccati function as follows:

$$w(t) = z(t)\left[\frac{r(t)(x^{(\alpha)}(t))^{\gamma}}{x^{\gamma}(t)e_{R(t)}(t,t_0)} + \psi(t)\right].$$

Clearly, $w(t) \geq 0$ and

$$\begin{aligned}
w^{(\alpha)}(t) &= \frac{z(t)}{x^{\gamma}(t)}\left[\frac{e_{R(t)}(t,t_0)[r(t)(x^{(\alpha)}(t))^{\gamma}]^{(\alpha)} + e_{R(t)}(t,t_0)p(t)(x^{(\alpha)}(t))^{\gamma}}{e_{R(t)}(t,t_0)e_{R(t)}(\sigma(t),t_0)}\right] \\
&\quad + \left[\frac{x^{\gamma}(t)z^{(\alpha)}(t) - z(t)(x^{\gamma}(t))^{(\alpha)}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right]\frac{r(\sigma(t))(x^{(\alpha)}(\sigma(t)))^{\gamma}}{e_{R(t)}(\sigma(t),t_0)} \\
&\quad + z(t)\psi^{(\alpha)}(t) + z^{(\alpha)}(t)\psi(\sigma(t)) \\
&\leq \frac{z(t)}{x^{\gamma}(t)}\left[\frac{-q(t)f(x(t))}{e_{R(t)}(\sigma(t),t_0)}\right] + \frac{z^{(\alpha)}(t)}{z(\sigma(t))}w(\sigma(t)) \\
&\quad - \frac{z(t)(x^{\gamma}(t))^{(\alpha)}r(\sigma(t))(x^{(\alpha)}(\sigma(t)))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))e_{R(t)}(\sigma(t),t_0)} + z(t)\psi^{(\alpha)}(t) \\
&\leq \frac{-Kq(t)z(t)}{e_{R(t)}(\sigma(t),t_0)} + \frac{z^{(\alpha)}(t)}{z(\sigma(t))}w(\sigma(t)) \\
&\quad - \frac{z(t)\gamma x^{\gamma-1}(t)x^{(\alpha)}(t)r(\sigma(t))(x^{(\alpha)}(\sigma(t)))^{\gamma}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))e_{R(t)}(\sigma(t),t_0)} + z(t)\psi^{(\alpha)}(t) \\
&\leq \frac{-Kq(t)z(t)}{e_{R(t)}(\sigma(t),t_0)} + \frac{z^{(\alpha)}(t)}{z(\sigma(t))}w(\sigma(t)) \\
&\quad - \gamma z(t)\mathfrak{U}(t,t_0)\left[\frac{w(\sigma(t))}{z(\sigma(t))} - \psi(\sigma(t))\right]^{1+\frac{1}{\gamma}} + z(t)\psi^{(\alpha)}(t). \qquad (3.3)
\end{aligned}$$

From Lemma 2.14, we have

$$\left[\frac{w(\sigma(t))}{z(\sigma(t)} - \psi(\sigma(t))\right]^{1+\frac{1}{\gamma}} \geq \frac{w^{1+\frac{1}{\gamma}}(\sigma(t))}{z^{1+\frac{1}{\gamma}}(\sigma(t))} + \frac{1}{\gamma}\psi^{1+\frac{1}{\gamma}}(\sigma(t)) - \left(1+\frac{1}{\gamma}\right)\frac{\psi^{\frac{1}{\gamma}}(\sigma(t))w(\sigma(t))}{z(\sigma(t))}. \qquad (3.4)$$

From (3.3) and (3.4), we obtain that

$$\begin{aligned}
w^{(\alpha)}(t) &\leq \frac{-Kq(t)z(t)}{e_{R(t)}(\sigma(t),t_0)} + z(t)\psi^{(\alpha)}(t) - z(t)\mathfrak{U}(t,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(t)) \\
&\quad + \left[z^{(\alpha)}(t) + (\gamma+1)z(t)\mathfrak{U}(t,t_0)\psi^{\frac{1}{\gamma}}(\sigma(t))\right]\frac{w(\sigma(t))}{z(\sigma(t))} \\
&\quad - \gamma z(t)\mathfrak{U}(t,t_0)\frac{w^{1+\frac{1}{\gamma}}(\sigma(t))}{z^{1+\frac{1}{\gamma}}(\sigma(t))}. \qquad (3.5)
\end{aligned}$$

Let

$$\lambda = 1+\frac{1}{\gamma},\ \ X^{\lambda} = \gamma z(t)\mathfrak{U}(t,t_0)\frac{w^{1+\frac{1}{\gamma}}(\sigma(t))}{z^{1+\frac{1}{\gamma}}(\sigma(t))},$$

$$Y^{\lambda-1}=\gamma^{\frac{1}{\gamma+1}}\left[\frac{z^{(\alpha)}(t)+(\gamma+1)z(t)\mathfrak{U}(t,t_0)\psi^{\frac{1}{\gamma}}(\sigma(t))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(t)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(t,t_0)}\right].$$

Using (3.5) and Lemma 2.15, we have

$$\begin{aligned} w^{(\alpha)}(t) &\le \frac{-Kq(t)z(t)}{e_{R(t)}(\sigma(t),t_0)}+z(t)\psi^{(\alpha)}(t)-z(t)\mathfrak{U}(t,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(t)) \\ &\quad+\left[\frac{z^{(\alpha)}(t)+(\gamma+1)z(t)\mathfrak{U}(t,t_0)\psi^{\frac{1}{\gamma}}(\sigma(t))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(t)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(t,t_0)}\right]^{\gamma+1}. \end{aligned}$$

By taking α- fractional integral for the above inequality from t_2 to t, we get

$$\begin{aligned} &\int_{t_2}^{t}\left\{\frac{Kq(s)z(s)}{e_{R(s)}(\sigma(s),t_0)}-z(s)\psi^{(\alpha)}(s)+z(s)\mathfrak{U}(s,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(s))\right. \\ &\left.-\left[\frac{z^{(\alpha)}(s)+(\gamma+1)z(s)\mathfrak{U}(s,t_0)\psi^{\frac{1}{\gamma}}(\sigma(s))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,t_0)}\right]^{\gamma+1}\right\}\Delta^{\alpha}s\le w(t_2)-w(t)\le w(t_2)\le\infty, \end{aligned}$$

which is a contraction to (3.2). The proof is complete. □

Theorem 3.3. *Let $R(t)\in\mathscr{R}^{+}$. If $(C_2),(H_1)$ hold, (2.1) has an eventually positive solution x and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{e_{R(s)}(s,t_0)}{r(s)}\int_{t_3}^{s}\frac{q(\nu)}{e_{R(\nu)}(\sigma(\nu),t_0)}\Delta^{\alpha}\nu\right]^{\frac{1}{\gamma}}\Delta^{\alpha}s=\infty, \tag{3.6}$$

then there exists a sufficiently large $t^{*}\in[t_0,\infty)_{\mathbb{T}}$ such that $x^{(\alpha)}(t)>0$ on $[t^{*},\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t)=0$.

Proof. Proceeding as in the proof of Lemma 3.5, we obtain that $x^{(\alpha)}(t)$ is eventually of one sign. Then there exists a sufficiently large $t_2>t_1$ such that either $x^{(\alpha)}(t)>0$ or $x^{(\alpha)}(t)<0$ on $[t_2,\infty)_{\mathbb{T}}$. Suppose that $x^{(\alpha)}(t)<0$. By the hypothesis of the lemma, we get $\lim_{t\to\infty}x(t)=C\ge 0$, where C is a constant. If $C>0$, then $x(t)\ge C$ on $[t_3,\infty)_{\mathbb{T}}$ for $t_3>t_2$. Now, by applying α-fractional integral for (3.1) from t_3 to t and using (H_1), we have

$$\begin{aligned} \frac{r(t)(x^{(\alpha)}(t))^{\gamma}}{e_{R(t)}(t,t_0)} &= \frac{r(t_3)(x^{(\alpha)}(t_3))^{\gamma}}{e_{R(t)}(t_3,t_0)}-\int_{t_3}^{t}\frac{q(s)f(x(s))}{e_{R(s)}(\sigma(s),t_0)}\Delta^{\alpha}s \\ &\le -KC^{\gamma}\int_{t_3}^{t}\frac{q(s)}{e_{R(s)}(\sigma(s),t_0)}\Delta^{\alpha}s \\ x^{(\alpha)}(t) &\le -\left[KC^{\gamma}\frac{e_{R(t)}(t,t_0)}{r(t)}\int_{t_3}^{t}\frac{q(s)}{e_{R(s)}(\sigma(s),t_0)}\Delta^{\alpha}s\right]^{\frac{1}{\gamma}}. \end{aligned}$$

Taking α-fractional integral from t_3 to t gives

$$x(t)\le x(t_3)-KC^{\gamma}\int_{t_3}^{t}\left[\frac{e_{R(s)}(s,t_0)}{r(s)}\int_{t_3}^{s}\frac{q(\nu)}{e_{R(\nu)}(\sigma(\nu),t_0)}\Delta^{\alpha}\nu\right]^{\frac{1}{\gamma}}\Delta^{\alpha}s.$$

Letting $t \to \infty$ and using (3.6), we get $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction to $x(t) > 0$. Therefore $\lim_{t\to\infty} x(t) = C = 0$. Hence the proof is complete. □

In light of the above results and their proofs, we can formulate the following theorem which involves H function.

Theorem 3.4. *Let that $R(t) \in \mathscr{R}^+$. If (C_1) and,(H_1) hold and there exist two function $z(t) > 0$ and $\psi(t) \geq 0$ on $[T,\infty)_{\mathbb{T}}$ and $H \in \mathscr{H}$, for sufficiently large T such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \left\{ \int_T^t H(t,s) \left\{ \frac{Kq(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + z(s)\mathfrak{U}(s,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(s)) - \left[\frac{z^{(\alpha)}(s) + (\gamma+1)z(s)\mathfrak{U}(s,t_0)\psi^{\frac{1}{\gamma}}(\sigma(s))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,t_0)} \right]^{\gamma+1} \right\} \Delta^\alpha s \right\} = \infty, \tag{3.7}$$

then equatin (2.1) is oscillatory.

3.2. **Oscillation criteria for equation (2.1) when f is increasing.** To establish oscillation criteria, we make use of the following assumption:

(H_2) f' exists and $f'(x) \geq M$ for some $M > 0$ and for all $x \neq 0$; $\gamma = 1$.

Lemma 3.5. *Let $R(t) \in \mathscr{R}^+$. If $(C_1),(H_2)$ holds and (2.1) has an eventually positive solution $x(t)$, then there exists a sufficiently large $t^* \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$\left(\frac{r(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)} \right)^{(\alpha)} < 0 \quad \text{and} \quad x^{(\alpha)}(t) > 0 \text{ on } [t^*,\infty)_{\mathbb{T}}.$$

Proof. Since x is eventually a positive solution of (2.1), there exists a sufficiently large t_1 such that $x(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Now, we have

$$\begin{aligned} \left(\frac{r(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)} \right)^{(\alpha)} &= \frac{e_{R(t)}(t,t_0)(r(t)x^{(\alpha)}(t))^{(\alpha)} - r(t)x^{(\alpha)}(t)(e_{R(t)}(t,t_0))^{(\alpha)}}{e_{R(t)}(t,t_0)e_{R(t)}(\sigma(t),t_0)} \\ &= \frac{e_{R(t)}(t,t_0)(r(t)x^{(\alpha)}(t))^{(\alpha)} + p(t)x^{(\alpha)}(t)e_{R(t)}(t,t_0)}{e_{R(t)}(t,t_0)e_{R(t)}(\sigma(t),t_0)} \\ &= \frac{-q(t)f(x(t))}{e_{R(t)}(\sigma(t),t_0)} < 0. \end{aligned} \tag{3.8}$$

Since $R(t) \in \mathscr{R}^+$, we get that $e_{R(t)}(\sigma(t),t_0) > 0$ and $\frac{r(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$. Therefore, by the assumption of $r(t) > 0$ and $R(t) \in \mathscr{R}^+$, we obtain that $x^{(\alpha)}(t)$ is eventually of one sign. Suppose that $x^{(\alpha)}(t) < 0$ on $[t_2,\infty)_{\mathbb{T}}$ for sufficiently large $t_2 > t_1$. Then

$$x(t) - x(t_2) = \int_{t_2}^t \frac{r(s)x^{(\alpha)}(s)}{r(s)} \Delta^\alpha s \leq r(t_2)x^{(\alpha)}(t_2) \int_{t_2}^t \frac{\Delta^\alpha s}{r(s)}.$$

Letting $t \to \infty$ and using (C_1), we get $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction to $x(t) > 0$. Therefore, $x^{(\alpha)}(t) > 0$. □

Theorem 3.6. *Let $R(t) \in \mathscr{R}^+$. If $(C_1),(H_2)$ hold and there exists two function $z(t) > 0$ and $\psi(t) \geq 0$ on $[T,\infty)_{\mathbb{T}}$ such that*

$$\limsup_{t\to\infty} \int_T^t \left\{ \frac{q(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + Mz(s)\mathfrak{U}_1(s,t_0)\psi^2(\sigma(s)) - \left[\frac{z^{(\alpha)}(s)+2Mz(s)\mathfrak{U}_1(s,t_0)\psi(\sigma(s))}{2M^{\frac{1}{2}}z^{\frac{1}{2}}(s)\mathfrak{U}_1^{\frac{1}{2}}(s,t_0)}\right]^2 \right\}\Delta^\alpha s = \infty, \tag{3.9}$$

then equation (2.1) is oscillatory.

Proof. Assume on the contrary that x is a nonoscillatory solution of (2.1). Without loss of generality, assume that $x(t) > 0$ in $[t_1,\infty)_{\mathbb{T}}$ for $t_1 > t_0$. Using Lemma 3.5, one can see that there exists $t_2 > t_1$ such that $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Define a generalized Riccati function as follows:

$$w(t) = z(t)\left[\frac{r(t)x^{(\alpha)}(t)}{f(x(t))e_{R(t)}(t,t_0)} + \psi(t)\right].$$

Clearly, $w(t) \geq 0$ and

$$\begin{aligned}
w^{(\alpha)}(t) &= \frac{z(t)}{f(x(t))}\left[\frac{e_{R(t)}(t,t_0)[r(t)x^{(\alpha)}(t)]^{(\alpha)} + e_{R(t)}(t,t_0)p(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)e_{R(t)}(\sigma(t),t_0)}\right] \\
&\quad + \left[\frac{f(x(t))z^{(\alpha)}(t) - z(t)(f(x(t)))^{(\alpha)}}{f(x(t))f(x(\sigma(t)))}\right]\frac{r(\sigma(t))x^{(\alpha)}(\sigma(t))}{e_{R(t)}(\sigma(t),t_0)} \\
&\quad + z(t)\psi^{(\alpha)}(t) + z^{(\alpha)}(t)\psi(\sigma(t)) \\
&\leq \frac{z(t)}{f(x(t))}\left[\frac{-q(t)f(x(t))}{e_{R(t)}(\sigma(t),t_0)}\right] + \frac{z^{(\alpha)}(t)}{z(\sigma(t))}w(\sigma(t)) \\
&\quad - \frac{z(t)(f(x(t)))^{(\alpha)}r(\sigma(t))x^{(\alpha)}(\sigma(t))}{f(x(t))f(x(\sigma(t)))e_{R(t)}(\sigma(t),t_0)} + z(t)\psi^{(\alpha)}(t) \\
&\leq \frac{-q(t)z(t)}{e_{R(t)}(\sigma(t),t_0)} + \frac{z^{(\alpha)}(t)}{z(\sigma(t))}w(\sigma(t)) \\
&\quad - \frac{z(t)f'(x(t))x^{(\alpha)}(t)r(\sigma(t))x^{(\alpha)}(\sigma(t))}{f(x(t))f(x(\sigma(t)))e_{R(t)}(\sigma(t),t_0)} + z(t)\psi^{(\alpha)}(t) \\
&\leq \frac{-q(t)z(t)}{e_{R(t)}(\sigma(t),t_0)} + \frac{z^{(\alpha)}(t)}{z(\sigma(t))}w(\sigma(t)) \\
&\quad - Mz(t)\mathfrak{U}_1(t,t_0)\left[\frac{w(\sigma(t))}{z(\sigma(t))} - \psi(\sigma(t))\right]^2 + z(t)\psi^{(\alpha)}(t).
\end{aligned} \tag{3.10}$$

From Lemma 2.14, we have

$$\left[\frac{w(\sigma(t))}{z(\sigma(t))} - \psi(\sigma(t))\right]^2 \geq \frac{w^2(\sigma(t))}{z^2(\sigma(t))} + \psi^2(\sigma(t)) - 2\frac{\psi(\sigma(t))w(\sigma(t))}{z(\sigma(t))}. \tag{3.11}$$

From (3.10) and (3.11), we obtain that

$$\begin{aligned} w^{(\alpha)}(t) &\leq \frac{-q(t)z(t)}{e_{R(t)}(\sigma(t),t_0)} + z(t)\psi^{(\alpha)}(t) - Mz(t)\mathfrak{U}_1(t,t_0)\psi^2(\sigma(t)) \\ &\quad + \left[\frac{z^{(\alpha)}(t) + 2Mz(t)\mathfrak{U}_1(t,t_0)\psi(\sigma(t))}{2M^{\frac{1}{2}}z^{\frac{1}{2}}(t)\mathfrak{U}_1{}^{\frac{1}{2}}(t,t_0)}\right]^2. \end{aligned}$$

By taking α- fractional integral for the above inequality from t_2 to t, we get

$$\begin{aligned} &\int_{t_2}^{t}\Bigg\{\frac{q(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + Mz(s)\mathfrak{U}_1(s,t_0)\psi^2(\sigma(s)) \\ &- \left[\frac{z^{(\alpha)}(s) + 2Mz(s)\mathfrak{U}_1(s,t_0)\psi^2(\sigma(s))}{2M^{\frac{1}{2}}z^{\frac{1}{2}}(s)\mathfrak{U}_1{}^{\frac{1}{2}}(s,t_0)}\right]^2\Bigg\}\Delta^\alpha s \\ &\leq w(t_2) - w(t) \\ &\leq w(t_2) \\ &\leq \infty, \end{aligned}$$

which is a contraction to (3.9). The proof is complete. □

Theorem 3.7. *Let $R(t) \in \mathscr{R}^+$. If $(C_2),(H_2)$ hold, (2.1) has an eventually positive solution $x(t)$ and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{e_{R(s)}(s,t_0)}{r(s)}\int_{t_3}^{s}\frac{q(\nu)}{e_{R(\nu)}(\sigma(\nu),t_0)}\Delta^\alpha \nu\right]\Delta^\alpha s = \infty, \tag{3.12}$$

then there exists a sufficiently large $t^* \in [t_0,\infty)_{\mathbb{T}}$ such that $x^{(\alpha)}(t) > 0$ on $[t^*,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Lemma 3.5, we obtain that $x^{(\alpha)}(t)$ is eventually of one sign. Then there exists a sufficiently large $t_2 > t_1$ such that either $x^{(\alpha)}(t) > 0$ or $x^{(\alpha)}(t) < 0$ on $[t_2,\infty)_{\mathbb{T}}$. Suppose that $x^{(\alpha)}(t) < 0$. By the hypothesis of the lemma, we get $\lim_{t\to\infty} x(t) = C \geq 0$, where C is a constant. If $C > 0$, then $x(t) \geq C$ on $[t_3,\infty)_{\mathbb{T}}$ for $t_3 > t_2$ and by (H_2), $f(x(t)) > M_1$ on $[t_3,\infty)_{\mathbb{T}}$ for some $M_1 > 0$. Now, by applying α-fractional integral for (3.1) from t_3 to t , we have

$$\begin{aligned} \frac{r(t)x^{(\alpha)}(t)}{e_{R(t)}(t,t_0)} &= \frac{r(t_3)x^{(\alpha)}(t_3)}{e_{R(t)}(t_3,t_0)} - \int_{t_3}^{t}\frac{q(s)f(x(s))}{e_{R(s)}(\sigma(s),t_0)}\Delta^\alpha s \\ &\leq -M_1\int_{t_3}^{t}\frac{q(s)}{e_{R(s)}(\sigma(s),t_0)}\Delta^\alpha s \\ x^{(\alpha)}(t) &\leq -\left[M_1\frac{e_{R(t)}(t,t_0)}{r(t)}\int_{t_3}^{t}\frac{q(s)}{e_{R(s)}(\sigma(s),t_0)}\Delta^\alpha s\right]. \end{aligned}$$

Taking α-fractional integral from t_3 to t gives

$$x(t) \leq x(t_3) - M_1\int_{t_3}^{t}\left[\frac{e_{R(s)}(s,t_0)}{r(s)}\int_{t_3}^{s}\frac{q(\nu)}{e_{R(\nu)}(\sigma(\nu),t_0)}\Delta^\alpha \nu\right]\Delta^\alpha s.$$

Letting $t \to \infty$ and using (3.12), we get $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction to $x(t) > 0$. Therefore $\lim_{t\to\infty} x(t) = C = 0$. Hence the proof is complete. □

In view of the above results, we can also formulate the following theorem which involves H function.

Theorem 3.8. *Let $R(t) \in \mathscr{R}^+$. If $(C_1),(H_2)$ holds and there exists two function $z(t) > 0$, $\psi(t) \geq 0$ on $[T,\infty)_{\mathbb{T}}$ and $H \in \mathscr{H}$, for sufficiently large T such that*

$$\limsup_{t\to\infty} \quad \frac{1}{H(t,T)} \left\{ \int_T^t H(t,s) \left\{ \frac{q(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + Mz(s)\mathfrak{U}_1(s,t_0)\psi^2(\sigma(s)) \right.\right.$$
$$\left.\left. - \left[\frac{z^{(\alpha)}(s) + 2Mz(s)\mathfrak{U}_1(s,t_0)\psi(\sigma(s))}{2M^{\frac{1}{2}} z^{\frac{1}{2}}(s)\mathfrak{U}_1{}^{\frac{1}{2}}(s,t_0)} \right]^2 \right\} \Delta^\alpha s \right\} = \infty, \tag{3.13}$$

then equation (2.1) is oscillatory.

The theoretical findings of this paper are examined by the following examples.

Example 3.9. Consider the nonlinear conformable fractional damped dynamic equation

$$\left(t^{-1}\left(x^{(\frac{1}{2})}(t)\right)^\gamma\right)^{(\frac{1}{2})} + t^{-3}\left(x^{(\frac{1}{2})}(t)\right)^\gamma + t^{-(\frac{\gamma+1}{2})}(x(t))^\gamma(1+|x(t)|) = 0,\ t \in [1,\infty), \tag{3.14}$$

where, $\mathbb{T} = \mathbb{R}$, $\alpha = \frac{1}{2}$, $r(t) = t^{-1}$, $p(t) = t^{-3}$, $q(t) = t^{-(\frac{\gamma+1}{2})}$, $t_0 = 1$ and

$$f(x(t)) = (x(t))^\gamma (1+|x(t)|).$$

It is clear that

$$\frac{f(x)}{x^\gamma} \geq 1 = K,$$

$\sigma(t) = t$, $\mu(t) = 0$, $R(t) = -t^{-\frac{5}{2}}$,

$$\int_{t_0}^\infty \frac{\Delta^\alpha s}{r(s)} = \int_{t_0}^\infty \frac{s^{-\frac{1}{2}} ds}{s^{-1}} = \int_{t_0}^\infty s^{\frac{1}{2}} ds = \infty,$$

$$\begin{aligned} 1 &\geq e_{R(t)}(t,t_0) \\ &= \exp\left(\int_{t_0}^t R(s)ds\right) \\ &\geq 1 + \int_1^t R(s)ds \\ &= 1 - \int_1^t s^{-\frac{5}{2}} ds \\ &= 1 + \frac{2}{3}\left(t^{-\frac{3}{2}} - 1\right) \\ &\geq \frac{1}{3} \end{aligned}$$

and $\mathfrak{U}_1(t,t_0) \to \infty$ as $t \to \infty$. Let $z(t) = t^{\frac{\gamma}{2}}$ and $\psi(t) = 0$. Since $\mathfrak{U}(t,t_0) \to \infty$ as $t \to \infty$, there exists a $T^* > T$ such that $\mathfrak{U}(t,t_0) > 1$ for $t \in [T^*, \infty)$. Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \int_T^t \Bigg\{ \frac{Kq(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + z(s)\mathfrak{U}(s,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(s)) \\
&\quad - \left[\frac{z^{(\alpha)}(s) + (\gamma+1)z(s)\mathfrak{U}(s,t_0)\psi^{\frac{1}{\gamma}}(\sigma(s))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,t_0)} \right]^{\gamma+1} \Bigg\} \Delta^\alpha s \\
&= \limsup_{t\to\infty} \int_T^t \left\{ K\frac{s^{-(\frac{\gamma+1}{2})}s^{\frac{\gamma}{2}}}{e_{R(s)}(s,t_0)} - \left[\frac{\frac{\gamma}{2}s^{\frac{\gamma-2}{2}}s^{\frac{1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,t_0)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} ds \\
&> \limsup_{t\to\infty} \int_T^t \left\{ \frac{s^{-\frac{1}{2}}}{e_{R(s)}(s,t_0)} - \left[\frac{\frac{\gamma}{2}s^{\frac{\gamma-1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} ds \\
&= \limsup_{t\to\infty} \int_T^{T^*} \left\{ \frac{s^{-\frac{1}{2}}}{e_{R(s)}(s,t_0)} - \left[\frac{\frac{\gamma}{2}s^{\frac{\gamma-1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} ds \\
&\quad + \limsup_{t\to\infty} \int_{T^*}^t \left\{ \frac{s^{-\frac{1}{2}}}{e_{R(s)}(s,t_0)} - \left[\frac{\frac{\gamma}{2}s^{\frac{\gamma-1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} ds \\
&> \limsup_{t\to\infty} \int_T^{T^*} \left\{ \frac{s^{-\frac{1}{2}}}{e_{R(s)}(s,t_0)} - \left[\frac{\frac{\gamma}{2}s^{\frac{\gamma-1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} ds \\
&\quad + \limsup_{t\to\infty} \int_{T^*}^t \left[1 - \left(\frac{\gamma}{2(\gamma+1)} \right)^{\gamma+1} \right] s^{-1} ds = \infty.
\end{aligned}$$

From Theorem 3.2, we obtain that equation (3.14) is oscillatory or every solution tends to zero.

Example 3.10. Consider the nonlinear conformable fractional damped dynamic equation

$$\Delta^{(\frac{1}{2})}\left(t^{-\frac{1}{2}}\left(\Delta^{(\frac{1}{2})}x(t)\right)^\gamma\right) + t^{-\frac{5}{2}}\left(\Delta^{(\frac{1}{2})}x(t)\right)^\gamma + t^{-(\frac{\gamma+1}{2})}(x(t))^\gamma\left(K + e^{x(t)}\right) = 0, \tag{3.15}$$

where, $t \in [2,\infty)_{\mathbb{Z}}$, $\mathbb{T} = \mathbb{Z}$, $\alpha = \frac{1}{2}$, $r(t) = t^{-\frac{1}{2}}$, $p(t) = t^{-\frac{5}{2}}$, $q(t) = t^{-(\frac{\gamma+1}{2})}$, $t_0 = 2$, and

$$f(x(t)) = (x(t))^\gamma\left(K + e^{x(t)}\right).$$

Therefore, $\frac{f(x)}{x^\gamma} \geq K$, $\sigma(t) = t+1$, $\mu(t) = 1$, $R(t) = -t^{-\frac{5}{2}}$,

$$1 + \mu(t)R(t) = 1 - t^{-\frac{5}{2}} \geq 1 - \frac{1}{2} > 0,$$

which implies $R(t) \in \mathscr{R}^+$. Also,

$$\int_{t_0}^\infty \frac{\Delta^\alpha s}{r(s)} = \int_2^\infty \frac{s^{\alpha-1}\Delta s}{r(s)} = \sum_{s=2}^\infty \frac{s^{\alpha-1}}{s^{-\frac{1}{2}}} = \sum_{s=2}^\infty 1 = \infty,$$

$$
\begin{aligned}
1 > e_{R(t)}(t,t_0) &= \exp\left(\int_{t_0}^{t} R(s)\Delta s\right) \\
&\geq 1 + \int_{2}^{t} R(s)\Delta s = 1 - \sum_{s=2}^{t-1} s^{-\frac{5}{2}} \\
&\geq 1 - \int_{1}^{t-1} s^{-\frac{5}{2}} ds = 1 + \frac{2}{3}\left((t-1)^{-\frac{3}{2}} - 1\right) \\
&> \frac{1}{3}
\end{aligned}
$$

and $\mathfrak{U}(t,t_0) \to \infty$ as $t \to \infty$. Let $z(t) = t^{\left(\frac{\gamma+1}{2}\right)}$ and $\psi(t) = 0$. Since $\mathfrak{U}(t,t_0) \to \infty$ as $t \to \infty$, there exists a $T^* > T$ such that $\mathfrak{U}(t,t_0) > 1$ for $t \in [T^*, \infty)$. By using the inequality

$$
(t+1)^{\gamma} - t^{\gamma} \leq \gamma (t+1)^{\gamma-1} < \gamma 2^{\gamma-1} t^{\gamma-1}, t \geq T^*,
$$

we have

$$
\begin{aligned}
&\limsup_{t\to\infty} \int_{T}^{t} \left\{ \frac{Kq(s)z(s)}{e_{R(s)}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + z(s)\mathfrak{U}(s,t_0)\psi^{1+\frac{1}{\gamma}}(\sigma(s)) \right. \\
&\quad \left. - \left[\frac{z^{(\alpha)}(s) + (\gamma+1)z(s)\mathfrak{U}(s,t_0)\psi^{\frac{1}{\gamma}}(\sigma(s))}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,t_0)} \right]^{\gamma+1} \right\} \Delta^{\alpha} s \\
&= \limsup_{t\to\infty} \int_{T}^{t} \left\{ \frac{K s^{-\left(\frac{\gamma+1}{2}\right)} s^{\left(\frac{\gamma+1}{2}\right)}}{e_{R(s)}(\sigma(s),2)} - \left[\frac{(z(s+1)-z(s))s^{\frac{1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)\mathfrak{U}^{\frac{\gamma}{\gamma+1}}(s,2)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} \Delta s \\
&> \limsup_{t\to\infty} \sum_{s=T}^{t-1} \left\{ \frac{K}{e_{R(s)}(\sigma(s),2)} - \left[\frac{(\gamma+1)2^{\frac{\gamma-1}{2}} s^{\frac{\gamma-1}{2}} s^{\frac{1}{2}}}{(\gamma+1)z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} \\
&= \limsup_{t\to\infty} \sum_{s=T}^{T^*} \left\{ \frac{K}{e_{R(s)}(\sigma(s),2)} - \left[\frac{2^{\frac{\gamma-1}{2}} s^{\frac{\gamma}{2}}}{z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} \\
&\quad + \limsup_{t\to\infty} \sum_{s=T^*}^{t-1} \left\{ \frac{K}{e_{R(s)}(\sigma(s),2)} - \left[\frac{2^{\frac{\gamma-1}{2}} s^{\frac{\gamma}{2}}}{z^{\frac{\gamma}{\gamma+1}}(s)} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} \\
&> \limsup_{t\to\infty} \sum_{s=T}^{T^*} \left\{ \frac{K}{e_{R(s)}(\sigma(s),2)} - \left[2^{\frac{\gamma-1}{2}} \right]^{\gamma+1} \right\} s^{-\frac{1}{2}} \\
&\quad + \limsup_{t\to\infty} \sum_{s=T^*}^{t-1} \left[K - \left(2^{\frac{\gamma-1}{2}} \right)^{\gamma+1} \right] \frac{1}{\sqrt{s}} = \infty
\end{aligned}
$$

provided $K > \left(2^{\frac{\gamma-1}{2}}\right)^{\gamma+1}$. From Theorem 3.2, equation (3.14) is oscillatory or every solution tends to zero.

Example 3.11. Consider the nonlinear conformable fractional damped dynamic equation

$$
\left(t\left(x^{\left(\frac{1}{2}\right)}(t)\right)\right)^{\left(\frac{1}{2}\right)} + t^{-1}\left(x^{\left(\frac{1}{2}\right)}(t)\right) + t^{\frac{5}{2}} x(t) = 0, \; t \in [1,\infty), \tag{3.16}
$$

where, $\mathbb{T}=\mathbb{R}$, $\alpha=\frac{1}{2}$, $r(t)=t$, $p(t)=t^{-1}$, $q(t)=t^{\frac{5}{2}}$, $f(x(t))=x(t)$, and $t_0=1$. Then, $f'(x)=1=M$, $\sigma(t)=t$, $\mu(t)=0$, $R(t)=-t^{-\frac{5}{2}}$ and

$$\int_{t_0}^{\infty}\frac{\Delta^{\alpha}s}{r(s)}=\int_{t_0}^{\infty}\frac{s^{-\frac{1}{2}}ds}{s}=\int_{t_0}^{\infty}s^{-\frac{1}{2}}ds<\infty,$$

$$\begin{aligned}1\geq e_{R(t)}(t,t_0)&=\exp\left(\int_{t_0}^{t}R(s)ds\right)\\&\geq 1+\int_{1}^{t}R(s)ds=1-\int_{1}^{t}s^{-\frac{5}{2}}ds=1+\frac{2}{3}\left(t^{-\frac{3}{2}}-1\right)\\&\geq\frac{1}{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{e_{R(s)}(s,t_0)}{r(s)}\int_{t_3}^{s}\frac{q(v)}{e_{R(v)}(\sigma(v),t_0)}\Delta^{\alpha}v\right]\Delta^{\alpha}s\\&>\limsup_{t\to\infty}\int_{2}^{t}\left[\frac{1}{3r(s)}\int_{3}^{s}q(v)\Delta^{\alpha}v\right]\Delta^{\alpha}s\\&=\limsup_{t\to\infty}\int_{2}^{t}\left[\frac{1}{3s}\int_{3}^{s}v^{2}dv\right]\Delta^{\alpha}s\\&=\limsup_{t\to\infty}\int_{2}^{t}\left[\frac{s^{2}}{9}-\frac{1}{s}\right]s^{-\frac{1}{2}}ds\to\infty \text{ as } t=\infty.\end{aligned}$$

From Theorem 3.7, we obtain that equation (3.16) is oscillatory or every solution tends to zero.

4. A CONCLUDING REMARK

In this paper, we discussed the oscillation and oscillatory behavior of a new class of nonlinear damped dynamic equations on time scales within conformable fractional derivative. The obtained results are new in the sense that they provide sufficient conditions for the oscillation of the proposed equations under different restrictive assumptions. Indeed, the oscillatory behavior of the equations considered in the above examples can not be commented by existing results in the literatures. Therefore, our theoretical findings in this paper improve and complement these results. We believe that the results of this paper are of great significance and could provide affirmative comments on the oscillation of some important equations such as conformable fractional Euler type dynamic equation of the form

$$(x^{(\alpha)}(t))^{(\alpha)}+\frac{a}{t^{\alpha}}x^{(\alpha)}(t)+\frac{b}{t^{2\alpha}}x(t)=0,\ \ t\in[t_0,\infty)_{\mathbb{T}}. \tag{4.1}$$

In this regard, we can formulate similar result for equation (4.1): Let $\frac{-a}{t} \in \mathscr{R}^+$. If there exist two function $z(t) > 0$ and $\psi(t) \geq 0$ on $[T,\infty)_{\mathbb{T}}$ such that

$$\limsup_{t\to\infty} \int_T^t \left\{ \frac{\frac{b}{s^{2\alpha}} z(s)}{e_{(\frac{-a}{s})}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + z(s)e_{(\frac{-a}{s})}(s,t_0)\psi^2(\sigma(s)) - \left[\frac{z^{(\alpha)}(s) + 2z(s)e_{(\frac{-a}{s})}(s,t_0)\psi(\sigma(s))}{2z^{\frac{1}{2}}(s)e^{\frac{1}{2}}_{(\frac{-a}{s})}(s,t_0)} \right]^2 \right\} \Delta^\alpha s = \infty, \tag{4.2}$$

then equation (4.1) is oscillatory. In particular, if we consider the following concrete conformable fractional Euler type dynamic equation

$$\left(x^{(\frac{1}{2})}(t)\right)^{(\frac{1}{2})} + at^{-\frac{1}{2}}x^{(\frac{1}{2})}(t) + bt^{-1}x(t) = 0, \tag{4.3}$$

where $t \in [1,\infty)_{\mathbb{Z}}$, $\mathbb{T} = \mathbb{R}$, $\alpha = \frac{1}{2}$ and $t_0 = 2$. Then $\sigma(t) = t+1$, $\mu(t) = 1$, and $e_{\frac{-a}{s}}(t,t_0) = exp\left(\int_{t_0}^t \frac{-a}{s} ds\right) = t^{-a}$. If we let $z(t) = t^{\frac{1}{2}}$ and $\psi(t) = 0$, then

$$\begin{aligned}
&\limsup_{t\to\infty} \int_T^t \left\{ \frac{\frac{b}{s^{2\alpha}} z(s)}{e_{(\frac{-a}{s})}(\sigma(s),t_0)} - z(s)\psi^{(\alpha)}(s) + z(s)e_{(\frac{-a}{s})}(s,t_0)\psi^2(\sigma(s)) - \left[\frac{z^{(\alpha)}(s) + 2z(s)e_{(\frac{-a}{s})}(s,t_0)\psi(\sigma(s))}{2z^{\frac{1}{2}}(s)e^{\frac{1}{2}}_{(\frac{-a}{s})}(s,t_0)} \right]^2 \right\} \Delta^\alpha s \\
&= \limsup_{t\to\infty} \int_T^t \left\{ \frac{bs^{\frac{1}{2}}}{s^{2\alpha-a}} - \frac{(z'(s))^2 s^{2(1-\alpha)}}{4s^{\frac{1}{2}}s^{-a}} \right\} s^{-\alpha} ds \\
&= \limsup_{t\to\infty} \int_T^t \left\{ \frac{bs^{\frac{1}{2}}}{s^{1-a}} - \frac{1}{16s^{\frac{1}{2}}s^{-a}} \right\} s^{-\frac{1}{2}} ds \\
&= \limsup_{t\to\infty} \int_T^t (b - \frac{1}{16}) \frac{1}{s^{1-a}} ds = \infty
\end{aligned}$$

provided that $b > \frac{1}{16}$ and $a < 1$. Hence one can conclude that equation (4.3) is oscillatory. The above discussion demonstrates significant applicability of the obtained results in this paper. In addition, one can conclude that the established results provide a practical platform that facilitates the investigation of oscillation and oscillatory behavior results for both fractional differential and difference equations simultaneously.

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