J. Nonlinear Funct. Anal. 2020 (2020), Article ID 11 https://doi.org/10.23952/jnfa.2020.11

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

A NEW SHRINKING ITERATIVE SCHEME FOR D-ACCRETIVE MAPPINGS WITH APPLICATIONS TO CAPILLARITY SYSTEMS

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Abstract. A new shrinking iterative scheme is studied for common zeros of a countable family of m-d-accretive mappings. Strong convergence theorems are obtained in a real Banach space. Moreover, the applications to capillarity systems are considered to support m-d-accretive mappings.

Keywords. Banach space; Fixed point; d-accretive mapping; Strong convergence; Zeros; Capillarity systems.

1. Introduction and Preliminaries

Assume that E is a real Banach space with E^* being its dual space. We use $\langle x, f\rangle$ to denote the value of $f \in E^*$ at $x \in E$. "$\to$" and "$\rightharpoonup$" denote strong and weak convergence either in E or E^*, respectively.

Recall that a function $\delta_E(\varepsilon) : (0,2] \to [0,1]$ is called the modulus of convexity of E if it is defined by $\delta_E(\varepsilon) = \inf\{\frac{2-\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}$, $0 \leq \varepsilon \leq 2$. E is said to be uniformly convex iff $\delta_E(0) = 0$, and $\delta_E(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Recall that a function $\rho_E : [0,\infty) \to [0,\infty)$ is called the modulus of smoothness of E if it is defined by

$$\rho_E(t) = \sup\{\frac{\|x+y\|+\|x-y\|-2}{2} : x \in B_E, \|y\| \leq t\}.$$

E is said to be uniformly smooth iff $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. It is known that E^* is uniformly convex iff E is uniformly smooth.

Further, one recalls that the normalized duality mapping $J_{E^*} : E^* \to 2^E$ is defined by

$$J_{E^*}(u) = \{x \in E : \langle x, u\rangle = \|x\|^2 = \|u\|^2\},\ u \in E^*.$$

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Received September 20, 2019; Accepted March 16, 2020.

Similarly, one also has the normalized duality mapping $J_E : E \to 2^{E^*}$ defined by

$$J_E(u) = \{x \in E^* : \langle x, u\rangle = \|x\|^2 = \|u\|^2\},\ u \in E.$$

One usually use j_{E^*} and j_E to denote the single-valued normalized duality mappings and one also knows that normalized duality mappings are single-valued in smooth Banach spaces.

Let $A : D(A) \sqsubseteq E \to E$ be a mapping. In this paper, we use $N(A)$ to denote the set of zeros of A. That is, $N(A) = \{x \in D(A) : Ax = 0\}$. Recall that

(1) A is said to be d-accretive iff, for all $x, y \in D(A)$, $\langle Ax - Ay, j_E(x) - j_E(y)\rangle \geq 0$, where $j_E(x) \in J_E(x), j_E(y) \in J_E(y)$;

(2) A is said to be m-d-accretive iff A is d-accretive and $R(I + \lambda A) = E$, for $\forall \lambda > 0$;

(3) A is said to be accretive iff, for all $x, y \in D(A)$, $\langle Ax - Ay, j_E(x - y)\rangle \geq 0$, where $j_E(x - y) \in J_E(x - y)$;

(4) A is said to be m-accretive iff A is accretive and $R(I + \lambda A) = E$, for $\forall \lambda > 0$.

It is easy to see that in a non-Hilbertian Banach space, d-accretive mappings and accretive mappings are two different types of nonlinear mappings.

Let $T : E \to E$ be a mapping. We use $F(T)$ to denote the set of fixed points of T. That is, $F(T) = \{x \in D(T) : Tx = x\}$. Recall that T is nonexpansive iff $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in E$, and T is pseudocontractive iff $\langle Tx - Ty, j_E(x - y)\rangle \leq \|x - y\|^2$, $\forall x, y \in E$.

Recall that a mapping $S \subset E^* \times E$ is said to be monotone iff $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, for $\forall y_i \in D(S)$, $\forall x_i \in Sy_i, i = 1, 2$. A monotone mapping S is called maximal monotone iff $R(J_{E^*} + \lambda S) = E$, $\forall \lambda > 0$. Recall that the Lyapunov functional $\phi : E^* \times E^* \to R^+$ [1] is defined as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle j_{E^*}(y), x\rangle + \|y\|^2,\ \forall x, y \in E^*, j_{E^*}(y) \in J_{E^*}(y).$$

Let $\{K_n^*\}$ be a sequence of nonempty closed and convex subsets of E^*. One recalls from [2] that

(1) s-$\liminf K_n^*$, which is called the strong lower limit of $\{K_n^*\}$, is defined as the set of all $x \in E^*$ such that there exists $x_n \in K_n^*$ for almost all n and it tends to x as $n \to \infty$ in the norm.

(2) w-$\limsup K_n^*$, which is called the weak upper limit of $\{K_n^*\}$, is defined as the set of all $x \in E^*$ such that there exists a subsequence $\{K_{n_m}^*\}$ of $\{K_n^*\}$ and $x_{n_m} \in K_{n_m}^*$ for every n_m and it tends to x as $n_m \to \infty$ in the weak topology;

(3) If s-$\liminf K_n^* = w$-$\limsup K_n^*$, then the common value is denoted by $\lim K_n^*$.

Let K be a nonempty convex and closed subset of space E. Let Q be a mapping of E onto K. Then Q is said to be sunny [3, 4, 5] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$. A mapping $Q : E \to K$ is said to be a retraction if $Q(z) = z$ for every $z \in K$. If E is a uniformly smooth and uniformly convex Banach space, then the sunny generalized non-expansive retraction of E onto K is uniquely decided, which is denoted by R_K.

Recall that [6, 7] if E^* is a real uniformly smooth and uniformly convex Banach space and K^* is a nonempty, closed and convex subset of E^*, then we have the facts (1) for each $x \in E^*$, there exists a unique element $v \in K^*$ such that $\|x - v\| = \inf\{\|x - y\| : y \in K^*\}$. Such an element v is denoted by $P_{K^*}x$ and P_{K^*} is called the metric projection of E^* onto K^*; (2) for each $x \in E^*$, there exists a unique element $x_0 \in K^*$ satisfying $\phi(x_0, x) = \inf\{\phi(z, x) : z \in K^*\}$. In this case, $\forall x \in E^*$, one defines $\Pi_{K^*} : E^* \to K^*$ by $\Pi_{K^*}x = x_0$, and Π_{K^*} is called the generalized projection from E^* onto K^*.

Numerous results on iterative methods for approximating zeros of m-accretive mappings have been obtained during past 20 years, see for examples [8, 9, 10, 11, 12, 13] and the references therein. However, less research investigation has been done on d-accretive mappings. The class of d-accretive mappings is also worth studying since it has a close relation with practical problems, see, e.g., [14, 15].

In particular, Wei, Liu and Agarwal [14] presented the following block projection iterative scheme for a finite family of m-d-accretive mappings $\{A_i\}_{i=1}^m \subset E \times E$:

$$\begin{cases} x_1 \in E, \\ u_n = \sum_{i=1}^m a_{n,i}[\alpha_{n,i}x_n + (1-\alpha_{n,i})(I + r_{n,i}A_i)^{-1}x_n], \\ v_{n+1} = \sum_{i=1}^m b_{n,i}[\beta_{n,i}x_n + (1-\beta_{n,i})(I + s_{n,i}A_i)^{-1}y_n], \\ H_1 = E, \\ H_{n+1} = \{z \in H_n : \omega(v_n, z) \le \omega(x_n, z)\}, \\ x_{n+1} = R_{H_{n+1}}x_1, \quad n \in \mathbb{N}. \end{cases} \tag{1.1}$$

Under mild assumptions, the sequence $\{x_n\}$ generated by (1.1) was proved to be strongly convergent to an element in $\bigcap_{i=1}^m N(A_i)$, where $R_{H_{n+1}}$ is the sunny generalized non-expansive retraction from E onto H_{n+1} and $\omega : E \times E \to R^+$ is the Lyapunov functional. Moreover, the following nonlinear elliptic boundary value problem was also presented in [14] to support the m-d-accretive mappings:

$$\begin{cases} -div(\alpha(grad u)) + |u|^{p-2}u + g(x, u(x)) = f(x), \ a.e. \ in \ \Omega, \\ -\langle \vartheta, \alpha(grad u)\rangle \in \beta_x(u(x)), \ a.e. \ in \ \Gamma. \end{cases} \tag{1.2}$$

In this paper, we investigate two groups of countable families of m-d-accretive mappings $\{A_i\}_{i=1}^\infty$ and $\{B_i\}_{i=1}^\infty$ via a new shrinking iterative schemes and present an example of capillarity systems to enrich the background of m-d-accretive mappings, which is one of the highlights. The other highlight is that we construct two key groups of sets $\{U_n\}$ and $\{X_n\}$ and choose the iterative elements $\{y_n\}$ and $\{u_n\}$ arbitrarily in two subsets of $\{U_n\}$ and $\{X_n\}$, respectively. The following lemmas are important for our main results.

Lemma 1.1. [6] *Let E be a real uniformly smooth and uniformly convex Banach space. Then the normalized duality mappings $J_E : E \to 2^{E^*}$ and $J_{E^*} : E^* \to 2^E$ have the following properties:*

(i) both J_E and J_{E^*} are single-valued and surjective;

(ii) $J_{E^*} = J_E^{-1}$;

(iii) both J_E and J_{E^*} are uniformly continuous on each bounded subset of E or E^*, respectively;

(iv) for $x \in E$ and $k \in (0, +\infty)$, $J_E(kx) = kJ_E(x)$; for $u \in E^*$ and $k \in (0, +\infty)$, $J_{E^*}(ku) = kJ_{E^*}(u)$.

Lemma 1.2. [16] *Let $S \subset E^* \times E$ be a maximal monotone mapping. Then*

(1) $N(S)$ is a closed and convex subset of E^*;

(2) if $y_n \to y$ and $x_n \in Sy_n$ with $x_n \rightharpoonup x$, or $y_n \rightharpoonup y$ and $x_n \in Sy_n$ with $x_n \to x$, then $y \in D(S)$ and $x \in Sy$.

Lemma 1.3. [17] *Let E be a real uniformly smooth and uniformly convex Banach space. If $\lim K_n^*$ exists and is not empty, then $\{P_{K_n^*}x\}$ converges strongly to $P_{\lim K_n^*}x$ for every $x \in E^*$.*

Lemma 1.4. [2] *Let $\{K_n^*\}$ be a decreasing sequence of closed and convex subsets of E^*, i.e. $K_n^* \subset K_m^*$ if $n \ge m$. Then $\{K_n^*\}$ converges in E^* and $\lim K_n^* = \bigcap_{n=1}^\infty K_n^*$.*

Lemma 1.5. *Let E be a real uniformly smooth and uniformly convex Banach space and $B \subset E \times E$ be an m-d-accretive mapping with $N(B) \neq \emptyset$. Then for $\forall x \in E^*$, $\forall z \in N(B)$ and $\forall r > 0$, one has:*

$$\phi(J_E z, (J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x) + \phi((J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x, x) \leq \phi(J_E z, x).$$

Proof. Observe that

$$J_{E^*}(J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x + rB[J_{E^*}(J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x] = J_{E^*} x, \quad \forall x \in E^*, r > 0.$$

Since B is m-d-accretive, we have

$$\langle (J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x - J_E z, BJ_{E^*}(J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x \rangle \geq 0.$$

Using the definition of Lyapunov functional, we have

$$\begin{aligned}
&\phi(J_E z, x) - \phi((J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x, x) - \phi(J_E z, (J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x) \\
&= 2\langle J_E z, J_{E^*}(J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x - J_{E^*} x \rangle \\
&\quad - 2\langle (J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x, J_{E^*}(J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x - J_{E^*} x \rangle \\
&= 2r\langle (J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x - J_E z, BJ_{E^*}(J_{E^*} + rBJ_{E^*})^{-1} J_{E^*} x \rangle \geq 0,
\end{aligned}$$

which concludes the desired conclusion. This completes the proof. □

Lemma 1.6. [7] *Let E be a real uniformly smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 1.7. [18] *Let E be a real uniformly convex Banach space and $r \in (0, +\infty)$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|kx + (1-k)y\|^2 \leq k\|x\|^2 + (1-k)\|y\|^2 - k(1-k)g(\|x - y\|),$$

for $k \in [0, 1]$, $x, y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r$.

Lemma 1.8. [1] *Let E^* be a real strictly convex and smooth Banach space and K^* a nonempty closed and convex subset of E^*. Then, $\forall x \in E^*, y \in K^*$, $\phi(y, \Pi_{K^*} x) + \phi(\Pi_{K^*} x, x) \leq \phi(y, x)$.*

2. The iterative schemes

In this section, unless otherwise stated, we always assume that:

(1) E is a real uniformly smooth and uniformly convex Banach space;

(2) $A_i, B_i \subset E \times E$ are m-d-accretive mappings, for each $i \in \mathbb{N}$ with $(\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$;

(3) $J_E : E \to E^*$ and $J_{E^*} : E^* \to E$ are normalized duality mappings;

(4) $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $[0, 1)$, $\{\lambda_n\}$ and $\{\delta_n\}$ are real number sequences in $(0, +\infty)$, $\{r_{n,i}\}$ and $\{s_{n,i}\}$ are real number sequences in $(0, +\infty)$, $\{a_{n,i}\}$ and $\{b_{n,i}\}$ are real number sequence in $(0, 1)$ with $\sum_{i=1}^{\infty} a_{n,i} = \sum_{i=1}^{\infty} b_{n,i} = 1$, for $i, n \in \mathbb{N}$;

(5) $\{e_n\}$, and $\{\varepsilon_n\} \subset E^*$ are the computational errors.

Theorem 2.1. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ y_n \in V_{n+1}, \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n \|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2}\}, \\ Y_{n+1} = \{p \in X_{n+1} : \|u_1 - p\|^2 \leq \|P_{X_{n+1}}(u_1) - u_1\|^2 + \delta_{n+1}\}, \\ u_{n+1} \in Y_{n+1}, \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.1}$$

If (i) $\inf_n r_{n,i} \geq 0, inf_n s_{n,i} \geq 0$ *for* $i \in \mathbb{N}$; (ii) $\lambda_n \to 0, \delta_n \to 0$, *as* $n \to \infty$; (iii) $0 \leq \sup_n \alpha_n < 1$ *and* $0 \leq \sup_n \beta_n < 1$; (iv) $e_n \to 0, \varepsilon_n \to 0$ *as* $n \to \infty$, *then* $\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$, *as* $n \to \infty$.

Proof. We split the proof into nine steps.

Step 1. $(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*})) \neq \emptyset$.

Since $(\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$, we see there exists $x_0 \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$ such that $A_i x_0 = B_i x_0$, for $i \in \mathbb{N}$. For x_0, using Lemma 1.1, one sees that there exists $y_0 \in E^*$ such that $x_0 = J_{E^*} y_0$, which implies $A_i J_{E^*} y_0 = B_i J_{E^*} y_0$, for $i \in \mathbb{N}$. Therefore,

$$(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*})) \neq \emptyset.$$

Step 2. $(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*})) \subset U_n \bigcap X_n$, for $n \in \mathbb{N}$.

It is obviously true for $n = 1$. Suppose that it is true for $n = k$. If $n = k+1$, for $\forall p \in (\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))$, then one obtains from Lemma 1.5 and (2.1) that

$$\begin{aligned} &\phi(p, v_{k+1}) \\ &\leq \alpha_{k+1}\phi(p, u_{k+1}) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_{k+1,i}\phi(p, (J_{E^*} + r_{k+1,i} A_i J_{E^*})^{-1} J_{E^*}(u_{k+1} + e_{k+1})) \\ &\leq \alpha_{k+1}\phi(p, u_{k+1}) + (1-\alpha_{k+1})\phi(p, u_{k+1} + e_{k+1}). \end{aligned}$$

From the definition of Lyapunov functional, one has

$$\begin{aligned} &\langle J_{E^*} v_{k+1} - \alpha_{k+1} J_{E^*} u_{k+1} - (1-\alpha_{k+1}) J_{E^*}(u_{k+1} + e_{k+1}), p\rangle \\ &\geq \frac{\|v_{k+1}\|^2 - \alpha_{k+1}\|u_{k+1}\|^2 - (1-\alpha_{k+1})\|u_{k+1} + e_{k+1}\|^2}{2}, \end{aligned}$$

which implies $p \in U_{k+2}$. Using Lemma 1.5 and (2.1) again, one has

$$\begin{aligned}
&\phi(p, z_{k+1}) \\
&\leq \beta_{k+1}\phi(p, u_{k+1}) + (1-\beta_{k+1})\sum_{i=1}^{\infty} b_{k+1,i}\phi(p, (J_{E^*} + s_{k+1,i}B_iJ_{E^*})^{-1}J_{E^*}(y_{k+1}+\varepsilon_{k+1})) \\
&\leq \beta_{k+1}\phi(p, u_{k+1}) + (1-\beta_{k+1})\phi(p, y_{k+1}+\varepsilon_{k+1}).
\end{aligned}$$

From the definition of Lyapunov functional, we have

$$\begin{aligned}
&\langle J_{E^*}z_{k+1} - \beta_{k+1}J_{E^*}u_{k+1} - (1-\beta_{k+1})J_{E^*}(y_{k+1}+\varepsilon_{k+1}), p\rangle \\
&\geq \tfrac{\|z_{k+1}\|^2 - \beta_{k+1}\|v_{k+1}\|^2 - (1-\beta_{k+1})\|y_{k+1}+\varepsilon_{k+1}\|^2}{2},
\end{aligned}$$

which implies that $p \in X_{k+2}$. Then by induction, one has $p \in U_n \bigcap X_n$, for $n \in \mathbb{N}$.

Step 3. $P_{U_n}(u_1) \to P_{\bigcap_{m=1}^{\infty} U_m}(u_1)$, $P_{X_n}(u_1) \to P_{\bigcap_{m=1}^{\infty} X_m}(u_1)$, as $n \to \infty$.

It is easy to see from the definitions of U_n and X_n in (2.1) that both U_n and X_n are closed and convex subsets of E^*, for each $n \in \mathbb{N}$.

Therefore, Lemmas 1.3 and 1.4 imply that $P_{U_n}(u_1) \to P_{\bigcap_{m=1}^{\infty} U_m}(u_1)$, $P_{X_n}(u_1) \to P_{\bigcap_{m=1}^{\infty} X_m}(u_1)$ as $n \to \infty$.

Step 4. $P_{\bigcap_{m=1}^{\infty} U_m}(u_1) = P_{\bigcap_{m=1}^{\infty} X_m}(u_1)$.

From (2.1), $X_{n+1} \subset U_{n+1}$ ensures that $\bigcap_{m=1}^{\infty} X_m \subset \bigcap_{m=1}^{\infty} U_m$. On the other hand, $U_1 = E^*$ and $U_{n+1} \subset X_n$ imply that

$$\bigcap_{m=1}^{\infty} U_m = \bigcap_{m=1}^{\infty} U_{m+1} \subset \bigcap_{m=1}^{\infty} X_m.$$

Therefore, $\bigcap_{m=1}^{\infty} U_m = \bigcap_{m=1}^{\infty} X_m$ and $P_{\bigcap_{m=1}^{\infty} U_m}(u_1) = P_{\bigcap_{m=1}^{\infty} X_m}(u_1)$.

Step 5. Both $\{y_n\}$ and $\{u_n\}$ are well-defined.

In fact, it suffices to show that $V_n \neq \emptyset$ and $Y_n \neq \emptyset$, for each $n \in \mathbb{N}$. Since

$$\|P_{U_{n+1}}(u_1) - u_1\| = \inf_{q \in U_{n+1}} \|q - u_1\|,$$

we find that, for λ_{n+1}, there exists $k_n \in U_{n+1}$ such that

$$\|u_1 - k_n\|^2 \leq (\inf_{q \in U_{n+1}} \|q - u_1\|)^2 + \lambda_{n+1} = \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}.$$

Therefore, $V_n \neq \emptyset$. Similarly, $Y_n \neq \emptyset$, for each $n \in \mathbb{N}$. So both $\{y_n\}$ and $\{u_n\}$ are well-defined.

Step 6. Both $\{y_n\}$ and $\{u_n\}$ are bounded.

Since $y_n \in V_{n+1}$, we have

$$\|u_1 - y_n\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}.$$

Since $\{P_{U_n}(u_1)\}$ is convergent and $\lambda_n \to 0$, we conclude that $\{y_n\}$ is bounded. Similarly, $\{u_n\}$ is also bounded.

Step 7. $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$ and $y_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$, as $n \to \infty$.

Since $y_n \in V_{n+1} \subset U_{n+1}$ and U_n is convex, we have, for $\forall k \in (0,1)$, $kP_{U_{n+1}}(u_1) + (1-k)y_n \in U_{n+1}$. Thus

$$\|P_{U_{n+1}}(u_1) - u_1\| \leq \|kP_{U_{n+1}}(u_1) + (1-k)y_n - u_1\|.$$

Using Lemma 1.7, we have

$$\begin{aligned}&\|P_{U_{n+1}}(u_1)-u_1\|^2\\&\leq \|kP_{U_{n+1}}(u_1)+(1-k)y_n-u_1\|^2\\&\leq k\|P_{U_{n+1}}(u_1)-u_1\|^2+(1-k)\|y_n-u_1\|^2-k(1-k)g(\|P_{U_{n+1}}(u_1)-y_n\|).\end{aligned}$$

Therefore, $kg(\|P_{U_{n+1}}(u_1)-y_n\|)\leq \|y_n-u_1\|^2-\|P_{U_{n+1}}(u_1)-u_1\|^2\leq \lambda_{n+1}\to 0$, as $n\to\infty$. Then $y_n-P_{U_{n+1}}(u_1)\to 0$, as $n\to\infty$. Combining Steps 3 and 4, we have $y_n\to P_{\bigcap_{n=1}^{\infty}U_n}(u_1)=P_{\bigcap_{n=1}^{\infty}X_n}(u_1)$, as $n\to\infty$. Since $u_{n+1}\in Y_{n+1}\subset X_{n+1}$ and X_n is convex, we have, for each $k\in(0,1)$, $kP_{X_{n+1}}(u_1)+(1-k)u_{n+1}\in X_{n+1}$. Thus

$$\|P_{X_{n+1}}(u_1)-u_1\|\leq \|kP_{X_{n+1}}(u_1)+(1-k)u_{n+1}-u_1\|.$$

Using Lemma 1.7 again, we have

$$\begin{aligned}&\|P_{X_{n+1}}(u_1)-u_1\|^2\\&\leq \|kP_{X_{n+1}}(u_1)+(1-k)u_{n+1}-u_1\|^2\\&\leq k\|P_{X_{n+1}}(u_1)-u_1\|^2+(1-k)\|u_{n+1}-u_1\|^2-k(1-k)g(\|P_{X_{n+1}}(u_1)-u_{n+1}\|).\end{aligned}$$

Therefore, $kg(\|P_{X_{n+1}}(u_1)-u_{n+1}\|)\leq \|u_{n+1}-u_1\|^2-\|P_{X_{n+1}}(u_1)-u_1\|^2\leq \delta_{n+1}\to 0$, as $n\to\infty$. Combining with Steps 3 and 4, we have $u_n\to P_{\bigcap_{n=1}^{\infty}X_n}(u_1)=P_{\bigcap_{n=1}^{\infty}U_n}(u_1)$, as $n\to\infty$.

Step 8. $P_{\bigcap_{n=1}^{\infty}U_n}(u_1)=P_{\bigcap_{n=1}^{\infty}X_n}(u_1)\in(\bigcap_{i=1}^{\infty}N(A_iJ_{E^*}))\bigcap(\bigcap_{i=1}^{\infty}N(B_iJ_{E^*}))$.

For $\forall q\in(\bigcap_{i=1}^{\infty}N(A_iJ_{E^*}))\bigcap(\bigcap_{i=1}^{\infty}N(B_iJ_{E^*}))$, using Lemma 1.5 and (2.1), we have

$$\begin{aligned}&\phi(q,v_n)\\&\leq \alpha_n\phi(q,u_n)+(1-\alpha_n)\sum_{i=1}^{\infty}a_{n,i}\phi(q,(J_{E^*}+r_{n,i}A_iJ_{E^*})^{-1}J_{E^*}(u_n+e_n))\\&\leq \alpha_n\phi(q,u_n)+(1-\alpha_n)\sum_{i=1,i\neq i_0}^{\infty}a_{n,i}\phi(q,(J_{E^*}+r_{n,i}A_iJ_{E^*})^{-1}J_{E^*}(u_n+e_n))\\&\quad+(1-\alpha_n)a_{n,i_0}\phi(q,(J_{E^*}+r_{n,i_0}A_{i_0}J_{E^*})^{-1}J_{E^*}(u_n+e_n))\\&\leq \alpha_n\phi(q,u_n)+(1-\alpha_n)\sum_{i=1,i\neq i_0}^{\infty}a_{n,i}\phi(q,(J_{E^*}+r_{n,i}A_iJ_{E^*})^{-1}J_{E^*}(u_n+e_n))\\&\quad+(1-\alpha_n)a_{n,i_0}[\phi(q,u_n+e_n)-\phi((J_{E^*}+r_{n,i_0}A_{i_0}J_{E^*})^{-1}J_{E^*}(u_n+e_n),u_n+e_n)]\\&\leq \alpha_n\phi(q,u_n)+(1-\alpha_n)\phi(q,u_n+e_n)\\&\quad-(1-\alpha_n)a_{n,i_0}\phi((J_{E^*}+r_{n,i_0}A_{i_0}J_{E^*})^{-1}J_{E^*}(u_n+e_n),u_n+e_n).\end{aligned}$$

Thus

$$\begin{aligned}&(1-\alpha_n)a_{n,i_0}\phi((J_{E^*}+r_{n,i_0}A_{i_0}J_{E^*})^{-1}J_{E^*}(u_n+e_n),u_n+e_n)\\&\leq \alpha_n\phi(q,u_n)+(1-\alpha_n)\phi(q,u_n+e_n)-\phi(q,v_n),\end{aligned}$$

which ensures from Lemma 1.6 and $0\leq sup_n\alpha_n<1$ that

$$\lim_{n\to\infty}((J_{E^*}+r_{n,i_0}A_{i_0}J_{E^*})^{-1}J_{E^*}(u_n+e_n)-(u_n+e_n))=0.$$

Setting $\xi_{n,i_0}=(J_{E^*}+r_{n,i_0}A_{i_0}J_{E^*})^{-1}J_{E^*}(u_n+e_n)$, we have

$$J_{E^*}\xi_{n,i_0}+r_{n,i_0}A_{i_0}J_{E^*}\xi_{n,i_0}=J_{E^*}(u_n+e_n).$$

Note that $\xi_{n,i_0} \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, $e_n \to 0$ and $inf_n r_{n,i_0} > 0$. Using Lemma 1.1, we have that $A_{i_0} J_{E^*} \xi_{n,i_0} \to 0$, as $n \to \infty$. It is not difficult to check that $A_{i_0} J_{E^*} \subset E^* \times E$ is maximal monotone. So, Lemma 1.2 implies that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in N(A_{i_0} J_{E^*})$. Repeating the above, we can see that, for $\forall i \in \mathbb{N}$, $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in N(A_i J_{E^*})$. It follows that $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} N(A_i J_{E^*})$. Similarly, $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) \in \bigcap_{i=1}^{\infty} N(B_i J_{E^*})$.

Step 9. $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1) = P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1)$.
From Step 8, we see that

$$\|P_{\bigcap_{n=1}^{\infty} U_n}(u_1) - u_1\| \geq \|P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) - u_1\|.$$

From Step 1, we see that

$$\|P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) - u_1\| \geq \|P_{\bigcap_{n=1}^{\infty} U_n}(u_1) - u_1\|.$$

Therefore,

$$\|P_{\bigcap_{n=1}^{\infty} U_n}(u_1) - u_1\| = \|P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) - u_1\|.$$

Since $P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ is unique, we have $P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1)$. Using Lemma 1.1, we have $\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(B_i))$, as $n \to \infty$. This completes the proof. □

From Theorem 2.1, we have the following results.

Corollary 2.2. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n) \sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p \rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\}, \\ y_n = P_{U_{n+1}}(u_1), \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n) \sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p \rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n \|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2}\}, \\ Y_{n+1} = \{p \in X_{n+1} : \|u_1 - p\|^2 \leq \|P_{X_{n+1}}(u_1) - u_1\|^2 + \delta_{n+1}\}, \\ u_{n+1} \in Y_{n+1}, \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.2}$$

If the assumptions (i), (iii) and (iv) of Theorem 2.1 hold and $(ii)'$ $\delta_n \to 0$, as $n \to 0$, then $\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(B_i))$, as $n \to \infty$.

Proof. Putting $y_n = P_{U_{n+1}}(u_1)$ in Theorem 2.1, we have scheme (2.2). We only need modify Steps 6 and 7 in Theorem 2.1 to get the results.

Step 6. Both $\{y_n\}$ and $\{u_n\}$ are bounded.
Since $y_n = P_{U_{n+1}}(u_1)$, we have, $\forall q \in (\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*})) \subset U_{n+1}$, $\|u_1 - y_n\| \leq \|q - u_1\|$, which implies that $\{y_n\}$ is bounded. Since $u_{n+1} \in Y_{n+1}$, we have that

$$\|u_1 - u_{n+1}\|^2 \leq \|P_{X_{n+1}}(u_1) - u_1\|^2 + \delta_{n+1}.$$

Since $\{P_{X_n}(u_1)\}$ is convergent and $\delta_n \to 0$, we obtain that $\{u_n\}$ is bounded.

Step 7. $y_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$ and $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$, as $n \to \infty$.

It follows from Lemmas 1.3 and 1.4 that $y_n = P_{U_{n+1}}(u_1) \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$, as $n \to \infty$. Following Step 7 in Theorem 2.1, we have $u_n \to P_{\bigcap_{n=1}^{\infty} X_n}(u_1) = P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$. This completes the proof. □

Similarly, we have the following two results.

Corollary 2.3. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ y_n \in V_{n+1}, \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n \|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2}\}, \\ u_{n+1} = P_{X_{n+1}}(u_1), \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.3}$$

If the assumptions (i), (iii) and (iv) of Theorem 2.1 hold and $(ii)''$ $\lambda_n \to 0$, as $n \to 0$, then $\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(B_i))$*, as $n \to \infty$.*

Proof. By taking $u_{n+1} = P_{X_{n+1}}(u_1)$ in (2.1), we have scheme (2.2). Similarly, we can obtain the desired result immediately. □

Corollary 2.4. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\}, \\ y_n = P_{U_{n+1}}(u_1), \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n \|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2}\}, \\ u_{n+1} = P_{X_{n+1}}(u_1), \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.4}$$

If the assumptions (i), (iii) and (iv) of Theorem 2.1, then $\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*})) \bigcap (\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(B_i))$*, as $n \to \infty$.*

Proof. By taking $y_n = P_{U_{n+1}}(u_1)$ and $u_{n+1} = P_{X_{n+1}}(u_1)$ in (2.1), we have scheme (2.4). Similarly, we can obtain the desired result immediately. □

Corollary 2.5. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*}u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i}A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*}v_n - \alpha_n J_{E^*}u_n - (1-\alpha_n)J_{E^*}(u_n+e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n\|u_n\|^2 - (1-\alpha_n)\|u_n+e_n\|^2}{2}\}, \\ y_n = \Pi_{U_{n+1}}(u_n), \\ z_n = J_E[\beta_n J_{E^*}u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i}B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*}z_n - \beta_n J_{E^*}u_n - (1-\beta_n)J_{E^*}(y_n+\varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n\|v_n\|^2 - (1-\beta_n)\|y_n+\varepsilon_n\|^2}{2}\}, \\ Y_{n+1} = \{p \in X_{n+1} : \|u_1 - p\|^2 \leq \|P_{X_{n+1}}(u_1) - u_1\|^2 + \delta_{n+1}\}, \\ u_{n+1} \in Y_{n+1}, \\ \overline{u_n} = J_{E^*}u_n,\ n \in \mathbb{N}. \end{cases} \tag{2.5}$$

If the assumptions that (i), (iii) and (iv) of Theorem 2.1 and $(ii)'$ $\delta_n \to 0$, as $n \to 0$, then $\overline{u_n} \to J_{E^*}P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$, as $n \to \infty$.

Proof. Putting $y_n = \Pi_{U_{n+1}}(u_n)$ in Theorem 2.1, we have scheme (2.5). We only need modify Steps 6 and 7 in Theorem 2.1 to get the results.

Step 6. Both $\{y_n\}$ and $\{u_n\}$ are bounded.
From Theorem 2.1, it is easy to see that $\{u_n\}$ is bounded. Since $y_n = \Pi_{U_{n+1}}(u_n)$, we have

$$\forall q \in (\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*})) \subset U_{n+1}.$$

Using Lemma 1.8, one has

$$\phi(q, y_n) + \phi(y_n, u_n) \leq \phi(q, u_n).$$

Thus $\{\phi(q, y_n)\}$ is bounded. Since

$$\phi(q, y_n) \geq (\|y_n\| - \|q\|)^2,$$

we have that $\{y_n\}$ is bounded.

Step 7. $y_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$ and $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1) = P_{\bigcap_{n=1}^{\infty} X_n}(u_1)$ as $n \to \infty$.
From Theorem 2.1, we have $u_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$, as $n \to \infty$. Since

$$u_{n+1} \in Y_{n+1} \subset X_{n+1} \subset U_{n+1},$$

we obtain from Lemma 1.8 that

$$\phi(u_{n+1}, y_n) + \phi(y_n, u_n) \leq \phi(u_{n+1}, u_n) \to 0$$

as $n \to \infty$. Thus $\phi(y_n, u_n) \to 0$, which implies from Lemma 1.6 that $y_n - u_n \to 0$. So $y_n \to P_{\bigcap_{n=1}^{\infty} U_n}(u_1)$ as $n \to \infty$. This completes the proof. □

Corollary 2.6. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\}, \\ V_{n+1} = \{p \in U_{n+1} : \|u_1 - p\|^2 \leq \|P_{U_{n+1}}(u_1) - u_1\|^2 + \lambda_{n+1}\}, \\ y_n \in V_{n+1}, \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n \|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2}\}, \\ u_{n+1} = \Pi_{X_{n+1}}(y_n), \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.6}$$

If the assumptions that (i), (iii) and (iv) of Theorem 2.1 and $(ii)'$ $\lambda_n \to 0$, as $n \to 0$, then $\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$, as $n \to \infty$.

Proof. By taking $u_n = \Pi_{X_{n+1}}(y_n)$ in (2.1), we have scheme (2.6). Similarly, we can obtain the desired result immediately. □

Corollary 2.7. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n \|u_n\|^2 - (1-\alpha_n)\|u_n + e_n\|^2}{2}\}, \\ y_n = \Pi_{U_{n+1}}(u_n), \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n \|v_n\|^2 - (1-\beta_n)\|y_n + \varepsilon_n\|^2}{2}\}, \\ u_{n+1} = P_{X_{n+1}}(y_n), \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.7}$$

If the assumptions that (i), (iii) and (iv) of Theorem 2.1, then
$\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$, as $n \to \infty$.

Proof. By taking $y_n = \Pi_{U_{n+1}}(u_n)$ and $u_{n+1} = P_{X_{n+1}}(u_1)$ in (2.1), we have scheme (2.7). Similarly, we can obtain the desired result immediately. □

Corollary 2.8. *Let $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in E^*, e_1 \in E^*, \varepsilon_1 \in E^*, \\ v_n = J_E[\alpha_n J_{E^*} u_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_{E^*}(J_{E^*} + r_{n,i} A_i J_{E^*})^{-1} J_{E^*}(u_n + e_n)], \\ U_1 = E^* = V_1, \\ U_{n+1} = \{p \in X_n : \langle J_{E^*} v_n - \alpha_n J_{E^*} u_n - (1-\alpha_n) J_{E^*}(u_n + e_n), p\rangle \\ \qquad \geq \frac{\|v_n\|^2 - \alpha_n\|u_n\|^2 - (1-\alpha_n)\|u_n+e_n\|^2}{2}\}, \\ y_n = P_{U_{n+1}}(u_1), \\ z_n = J_E[\beta_n J_{E^*} u_n + (1-\beta_n)\sum_{i=1}^{\infty} b_{n,i} J_{E^*}(J_{E^*} + s_{n,i} B_i J_{E^*})^{-1} J_{E^*}(y_n + \varepsilon_n)], \\ X_{n+1} = \{p \in U_{n+1} : \langle J_{E^*} z_n - \beta_n J_{E^*} u_n - (1-\beta_n) J_{E^*}(y_n + \varepsilon_n), p\rangle \\ \qquad \geq \frac{\|z_n\|^2 - \beta_n\|v_n\|^2 - (1-\beta_n)\|y_n+\varepsilon_n\|^2}{2}\}, \\ u_{n+1} = \Pi_{X_{n+1}}(y_n), \\ \overline{u_n} = J_{E^*} u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.8}$$

If the assumptions that (i), (iii) and (iv) of Theorem 2.1, then
$\overline{u_n} \to J_{E^*} P_{(\bigcap_{i=1}^{\infty} N(A_i J_{E^*}))\bigcap(\bigcap_{i=1}^{\infty} N(B_i J_{E^*}))}(u_1) \in (\bigcap_{i=1}^{\infty} N(A_i))\bigcap(\bigcap_{i=1}^{\infty} N(B_i))$*, as* $n \to \infty$.

Proof. By taking $y_n = P_{U_{n+1}}(u_1)$ and $u_{n+1} = \Pi_{X_{n+1}}(y_n)$ in (2.1), we have scheme (2.8). Similarly, we can obtain the desired result immediately. □

Remark 2.9. From the Corollaries, we see that (2.1) includes the traditional projection iterative schemes that involve $P_{X_{n+1}}(u_1)$ (or $P_{U_{n+1}}(u_1)$) and $\Pi_{U_{n+1}}(u_n)$ (or $\Pi_{X_{n+1}}(y_n)$) for the discussion of accretive-type mappings, e.g., [10, 15, 19]. In addition, In (2.1), for each iterative step n, the iterative elements y_n and u_n can be chosen arbitrarily within two sets. This helps us to get one of the iterative sequences from the infinite ones more flexibly to meet the needs for a special case.

3. Connections with capillarity systems

In a Hilbert space, m-d-accretive mappings and m-accretive mappings are the same, while they are different in a non-Hilbertian Banach space. In this section, we present a new m-d-accretive mapping in a Banach space.

Definition 3.1. [16] Recall that a mapping $S : D(S) = E \to E^*$ is said to be a hemi-continuous mapping if $S(x+ty) \rightharpoonup Sx$ as $t \to 0$, for $\forall x, y \in E$.

Lemma 3.2. [16] *If $B : E \to 2^{E^*}$ is an everywhere defined monotone and hemi-continuous mapping, then B is maximal monotone.*

3.1. m-accretive mappings and capillarity systems.

The capillarity equation is an important equation appeared in the capillarity phenomenon (see [20]) and the following capillarity systems were studied in [21] as an example of m-accretive mappings in the Hilbert space $L^2(\Omega)$:

$$\begin{cases} -div[(1+\frac{|grad(u^{(i)}(x))|^{p_i}}{\sqrt{1+|grad(u^{(i)}(x))|^{2p_i}}})|grad(u^{(i)}(x))|^{p_i-2}grad(u^{(i)}(x))] \\ +\lambda_i(|u^{(i)}(x)|^{q_i-2}u^{(i)}(x)+|u^{(i)}(x)|^{r_i-2}u^{(i)}(x))+u^{(i)}(x)=f_i(x), \quad x\in\Omega, \\ -<\vartheta,(1+\frac{|grad(u^{(i)}(x))|^{p_i}}{\sqrt{1+|grad(u^{(i)}(x))|^{2p_i}}})|grad(u^{(i)}(x))|^{p_i-2}grad(u^{(i)}(x))>=0, \quad x\in\Gamma, \quad i\in\mathbb{N}, \end{cases} \tag{3.1}$$

where $|\cdot|$ and $<\cdot,\cdot>$ denote the norm and inner-product in R^n, respectively.

The study on (3.1) was based on the following assumptions in [21].

(1) Ω is bounded conical domain in R^n $(n \in \mathbb{N})$ with $\Gamma \in C^1$, ϑ is the the exterior normal derivative of Γ, λ_i is a positive number and $f_i(x) \in L^{p_i}(\Omega)$ is a given function, for $i \in \mathbb{N}$.

(2) For $i \in \mathbb{N}$, $\frac{2n}{n+1} < p_i < +\infty$. If $p_i \geq n$, then $1 \leq q_i, r_i < +\infty$. If $p_i < n$, then $1 \leq q_i, r_i \leq \frac{np_i}{n-p_i}$.

The following results were proved in [21].

Lemma 3.3. [21] *The mapping $U_i : W^{1,p_i}(\Omega) \to (W^{1,p_i}(\Omega))^*$ defined by*

$$\langle v, U_i u\rangle = \int_\Omega < (1 + \frac{|grad(u(x))|^{p_i}}{\sqrt{1+|grad(u(x))|^{2p_i}}})|grad(u(x))|^{p_i-2}grad(u(x)), grad(v(x)) > dx$$
$$+\lambda_i \int_\Omega |u(x)|^{q_i-2}u(x)v(x)dx + \lambda_i \int_\Omega |u(x)|^{r_i-2}u(x)v(x)dx,$$

for $\forall u, v \in W^{1,p_i}(\Omega)$, is everywhere defined, hemi-continuous, monotone and coercive, for each $i \in \mathbb{N}$.

Lemma 3.4. [21] *Define $B_i : L^2(\Omega) \to L^2(\Omega)$ by*

$$D(B_i) = \{u \in L^2(\Omega) |\ \exists\, f \in L^2(\Omega) \text{ such that } f \in U_i u\}.$$

For $u \in D(B_i)$, $B_i u = \{f \in L^2(\Omega) \mid f \in U_i u\}$. Then $B_i : L^2(\Omega) \to L^2(\Omega)$ is m-d-accretive, $\forall i \in \mathbb{N}$.

Remark 3.5. From Lemma 3.3, we see that an m-d-accretive mapping B_i is defined in a Hilbert space $L^2(\Omega)$ based on capillarity systems (3.1), $\forall i \in \mathbb{N}$.

3.2. **m-d-accretive mappings in a Banach space and capillarity systems.** A new m-d-accretive mapping, which is different from B_i in Lemma 3.3, will be defined based on capillarity systems (3.1) again.

Suppose $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $\frac{1}{q_i} + \frac{1}{q_i'} = 1$, and $\frac{1}{r_i} + \frac{1}{r_i'} = 1$, for $i \in \mathbb{N}$. We use $\|\cdot\|_{L^{p_i}(\Omega)}$, $\|\cdot\|_{L^{p_i'}(\Omega)}$, $\|\cdot\|_{W^{1,p_i}(\Omega)}$ and $\|\cdot\|_{W^{1,p_i'}(\Omega)}$ to denote the norms in $L^{p_i}(\Omega)$, $L^{p_i'}(\Omega)$, $W^{1,p_i}(\Omega)$ and $W^{1,p_i'}(\Omega)$, respectively.

In the following, we suppose $2 \leq p_i' < +\infty$ and $1 \leq q_i, r_i < +\infty$, $\forall i \in \mathbb{N}$.

Lemma 3.6. *The mapping $\widetilde{U}_i : W^{1,p_i'}(\Omega) \to (W^{1,p_i'}(\Omega))^*$ defined by*

$$\langle v, \widetilde{U}_i u\rangle = \int_\Omega < (1 + \frac{|grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i}}{\sqrt{1+|grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{2p_i}}})|$$
$$\times grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i-2}grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'}),$$
$$grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'}) > dx,$$

for $\forall u, v \in W^{1,p_i'}(\Omega)$, is everywhere defined, hemi-continuous and monotone, for each $i \in \mathbb{N}$. Then Lemma 3.2 implies that it is maximal monotone, for each $i \in \mathbb{N}$.

Proof. We split the proof into three steps.

Step 1. $\widetilde{U}_i$ is everywhere defined.

In fact, for $\forall u, v \in W^{1,p_i'}(\Omega)$, one has

$$
\begin{aligned}
&|\langle v, \widetilde{U}_i u\rangle| \\
&\leq 2\int_\Omega |grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i-1} \times |grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'})|dx \\
&\leq 2(p_i'-1)^{p_i}\|u\|_{p_i'}^{(2-p_i')(p_i-1)}\|v\|_{p_i'}^{2-p_i'}\int_\Omega |u|^{2-p_i}|gradu|^{p_i-1}|v|^{p_i'-2}|gradv|dx \\
&\leq 2(p_i'-1)^{p_i}\|u\|_{p_i'}^{(2-p_i')(p_i-1)}\|v\|_{p_i'}^{2-p_i'}(\int_\Omega |u|^{p_i'-p_i}|gradu|^{p_i}dx)^{\frac{1}{p_i'}} \\
&\quad \times (\int_\Omega |v|^{(p_i'-2)p_i}|gradv|^{p_i}dx)^{\frac{1}{p_i}} \\
&\leq 2(p_i'-1)^{p_i}\|u\|_{p_i'}^{(2-p_i')(p_i-1)}\|v\|_{p_i'}^{2-p_i}\|u\|_{p_i'}^{2-p_i}\||gradu|\|_{p_i'}^{\frac{p_i}{p_i'}}\||gradv|\|_{p_i'}\|v\|_{p_i'}^{\frac{p_i'-p_i}{p_i}} \\
&\leq 2(p_i'-1)^{p_i}\|u\|_{1,p_i'}^{p_i-1}\|v\|_{1,p_i'}.
\end{aligned}
$$

Therefore, $\widetilde{U}_i$ is everywhere defined.

Step 2. $\widetilde{U}_i$ is monotone.

For $\forall u, v \in W^{1,p_i'}(\Omega)$, one has

$$
\begin{aligned}
&\langle u-v, \widetilde{U}_i u - \widetilde{U}_i v\rangle \\
&= \int_\Omega < (1 + \frac{|grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i}}{\sqrt{1+|grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{2p_i}}}) \\
&\quad \times |grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i-2}grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'}) \\
&\quad - (1 + \frac{|grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'})|^{p_i}}{\sqrt{1+|grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'})|^{2p_i}}}) \\
&\quad \times |grad(|v|^{p_i'-1}sgnu\|v\|_{p_i'}^{2-p_i'})|^{p_i-2}grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'}), \\
&grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'}) - grad(|v|^{p_i'-1}sgnv\|v\|_{p_i'}^{2-p_i'}) > dx \geq 0.
\end{aligned}
$$

Therefore, $\widetilde{U}_i$ is monotone.

Step 3. $\widetilde{U}_i$ is hemi-continuous.

It suffices to show that, for $\forall u, v, w \in W^{1,p_i'}(\Omega)$ and $t \in [0,1]$, $\langle w, \widetilde{U}_i(u+tv) - \widetilde{U}_i u\rangle \to 0$ as $t \to 0$. In fact, using Lebesgue's dominated convergence theorem, one has

$$
\begin{aligned}
&|\langle w,\widetilde{U}_i(u+tv)-\widetilde{U}_iu\rangle| \\
&=\int_\Omega |(1+\frac{|grad(|u+tv|^{p_i'-1}sgn(u+tv)\|u+tv\|_{p_i'}^{2-p_i'})|^{p_i}}{\sqrt{1+|grad(|u+tv|^{p_i'-1}sgn(u+tv)\|u+tv\|_{p_i'}^{2-p_i'})|^{2p_i}}}) \\
&\quad\times|grad(|u+tv|^{p_i'-1}sgn(u+tv)\|u+tv\|_{p_i'}^{2-p_i'})|^{p_i-2} \\
&\quad\times grad(|u+tv|^{p_i'-1}sgn(u+tv)\|u+tv\|_{p_i'}^{2-p_i'}) \\
&\quad-(1+\frac{|grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i}}{\sqrt{1+|grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{2p_i}}})| \\
&\quad\times grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})|^{p_i-2}grad(|u|^{p_i'-1}sgnu\|u\|_{p_i'}^{2-p_i'})| \\
&\quad\times|grad(|w|^{p_i'-1}sgnw\|w\|_{p_i'}^{2-p_i'})|dx\to 0,
\end{aligned}
$$

as $t\to 0$. Therefore, $\widetilde{U}_i$ is hemi-continuous. This completes the proof. □

Lemma 3.7. *The mapping* $\widetilde{V}_i : W^{1,p_i'}(\Omega)\to(W^{1,p_i'}(\Omega))^*$ *defined by*

$$\langle v,\widetilde{V}_iu\rangle=\int_\Omega uvdx,$$

for $\forall u,v\in W^{1,p_i'}(\Omega)$*, is everywhere defined, hemi-continuous and monotone, for each* $i\in\mathbb{N}$*. Then Lemma 3.2 implies that it is maximal monotone, for each* $i\in\mathbb{N}$*.*

Proof. We split the proof into three steps.

Step 1. $\widetilde{V}_i$ is everywhere defined.

In fact, for $\forall u,v\in W^{1,p_i'}(\Omega)$, one has,

$$|\langle v,\widetilde{V}_iu\rangle|\le\int_\Omega |u||v|dx\le\|u\|_{p_i}\|v\|_{p_i'}\le\|u\|_{1,p_i'}\|v\|_{1,p_i'}.$$

Therefore, $\widetilde{V}_i$ is everywhere defined.

Step 2. $\widetilde{V}_i$ is monotone.

For $\forall u,v\in W^{1,p_i'}(\Omega)$, one has

$$\langle u-v,\widetilde{V}_iu-\widetilde{V}_iv\rangle=\int_\Omega (u-v)(u-v)dx\ge 0.$$

Therefore, $\widetilde{V}_i$ is monotone.

Step 3. $\widetilde{V}_i$ is hemi-continuous.

It suffices to show that for $\forall u,v,w\in W^{1,p_i'}(\Omega)$ and $t\in[0,1]$, $\langle w,\widetilde{V}_i(u+tv)-\widetilde{V}_iu\rangle\to 0$, as $t\to 0$. In fact, using Lebesgue's dominated convergence theorem, one has

$$|\langle w,\widetilde{V}_i(u+tv)-\widetilde{V}_iu\rangle|\le\int_\Omega |(u+tv)-u||w|dx=|t|\int_\Omega |v||w|dx\to 0,$$

as $t\to 0$. Therefore, $\widetilde{V}_i$ is hemi-continuous. This completes the proof. □

Remark 3.8. [16] There exists a maximal monotone extension of $\widetilde{U}_i$ from $L^{p_i'}(\Omega)$ to $L^{p_i}(\Omega)$, which is denoted by $\overline{U_i}$, for $i \in \mathbb{N}$. There exists a maximal monotone extension of $\widetilde{V}_i$ from $L^{p_i'}(\Omega)$ to $L^{p_i}(\Omega)$, which is denoted by $\overline{V_i}$, for $i \in \mathbb{N}$.

Lemma 3.9. [22] *For $2 \leq p_i' < +\infty$, the normalized duality mapping $J_i : L^{p_i'}(\Omega) \to L^{p_i}(\Omega)$ is defined by: $J_i u = |u|^{p_i'-1} sgnu \|u\|_{p_i'}^{2-p_i'}$, for $\forall u \in L^{p_i'}(\Omega)$ and $i \in \mathbb{N}$. And then, $J_i^{-1} : L^{p_i}(\Omega) \to L^{p_i'}(\Omega)$ is defined by: $J_i^{-1} u = |u|^{p_i-1} sgnu$, for $\forall u \in L^{p_i}(\Omega)$ and $i \in \mathbb{N}$.*

Theorem 3.10. *The mapping $A_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ defined by*

$$(A_i u)(x) = \overline{U_i} J_i^{-1} u(x), \quad \forall u(x) \in L^{p_i}(\Omega),$$

is m-d-accretive, $\forall i \in \mathbb{N}$.

Proof. Since $\overline{U_i}$ is monotone, we have, for $\forall u, v \in L^{p_i}(\Omega)$, $\langle A_i u - A_i v, J_i^{-1} u - J_i^{-1} v \rangle = \langle \overline{U_i} J_i^{-1} u - \overline{U_i} J_i^{-1} v, J_i^{-1} u - J_i^{-1} v \rangle \geq 0$, for $i \in \mathbb{N}$. For $\forall f(x) \in L^{p_i}(\Omega)$, there exists $u(x) \in L^{p_i'}(\Omega)$ such that $J_i u + \lambda \overline{U_i} u = f(x)$, $\forall i \in \mathbb{N}$. For this $u(x)$, Lemma 1.1 implies that there exists $u^*(x) \in L^{p_i}(\Omega)$ such that $u(x) = J_i^{-1} u^*(x)$, for $i \in \mathbb{N}$. Therefore, $u^*(x) + \lambda \overline{U_i} J_i^{-1} u^* = u^* + \lambda A_i u^* = f(x)$, which implies that $L^{p_i}(\Omega) \subset R(I + \lambda A_i)$, for $\forall \lambda > 0$ and $i \in \mathbb{N}$. That is, $L^{p_i}(\Omega) = R(I + \lambda A_i)$, for $\forall \lambda > 0$ and $i \in \mathbb{N}$. Then A_i is m-d-accretive, for $\forall \lambda > 0$ and $i \in \mathbb{N}$. This completes the proof. □

Theorem 3.11. *The mapping $C_i : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ defined by*

$$(C_i u)(x) = \overline{V_i} J_i^{-1} u(x), \quad \forall u(x) \in L^{p_i}(\Omega),$$

is m-d-accretive, for $i \in \mathbb{N}$.

Proof. From Theorem 3.10, the result follows immediately. This completes the proof. □

We can easily obtain the following results.

Theorem 3.12. *If $f_i(x) \equiv \lambda_i(|k|^{q_i-1} + |k|^{r_i-1}) sgnk + k$, where k represents constant in (3.1), then $\{u^{(i)}(x) \equiv k : i \in \mathbb{N}\}$ is the solution of capillarity systems (3.1).*

Theorem 3.13. *The mapping $\overline{C_i} : L^{p_i}(\Omega) \to L^{p_i}(\Omega)$ defined by*

$$(\overline{C_i} u)(x) = C_i u(x) - |k|^{p_i-1} sgnk, \quad \forall u(x) \in L^{p_i}(\Omega),$$

where k is as in Theorem 3.12, is also m-d-accretive, for $i \in \mathbb{N}$.

Theorem 3.14. *Under the assumption of Theorem 3.12, we have*

$$\{u^{(i)}(x) \equiv k : i \in \mathbb{N}\} \subset (\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(\overline{C_i})).$$

Proof. In fact, if $u^{(i)}(x) \equiv k$, then $A_i u^{(i)}(x) \equiv A_i k = \overline{U_i} J_i^{-1} k = \widetilde{U}_i J_i^{-1} k$, which ensures that $\langle v, \widetilde{U}_i J_i^{-1} k \rangle \equiv 0$, for $\forall v(x) \in W^{1,p_i}(\Omega)$, $i \in \mathbb{N}$. Therefore, $\widetilde{U}_i J_i^{-1} k \equiv 0$. So,

$$\{u^{(i)}(x) \equiv k : i \in \mathbb{N}\} \subset \bigcap_{i=1}^{\infty} N(A_i).$$

If $u^{(i)}(x) \equiv k$, then

$$\overline{C_i} u^{(i)}(x) \equiv \overline{C_i} k = C_i k - |k|^{p_i-1} sgnk = \overline{V_i} J_i^{-1} k - |k|^{p_i-1} sgnk = \widetilde{V}_i J_i^{-1} k - |k|^{p_i-1} sgnk.$$

Therefore,

$$\begin{aligned}\langle v, \widetilde{V}_i J_i^{-1}k - |k|^{p_i-1}sgnk\rangle &= \int_\Omega (J_i^{-1}k - |k|^{p_i-1}sgnk)vdx \\ &= \int_\Omega (|k|^{p_i-1}sgnk - |k|^{p_i-1}sgnk)vdx = 0.\end{aligned}$$

Then $\overline{C_i}k = 0$, which implies that $\{u^{(i)}(x) \equiv k : i \in \mathbb{N}\} \subset \bigcap_{i=1}^{\infty} N(\overline{C_i})$. This completes the proof. □

Remark 3.15. From Theorem 3.14, we see that the assumption that "$(\bigcap_{i=1}^{\infty} N(A_i)) \bigcap (\bigcap_{i=1}^{\infty} N(B_i)) \neq \emptyset$" in Theorem 2.1 is valid. Then Theorem 2.1 can be applied to approximate the common zeros of two infinite families of m-d-accretive mappings related to capillarity systems (3.1).

Acknowledgements

This paper was supported by the National Natural Science Foundation of China (11071053), Natural Science Foundation of Hebei Province (A2019207064), Key Project of Science and Research of Hebei Educational Department (ZD2019073), and Key Project of Science and Research of Hebei University of Economics and Business (2018ZD06).

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