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ON STANCU TYPE GENERALIZATION OF (p,q)-GAMMA OPERATORS

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Abstract. In this paper, we construct stancu type generalization of (p,q)-Gamma operators: (p,q)-Gamma-Stancu operators. We establish the auxiliary results on the moments and the central moments of the operators. We also discuss some local approximation properties of these operators by means of the modulus of the continuity and Peetre K-functional. Furthermore, we investigate Voronovskaja type theorem.

Keywords. (p,q)-Gamma-Stancu operators; Rate of convergence; Weighted approximation; Pointwise estimate; Voronovskaja type theorem.

1. Introduction

In 1968, Stancu [1] introduced the following generalization of classical Bernstein operators for all $0 \leq \alpha \leq \beta$ and $n \in \mathbb{N}$

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),\ x \in [0,1],$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0,1,\cdots,n$. Recently, Stancu generalization of many positive operators was widely studied; see, e.g., [2, 3, 4]. In [5], Karsli introduced the Gamma operators preserving x^2. In [6], Cai and Zeng studied and discussed approximation properties of q-Gamma operators. In [7], Zhao, Cheng and Zeng investigated stancu type generalization of q-Gamma operators. Since Mursaleen, Khan and Khan [8, 9] introduced (p,q)-calculus in approximation theory and constructed the (p,q)-Bernstein operators and (p,q)-Bernstein-Stancu operators [10, 11], many positive operators have been studied and discussed; see, e.g., [12, 13, 14, 15, 16, 17]. In [18], Cheng, Zhang and Cai introduced (p,q)-analogue of Gamma operators preserving x^2. Then, Stancu generalization of q-analogue operators or (p,q)-analogue

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operators have widely been discussed and investigated; see, e.g., [19, 20, 21, 22]. All results motivate us to construct the stancu type generalization of (p,q)-Gamma operators (1.1).

First, we recall and introduce some concepts from (p,q)-calculus (see [23]). For nonnegative integer s, the (p,q)-integer $[s]_{p,q}$ and (p,q)-factorial $[s]_{p,q}!$ are defined by

$$[s]_{p,q}=p^{s-1}+p^{s-2}q+p^{s-3}q^2+\cdots+pq^{s-2}+q^{s-1}=\begin{cases}\frac{p^s-q^s}{p-q}, & p\neq q\neq 1;\\ sp^{s-1}, & p=q\neq 1;\\ [s]_q, & p=1;\\ s, & p=q=1\end{cases}$$

and

$$[s]_{p,q}!=\begin{cases}[1]_{p,q}[2]_{p,q}\cdots[s]_{p,q}, & s\geq 1;\\ 1, & s=0.\end{cases}$$

The (p,q)-power basis is defined by

$$(x\oplus t)_{p,q}^s=\prod_{i=0}^{s-1}(p^ix+q^it),\ s\in\mathbb{N}_+$$

and

$$(x\ominus t)_{p,q}^s=\prod_{i=0}^{s-1}(p^ix-q^it),\ s\in\mathbb{N}_+.$$

The improper (p,q)-integral of f on $\mathbb{R}_+:=(0,\infty)$ is defined by (see [24])

$$\int_0^\infty f(x)\mathrm{d}_{p,q}x=(p-q)\sum_{s=-\infty}^{\infty}\frac{q^s}{p^{s+1}}f\left(\frac{q^s}{p^{s+1}}\right),\ 0<\frac{q}{p}<1.$$

The (p,q)-Gamma function is defined by

$$\Gamma_{p,q}(s+1)=\frac{(p\ominus q)_{p,q}^s}{(p-q)^s}=[s]_{p,q}!,\ 0<q<p\leq 1,\ s\in\mathbb{N}_+.$$

Aral and Gupta [25] defined the (p,q)-Beta function of the second kind for any $s,t\in\mathbb{N}_+$ by

$$B_{p,q}(s,t)=\int_0^\infty\frac{x^{s-1}}{(1\oplus px)_{p,q}^{s+t}}\mathrm{d}_{p,q}t$$

and proved the following relation with (p,q)-Gamma function:

$$B_{p,q}(s,t)=\frac{q\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{(p^{s+1}q^{s-1})^{\frac{s}{2}}\Gamma_{p,q}(s+t)}.$$

Meantime, they also proved that the (p,q)-Beta function does not satisfy commutativity property, i.e., $B_{p,q}(s,t)\neq B_{p,q}(t,s)$.

In [18], Cheng, Zhang and Cai introduced the following (p,q)-Gamma operators preserving x^2 with (p,q)-Beta function.

Definition 1.1. For $f\in C(\mathbb{R}_+)$, $0<q<p\leq 1$, $n\in\mathbb{N}$, the (p,q)-Gamma operators is defined by

$$G_n^{p,q}(f;x)=\frac{x^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}(q^{n+\frac{3}{2}})^{n+3}}{B_{p,q}(n+1,n+3)}\int_0^\infty\frac{t^n}{\left((pq)^{n+\frac{3}{2}}x\oplus t\right)_{p,q}^{2n+4}}f(t)\,\mathrm{d}_{p,q}t. \tag{1.1}$$

In case $p=1$, we obtain the q-Gamma operators [6]. In case $p=q=1$, we obtain the well known Gamma operators [5] immediately. Now, we construct Stancu type generalization of (p,q)-Gamma operators.

Definition 1.2. For $f\in C(\mathbb{R}_+)$, $0<q<p\le 1$, $0\le\alpha\le\beta$ and $n\in\mathbb{N}$, we define (p,q)-Gamma-Stancu operators $\mathfrak{G}_n^{p,q}(f;x)$ by

$$\mathfrak{G}_n^{p,q}(f;x)=\frac{x^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}(q^{n+\frac{3}{2}})^{n+3}}{B_{p,q}(n+1,n+3)}\int_0^\infty \frac{t^n}{\left((pq)^{n+\frac{3}{2}}x\oplus t\right)_{p,q}^{2n+4}}f\left(\frac{[n]_{p,q}t+\alpha}{[n]_{p,q}+\beta}\right)\mathrm{d}_{p,q}t. \quad (1.2)$$

The paper is organized as follows. In Section 1, we introduce the history of Stancu type operators and the basic concepts of (p,q)-calculus and construct (p,q)-Gamma-Stancu operators (1.2). In Section 2, we discuss the auxiliary results on the moments computation formulas, the second and fourth order central moments computation formulas and limit equalities. In Section 3, we mainly investigate the approximation properties of operators (1.2). We divide this section into five subsections. To be more precise, we discuss, in subsection 3.1, the local approximation by means of the modulus of the continuity and Peetre K-functional. In subsection 3.2 and subsection 3.3, the rate of convergence and weighted approximation for these operators are investigated. In subsection 3.4, we obtain point-wise estimate by using the Lipschitz type maximal function. In subsection 3.5, the last subsection, we establish the Voronovskaja type asymptotic formula.

2. Moment estimates

In this section, we will give some lemmas on moment estimates for operators (1.2), which are necessary to establish the approximation properties.

Lemma 2.1. [18, Lemma 2.1] *For $x\in\mathbb{R}_+$, $n\in\mathbb{N}$, $0<q<p\le 1$, and $k=0,1,\cdots,n+2$, we have*

$$G_n^{p,q}(t^k;x)=\frac{x^k(pq)^{k-\frac{k^2}{2}}[n+k]_{p,q}![n-k+2]_{p,q}!}{[n]_{p,q}![n+2]_{p,q}!}.$$

Lemma 2.2. *For $x\in\mathbb{R}_+$, $n\in\mathbb{N}$, $0<q<p\le 1$, $0\le\alpha\le\beta$ and $m=0,1,\cdots,n+2$, we have the following relation*

$$\mathfrak{G}_n^{p,q}(t^m;x)=\sum_{k=0}^{m}\binom{m}{k}\frac{[n]_{p,q}^k\alpha^{m-k}}{([n]_{p,q}+\beta)^m}G_n^{p,q}(t^k;x).$$

Proof. Observe that

$$\begin{aligned}
&\mathfrak{G}_n^{p,q}(t^m;x)\\
&=\frac{x^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}(q^{n+\frac{3}{2}})^{n+3}}{B_{p,q}(n+1,n+3)}\int_0^\infty \frac{t^n}{\left((pq)^{n+\frac{3}{2}}x\oplus t\right)_{p,q}^{2n+4}}\left(\frac{[n]_{p,q}t+\alpha}{[n]_{p,q}+\beta}\right)^m \mathrm{d}_{p,q}t\\
&=\frac{x^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}(q^{n+\frac{3}{2}})^{n+3}}{B_{p,q}(n+1,n+3)}\int_0^\infty \frac{t^n}{\left((pq)^{n+\frac{3}{2}}x\oplus t\right)_{p,q}^{2n+4}}\frac{\sum\limits_{k=0}^{m}\binom{m}{k}[n]_{p,q}^k t^k\alpha^{m-k}}{([n]_{p,q}+\beta)^m}\mathrm{d}_{p,q}t\\
&=\sum_{k=0}^{m}\binom{m}{k}\frac{[n]_{p,q}^k\alpha^{m-k}}{([n]_{p,q}+\beta)^m}G_n^{p,q}(t^k;x).
\end{aligned}$$

This completes this Lemma. □

From Lemma 2.1 and Lemma 2.2, we can write

$$\mathfrak{G}_n^{p,q}(1;x)=1,\ \mathfrak{G}_n^{p,q}(t;x)=\frac{\sqrt{pq}[n]_{p,q}[n+1]_{p,q}}{([n]_{p,q}+\beta)[n+2]_{p,q}}x+\frac{\alpha}{[n]_{p,q}+\beta}$$

and

$$\mathfrak{G}_n^{p,q}(t^2;x)=\frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2}x^2+\frac{2\alpha\sqrt{pq}[n]_{p,q}[n+1]_{p,q}}{([n]_{p,q}+\beta)^2[n+2]_{p,q}}x+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}.$$

Lemma 2.3. *Assume that the sequences (p_n) and (q_n) satisfy $0<q_n<p_n\le 1$ with $p_n\to 1$, $q_n\to 1$, $p_n^n\to a\in[0,1]$, $q_n^n\to b\in[0,1]$, and $[n]_{p_n,q_n}\to\infty$ as $n\to\infty$. Then, for all $x\in\mathbb{R}_+$,*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}(t-x;x)=\alpha-\left(\beta+\frac{a+b}{2}\right)x \tag{2.1}$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}\left((t-x)^2;x\right)=(a+b)x^2, \tag{2.2}$$

and

$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}\left((t-x)^4;x\right)=0. \tag{2.3}$$

Proof. We rewrite and expand $\mathfrak{G}_n^{p_n,q_n}(t^i;x), i=1,2,3,4$. Using Lemmas 2.1, 2.2 and the properties of (p,q)-integers, we have

$$\begin{aligned}
\mathfrak{G}_n^{p_n,q_n}(t;x)&=\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}G_n^{p_n,q_n}(t;x)+\frac{\alpha}{[n]_{p_n,q_n}+\beta}G_n^{p_n,q_n}(1;x)\\
&=\left(1-\frac{\beta}{[n]_{p_n,q_n}+\beta}\right)\left(\frac{q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}-\frac{q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\\
&=\left(\frac{q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}-\frac{q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}-\frac{q_n^{\frac{1}{2}}\beta}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x+\frac{\alpha}{[n]_{p_n,q_n}+\beta},
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{G}_n^{p_n,q_n}(t^2;x)\\
&=\frac{[n]_{p_n,q_n}^2}{([n]_{p_n,q_n}+\beta)^2}G_n^{p_n,q_n}(t^2;x)+\frac{2\alpha[n]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2}G_n^{p_n,q_n}(t;x)+\frac{\alpha^2}{([n]_{p_n,q_n}+\beta)^2}G_n^{p_n,q_n}(1;x)\\
&=\left(1-\frac{2\beta}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2\\
&\quad+\left(\frac{2\alpha}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)\left(\frac{q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}-\frac{q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}\right)x\\
&=\left(1-\frac{2\beta}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2+\left(\frac{2q_n^{\frac{1}{2}}\alpha}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{G}_n^{p_n,q_n}(t^3;x)&=\frac{[n]_{p_n,q_n}^3}{([n]_{p_n,q_n}+\beta)^3}G_n^{p_n,q_n}(t^3;x)+\frac{3\alpha[n]_{p_n,q_n}^2}{([n]_{p_n,q_n}+\beta)^3}G_n^{p_n,q_n}(t^2;x)\\
&\quad+\frac{3\alpha^2[n]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^3}G_n^{p_n,q_n}(t;x)+\frac{\alpha^3}{([n]_{p_n,q_n}+\beta)^3}G_n^{p_n,q_n}(1;x)\\
&=\left(1-\frac{3\beta}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)\left(\frac{p_n^{\frac{3}{2}}}{q_n^{\frac{3}{2}}}+\frac{q_n^{n-\frac{3}{2}}[3]_{p_n,q_n}}{p_n^{\frac{3}{2}}[n]_{p_n,q_n}}\right)x^3\\
&\quad+\left(\frac{3\alpha}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2\\
&=\left(\frac{p_n^{\frac{3}{2}}}{q_n^{\frac{3}{2}}}+\frac{q_n^{n-\frac{3}{2}}[3]_{p_n,q_n}}{p_n^{\frac{3}{2}}[n]_{p_n,q_n}}-\frac{3p_n^{\frac{3}{2}}\beta}{q_n^{\frac{3}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^3\\
&\quad+\left(\frac{3\alpha}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{G}_n^{p_n,q_n}(t^4;x)\\
&=\left(1-\frac{4\beta}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)\left(\frac{p_n^4}{q_n^4}+\frac{p_nq_n^{n-4}[3]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{q_n^{n-5}[5]_{p_n,q_n}}{p_n[n-1]_{p_n,q_n}}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^4\\
&+\left(\frac{4\alpha}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)\left(\frac{p_n^{\frac{3}{2}}}{q_n^{\frac{3}{2}}}+\frac{q_n^{n-\frac{3}{2}}[3]_{p_n,q_n}}{p_n^{\frac{3}{2}}[n]_{p_n,q_n}}\right)x^3\\
&=\left(\frac{p_n^4}{q_n^4}+\frac{p_nq_n^{n-4}[3]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{q_n^{n-5}[5]_{p_n,q_n}}{p_n[n-1]_{p_n,q_n}}-\frac{4p_n^4\beta}{q_n^4([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^4\\
&+\left(\frac{4p_n^{\frac{3}{2}}\alpha}{q_n^{\frac{3}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^3.
\end{aligned}$$

Thus, we can estimate the first, second and fourth order central moments

$$
\begin{aligned}
&\mathfrak{G}_n^{p_n,q_n}(t-x;x)\\
&=\left(-1+\frac{q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}-\frac{q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}-\frac{q_n^{\frac{1}{2}}\beta}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x\\
&\quad+\frac{\alpha}{[n]_{p_n,q_n}+\beta},
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{G}_n^{p_n,q_n}\left((t-x)^2;x\right)\\
&=\mathfrak{G}_n^{p_n,q_n}(t^2;x)-2x\mathfrak{G}_n^{p_n,q_n}(t;x)+x^2\mathfrak{G}_n^{p_n,q_n}(1;x)\\
&=\left(2-\frac{2q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}+\frac{2q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}-\frac{2\beta}{[n]_{p_n,q_n}+\beta}+\frac{2q_n^{\frac{1}{2}}\beta}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2\\
&+\left(\frac{2q_n^{\frac{1}{2}}\alpha}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}-\frac{2\alpha}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{G}_n^{p_n,q_n}\left((t-x)^4;x\right)\\
&=\mathfrak{G}_n^{p_n,q_n}(t^4;x)-4x\mathfrak{G}_n^{p_n,q_n}(t^3;x)+6x^2\mathfrak{G}_n^{p_n,q_n}(t^2;x)\\
&\quad-4x^3\mathfrak{G}_n^{p_n,q_n}(t;x)+x^4\mathfrak{G}_n^{p_n,q_n}(1;x)\\
&=\left(7+\frac{p_n^4}{q_n^4}-\frac{4p_n^{\frac{3}{2}}}{q_n^{\frac{3}{2}}}-\frac{4p_n^{\frac{1}{2}}}{q_n^{\frac{1}{2}}}+\frac{p_nq_n^{n-4}[3]_{p_n,q_n}}{[n]_{p_n,q_n}}+\frac{q_n^{n-5}[5]_{p_n,q_n}}{p_n[n-1]_{p_n,q_n}}-\frac{4q_n^{n-\frac{3}{2}}[3]_{p_n,q_n}}{p_n^{\frac{3}{2}}[n]_{p_n,q_n}}\right.\\
&\quad+\frac{4q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}-\frac{4p_n^4\beta}{q_n^4([n]_{p_n,q_n}+\beta)}+\frac{12p_n^{\frac{3}{2}}\beta}{q_n^{\frac{3}{2}}([n]_{p_n,q_n}+\beta)}-\frac{12\beta}{[n]_{p_n,q_n}+\beta}\\
&\quad\left.+\frac{4q_n^{\frac{1}{2}}\beta}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^4\\
&\quad+\left(-\frac{12\alpha}{[n]_{p_n,q_n}+\beta}-\frac{4\alpha}{[n]_{p_n,q_n}+\beta}+\frac{4p_n^{\frac{3}{2}}\alpha}{q_n^{\frac{3}{2}}([n]_{p_n,q_n}+\beta)}\right.\\
&\quad\left.+\frac{12q_n^{\frac{1}{2}}\alpha}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^3.
\end{aligned}
$$

Letting $n\to\infty$, the following limit equalities hold

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}(t-x;x)\\
&=\lim_{n\to\infty}[n]_{p_n,q_n}\left(-1+\frac{q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}-\frac{q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}-\frac{q_n^{\frac{1}{2}}\beta}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x+\alpha\\
&=-\left(\frac{a+b}{2}+\beta\right)x+\alpha,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}\left((t-x)^2;x\right)\\
&=\lim_{n\to\infty}[n]_{p_n,q_n}\left(2-\frac{2q_n^{\frac{1}{2}}}{p_n^{\frac{1}{2}}}+\frac{2q_n^{n+\frac{3}{2}}}{p_n^{\frac{1}{2}}[n+2]_{p_n,q_n}}-\frac{2\beta}{[n]_{p_n,q_n}+\beta}+\frac{2q_n^{\frac{1}{2}}\beta}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x^2\\
&+\lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{2q_n^{\frac{1}{2}}\alpha}{p_n^{\frac{1}{2}}([n]_{p_n,q_n}+\beta)}-\frac{2\alpha}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\right)x\\
&=(a+b)x^2,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}\left((t-x)^4;x\right)\\
&=\lim_{n\to\infty}(\frac{p_n^n-q_n^n}{p_n-q_n}\left(\frac{p_n^4-q_n^4}{q_n^4}-\frac{4p_n^{\frac{3}{2}}-4q_n^{\frac{3}{2}}}{q_n^{\frac{3}{2}}}-\frac{4p_n^{\frac{1}{2}}-4q_n^{\frac{1}{2}}}{q_n^{\frac{1}{2}}}\right)\\
&\quad+3b+5b-12b+4b-4\beta+12\beta-12\beta+4\beta)x^4+(-12\alpha-4\alpha+4\alpha+12\alpha)x^3\\
&=\lim_{n\to\infty}(a-b)\left([4]_{p_n,q_n}-2[3]_{p_n,q_n}+2\right)x^4\\
&=0.
\end{aligned}$$

The proof of this Lemma is completed. □

3. Approximation Properties

3.1. **Local approximation.** Let $C_B(\mathbb{R}_+)$ be the function space of all real-valued continuous and bounded functions f defined on the interval $\mathbb{R}_+$, endowed with the norm $\|f\|=\sup_{x\in\mathbb{R}_+}|f(x)|$. Further, for any $\delta\in\mathbb{R}_+$, the usual modulus of continuity, the second-order modulus of smoothness and the Peetre's K-functional are defined respectively by

$$\omega(f;\delta)=\sup_{0<|h|\leq\delta}\sup_{x\in(0,\infty)}|f(x+h)-f(x)|,$$

$$\omega_2(f;\delta)=\sup_{0<|h|\leq\delta}\sup_{x\in(0,\infty)}|f(x+2h)-2f(x+h)+f(x)|,$$

$$K(f;\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\},$$

where

$$W^2=\{g\in C_B(R_+):g',g''\in C_B(R_+)\}.$$

By [26, p. 177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K(f;\delta^2) \le C\omega_2(f;\delta). \tag{3.1}$$

Now, we establish a direct local approximation theorem for operators (1.2).

Theorem 3.1. *Let $f \in C_B(\mathbb{R}_+)$ and (p_n), (q_n) be the sequences defined in Lemma 2.3. Then, for all $n \in \mathbb{N}$, there exists an absolute positive $C_1 = 4C$ such that*

$$|\mathfrak{G}_n^{p_n,q_n}(f;x) - f(x)| \le C_1\omega_2\left(f;\sqrt{\left(A_{p_n,q_n}(x)\right)^2 + B_{p_n,q_n}(x)}\right) + \omega\left(f;|A_{p_n,q_n}(x)|\right),$$

where $A_{p_n,q_n}(x) = \mathfrak{G}_n^{p_n,q_n}(t-x;x)$ and $B_{p_n,q_n}(x) = \mathfrak{G}_n^{p_n,q_n}\left((t-x)^2;x\right)$.

Proof. For any $x \in \mathbb{R}_+$, we define the following new positive operators

$$\mathfrak{H}_n^{p_n,q_n}(f;x) = \mathfrak{G}_n^{p_n,q_n}(f;x) - f\left(A_{p_n,q_n}(x) + x\right) + f(x).$$

It is observed that $\mathfrak{H}_n^{p_n,q_n}(f;x)$ preserve linear functions. Let $x,t \in \mathbb{R}_+$ and $g \in W^2$. Using the Taylor's expansion formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u,$$

we can obtain

$$\begin{aligned}
&|\mathfrak{H}_n^{p_n,q_n}(g;x) - g(x)| \\
&\le \mathfrak{H}_n^{p_n,q_n}\left(\left|\int_x^t g''(u)(t-u)\mathrm{d}u\right|;x\right) + \left|\int_x^{x+A_{p_n,q_n}(x)} g''(u)\left(A_{p_n,q_n}(x) + x - u\right)\mathrm{d}u\right| \\
&\le \mathfrak{H}_n^{p_n,q_n}\left(\int_x^t |g''(u)|(t-u)\mathrm{d}u;x\right) + \int_x^{x+A_{p_n,q_n}(x)} |g''(u)|\left(A_{p_n,q_n}(x) + x - u\right)\mathrm{d}u \\
&\le \|g''\|\mathfrak{H}_n^{p_n,q_n}\left((t-x)^2;x\right) + \|g''\|\int_x^{x+A_{p_n,q_n}(x)} \left(A_{p_n,q_n}(x) + x - u\right)\mathrm{d}u \\
&\le \left(B_{p_n,q_n}(x) + (A_{p_n,q_n}(x))^2\right)\|g''\|.
\end{aligned}$$

We also have

$$|\mathfrak{H}_n^{p_n,q_n}(f;x)| \le |\mathfrak{G}_n^{p_n,q_n}(f;x)| + 2\|f\| \le 3\|f\|.$$

Therefore

$$\begin{aligned}
&|\mathfrak{H}_n^{p_n,q_n}(f;x) - f(x)| \\
&= \left|\mathfrak{H}_n^{p_n,q_n}(f;x) + f(A_{p_n,q_n}(x) + x) - 2f(x)\right| \\
&\le |\mathfrak{H}_n^{p_n,q_n}(f-g;x) - (f-g)(x)| + |\mathfrak{H}_n^{p_n,q_n}(g;x) - g(x)| + \left|f(A_{p_n,q_n}(x) + x) - f(x)\right| \\
&\le 4\|f-g\| + \left(B_{p_n,q_n}(x) + (A_{p_n,q_n}(x))^2\right)\|g''\| + \omega\left(f;\left|A_{p_n,q_n}(x)\right|\right)
\end{aligned}$$

Taking infimum on the right hand side over all $g \in W^2$ and using (3.1), we get

$$|\mathfrak{G}_n^{p_n,q_n}(f;x) - f(x)| \le C_1\omega_2\left(f;\sqrt{(A_{p_n,q_n}(x))^2 + B_{p_n,q_n}(x)}\right) + \omega\left(f;\left|A_{p_n,q_n}(x)\right|\right).$$

This complete the proof of this Theorem. ∎

Corollary 3.2. *Let $f \in C_B(\mathbb{R}_+)$ and (p_n), (q_n) be the sequences defined in Lemma 2.3. Then, for any finite interval $I \subset \mathbb{R}_+$, the sequence $\{\mathfrak{G}_n^{p_n,q_n}(f;x)\}$ converges to f on I uniformly.*

3.2. **Rate of convergence.** Let $B_w(\mathbb{R}_+)$ be the set of all functions f defined on $\mathbb{R}_+$ satisfying the condition $|f(x)| \leq C_f w(x)$, where $C_f > 0$ is a constant depending on f only and $w(x)$ is a weighted function. By $C_w(\mathbb{R}_+)$, we denote the subspace of all continuous functions belonging to $B_w(\mathbb{R}_+)$. Also, Let $C_w^0(\mathbb{R}_+)$ be the subspace of all functions $f \in C_w(\mathbb{R}_+)$, for which $\lim\limits_{x\to\infty} \frac{f(x)}{w(x)}$ is finite. The norm on $C_w^0(\mathbb{R}_+)$ is

$$\|f\|_w = \sup_{x\in\mathbb{R}_+} \frac{|f(x)|}{w(x)}.$$

We consider $w(x) = 1 + x^2$ in this section, define the weighted modulus of continuity by

$$\Omega(f;\delta) = \sup_{0<t\leq\delta, x\in\mathbb{R}_+} \frac{|f(x+t) - f(x)|}{w(x)w(t)}$$

and have the inequality $\Omega(f;\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(f;\delta)$ holds, $\lambda \in \mathbb{R}_+$ (see [29]). On the other hand, we denote the modulus of the continuity of f on the interval $(0,\mathrm{a}]$, $\mathrm{a} > 0$, by

$$\omega_{\mathrm{a}}(f;\delta) = \sup_{x,t\in(0,\mathrm{a}], |x-t|\leq\delta} |f(t) - f(x)|,\ \delta > 0.$$

Now, we establish the following theorem on the rate of convergence for the operators $\mathfrak{G}_n^{p,q}(f;x)$.

Theorem 3.3. *Let* $f \in C_w(\mathbb{R}_+)$, $0 < q < p \leq 1$ *and* $\mathrm{a} > 0$. *Then*

$$|\mathfrak{G}_n^{p,q}(f;x) - f(x)| \leq C_f(3 + 2\mathrm{a}^2)B_{p,q}(x) + 2\omega_{\mathrm{a}+1}(f;\sqrt{B_{p,q}(\mathrm{x})}).$$

Proof. For any $x \in (0,\mathrm{a}]$ and $t > \mathrm{a}+1$, we have

$$1 \leq (t-\mathrm{a})^2 \leq (t-x)^2.$$

Thus

$$\begin{aligned} |f(t) - f(x)| &\leq |f(t)| + |f(x)| \leq C_f(2 + t^2 + x^2) \\ &= C_f\left(2 + x^2 + (t - x + x)^2\right) \leq C_f\left(2 + 2x^2 + (t-x)^2\right) \\ &\leq C_f(3 + 2x^2)(t-x)^2 \leq C_f(3 + 2\mathrm{a}^2)(t-x)^2. \end{aligned} \tag{3.2}$$

For any $x \in (0,\mathrm{a}]$, $t \in (0,\mathrm{a}+1]$ and $\delta > 0$, we have

$$|f(t) - f(x)| \leq \omega_{\mathrm{a}+1}(|t-x|;x) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{\mathrm{a}+1}(f;\delta). \tag{3.3}$$

For (3.2) and (3.3), we get

$$|f(t) - f(x)| \leq C_f(3 + 2\mathrm{a}^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{\mathrm{a}+1}(f;\delta).$$

By Schwarz's inequality, for any $x \in (0, \mathrm{a}]$, we have

$$\begin{aligned}
&|\mathfrak{G}_n^{p,q}(f;x) - f(x)| \\
&\leq \mathfrak{G}_n^{p,q}(|f(t) - f(x)|;x) \\
&\leq C_f(3+2\mathrm{a}^2)\mathfrak{G}_n^{p,q}((t-x)^2;x) + \mathfrak{G}_n^{p,q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{\mathrm{a}+1}(f;\delta) \\
&\leq C_f(3+2\mathrm{a}^2)\mathfrak{G}_n^{p,q}((t-x)^2;x) + \omega_{\mathrm{a}+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\mathfrak{G}_n^{p,q}((t-x)^2;x)}\right) \\
&\leq C_f(3+2\mathrm{a}^2)B_{p,q}(x) + \omega_{\mathrm{a}+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{B_{p,q}(x)}\right).
\end{aligned}$$

By taking $\delta = \sqrt{A_{p,q}(\mathrm{x})}$, we end the proof of this Theorem immediately. □

3.3. **Weighted Approximation.** In this subsection, we discuss the following three theorems about weighted approximation for the operators $\mathfrak{G}_n^{p_n,q_n}(f;x)$.

Theorem 3.4. *Let $f \in C_w^0(\mathbb{R}_+)$ and the sequences (p_n), (q_n) satisfy $0 < q_n < p_n \leq 1$ with $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$, $q_n^n \to 1$, and $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$. Then there exists $N \in \mathbb{N}_+$ such that, for all $n > N$ and $\rho > 0$,*

$$\sup_{x\in\mathbb{R}_+} \frac{|\mathfrak{G}_n^{p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}+\rho}} \leq 64\sqrt{2}\Omega\left(f;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right)$$

holds.

Proof. Using the definition and property of weighted modulus of smoothness $\Omega(f;\delta)$, we can write

$$\begin{aligned}
|f(t) - f(x)| &\leq \left(1+(t-x)^2\right)(1+x^2)\Omega(f;|t-x|) \\
&\leq 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f;\delta)\left(1+(t-x)^2\right)(1+x^2) \\
&\leq \begin{cases} 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta), & |t-x| \leq \delta, \\ 4(1+\delta^2)(1+x^2)\Omega(f;\delta)\dfrac{|t-x|+|t-x|^3}{\delta}, & |t-x| > \delta. \end{cases}
\end{aligned} \tag{3.4}$$

Setting $\delta \in (0,1)$, for all $x, t \in \mathbb{R}_+$, we have that (3.4) can be rewritten

$$|f(t) - f(x)| \leq 16(1+x^2)\Omega(f;\delta)\left(1+\frac{|t-x|+|t-x|^3}{\delta}\right). \tag{3.5}$$

Using (2.2) and (2.3), we have that there exists $N \in \mathbb{N}_+$ such that, for any $n > N$,

$$\mathfrak{G}_n^{p_n,q_n}((t-x)^2;x) \leq \frac{2}{[n]_{p_n,q_n}}(1+x^2), \quad \mathfrak{G}_n^{p_n,q_n}((t-x)^4;x) \leq (1+x^2)^2.$$

By Schwarz's inequality, we obtain

$$\begin{aligned}
\mathfrak{G}_n^{p_n,q_n}(|t-x|;x) &\leq \sqrt{\mathfrak{G}_n^{p_n,q_n}((t-x)^2;x)} \\
&\leq \sqrt{2}\frac{\sqrt{1+x^2}}{\sqrt{[n]_{p_n,q_n}}}
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mathfrak{G}_n^{p_n,q_n}(|t-x|^3;x) &\leq \sqrt{\mathfrak{G}_n^{p_n,q_n}((t-x)^2;x)}\sqrt{\mathfrak{G}_n^{p_n,q_n}((t-x)^4;x)}\\ &\leq \sqrt{2}\frac{\sqrt{(1+x^2)^3}}{\sqrt{[n]_{p_n,q_n}}}.\end{aligned} \tag{3.7}$$

Since $\mathfrak{G}_n^{p_n,q_n}$ is linear and positive, using (3.5-3.7), we obtain

$$\begin{aligned}|\mathfrak{G}_n^{p_n,q_n}(f;x)-f(x)| &\leq 16(1+x^2)\Omega(f;\delta)\left(1+\frac{\mathfrak{G}_n^{p_n,q_n}\left(|t-x|+|t-x|^3;x\right)}{\delta}\right)\\ &\leq 16(1+x^2)\left(1+\frac{2\sqrt{2}\sqrt{(1+x^2)^3}}{\delta\sqrt{[n]_{p_n,q_n}}}\right)\Omega(f;\delta).\end{aligned}$$

Choosing $\delta=\frac{1}{\sqrt{[n]_{p_n,q_n}}}$, we obtain the conclusion immediately. □

Theorem 3.5. *Let (p_n), (q_n) be the sequences defined in Theorem 3.4. Then, for any $f\in C_w^0(\mathbb{R}_+)$,*

$$\lim_{n\to\infty}\|\mathfrak{G}_n^{p_n,q_n}(f;x)-f\|_w=0.$$

Proof. From weighted Korovkin theorem [27], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|\mathfrak{G}_n^{p_n,q_n}(t^k;x)-x^k\right\|_w=0,\quad k=0,1,2. \tag{3.8}$$

Since $\mathfrak{G}_n^{p_n,q_n}(1;x)=1$, we have that (3.8) is true for $k=0$. By using Lemma 2.3, we obtain

$$\begin{aligned}&\|\mathfrak{G}_n^{p_n,q_n}(t;x)-x\|_w\\ &=\sup_{x\in\mathbb{R}_+}\frac{\left|\mathfrak{G}_n^{p_n,q_n}(t;x)-x\right|}{1+x^2}\\ &\leq\left|\frac{\sqrt{p_nq_n}[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)[n+2]_{p_n,q_n}}-1\right|\sup_{x\in\mathbb{R}_+}\frac{x}{1+x^2}+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\sup_{x\in\mathbb{R}_+}\frac{1}{1+x^2}\\ &=\left|\frac{\sqrt{p_nq_n}[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)[n+2]_{p_n,q_n}}-1\right|+\frac{\alpha}{[n]_{p_n,q_n}+\beta}\\ &\to 0,\ n\to\infty,\end{aligned}$$

and

$$\begin{aligned}
&\left\|\mathfrak{G}_n^{p_n,q_n}(t^2;x)-x^2\right\|_w\\
&=\sup_{x\in\mathbb{R}_+}\frac{\left|\mathfrak{G}_n^{p_n,q_n}(t^2;x)-x^2\right|}{1+x^2}\\
&\leq \sup_{x\in\mathbb{R}_+}\frac{x^2}{1+x^2}\left|\frac{[n]_{p_n,q_n}^2}{([n]_{p_n,q_n}+\beta)^2}-1\right|+\sup_{x\in\mathbb{R}_+}\frac{x}{1+x^2}\frac{2\alpha\sqrt{p_nq_n}[n]_{p_n,q_n}[n+1]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2[n+2]_{p_n,q_n}}\\
&+\sup_{x\in\mathbb{R}_+}\frac{1}{1+x^2}\frac{\alpha^2}{([n]_{p_n,q_n}+\beta)^2}\\
&=\left|\frac{[n]_{p_n,q_n}^2}{([n]_{p_n,q_n}+\beta)^2}-1\right|+\frac{\alpha\sqrt{p_nq_n}[n]_{p,q}[n+1]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2[n+2]_{p_n,q_n}}+\frac{\alpha^2}{([n]_{p_n,q_n}+\beta)^2}\\
&\to 0,\ n\to\infty.
\end{aligned}$$

Thus, the proof of Theorem 3.5 is completed. □

Theorem 3.6. *Let (p_n) and (q_n) be the sequences defined in Theorem 3.4. Then, for any $f\in C_w^0(\mathbb{R}_+)$ and $\lambda>0$,*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}_+}\frac{\left|\mathfrak{G}_n^{p_n,q_n}(f;x)-f(x)\right|}{(1+x^2)^{1+\lambda}}=0.$$

Proof. Fixing $x_0\in\mathbb{R}_+$, we have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}_+}\frac{\left|\mathfrak{G}_n^{p_n,q_n}(f;x)-f(x)\right|}{(1+x^2)^{1+\lambda}}\\
&\leq \sup_{x\in(0,x_0]}\frac{\left|\mathfrak{G}_n^{p_n,q_n}(f;x)-f(x)\right|}{(1+x^2)^{1+\lambda}}+\sup_{x\in(x_0,\infty)}\frac{\left|\mathfrak{G}_n^{p_n,q_n}(f;x)-f(x)\right|}{(1+x^2)^{1+\lambda}}\\
&\leq \|\mathfrak{G}_n^{p_n,q_n}(f;x)-f\|_{C(0,x_0]}+\|f\|_w\sup_{x\in(x_0,\infty)}\frac{\left|\mathfrak{G}_n^{p_n,q_n}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\lambda}}+\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\lambda}}.
\end{aligned}$$

Since $|f(x)|\leq C_f(1+x^2)$, we have

$$\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\lambda}}\leq\frac{C_f\|f\|_w}{(1+x_0^2)^{\lambda}}.$$

For any $\varepsilon>0$, we can choose x_0 such that

$$\frac{C_f\|f\|_w}{(1+x_0^2)^{\lambda}}<\frac{\varepsilon}{3}. \tag{3.9}$$

From $x\in(x_0,\infty)$, we obtain

$$\|f\|_w\lim_{x\to\infty}\lim_{n\to\infty}\frac{\left|\mathfrak{G}_n^{p_n,q_n}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\lambda}}=\|f\|_w\lim_{x\to\infty}\frac{1}{(1+x^2)^{\lambda}}=0.$$

Hence we can choose N and x_0 such that, for any $n > N$

$$\sup_{x\in[x_0,\infty)} \|f\|_w \frac{\left|\mathfrak{G}_n^{p_n,q_n}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\lambda}} < \frac{\varepsilon}{3}. \tag{3.10}$$

Also, the first term of the above inequality tends to zero by Theorem 3.3, that is,

$$\|\mathfrak{G}_n^{p_n,q_n}(f;x) - f\|_{C(0,x_0]} < \frac{\varepsilon}{3}. \tag{3.11}$$

Thus, combining (3.9)-(3.11), we obtain the desired result easily. □

3.4. **Pointwise estimates.** In this subsection, we establish two pointwise estimates of operators (1.2). First, we give the relation between the local smoothness of f and local approximation. Note that $f \in C(\mathbb{R}_+)$ is in $\mathrm{Lip}_M(\gamma,E)$, $\gamma \in (0,1]$, $E \subset \mathbb{R}_+$ if it satisfies the condition

$$|f(t)| - f(x) \le M|t-x|^\gamma,\ t\in\mathbb{R}_+ \ \text{ and } \ x\in E,$$

where M is a constant depending on γ and f only.

Theorem 3.7. *Let $0<q<p\le 1$ and E be any bounded subset on $\mathbb{R}_+$. If $f \in C_B(\mathbb{R}_+)\bigcap \mathrm{Lip}_M(\gamma,E)$, then, for all $x\in\mathbb{R}_+$,*

$$|\mathfrak{G}_n^{p,q}(f;x) - f(x)| \le M\left((B_{p,q}(x))^{\frac{\gamma}{2}} + 2\mathrm{d}^\gamma(x;E)\right)$$

where $\mathrm{d}(x;E)$ denotes the distance between x and E defined by

$$\mathrm{d}(x;E) = \inf\{|t-x| : t\in E\}.$$

Proof. Let $\overline{E}$ be the closure of E. Using the properties of infimum, there is at least a point $t_0 \in \overline{E}$ such that $\mathrm{d}(x;E) = |x-t_0|$. By the triangle inequality

$$|f(t)-f(x)| \le |f(t)-f(t_0)| + |f(x)-f(t_0)|,$$

we can obtain

$$\begin{aligned}
|\mathfrak{G}_n^{p,q}(f;x) - f(x)| &\le \mathfrak{G}_n^{p,q}(|f(t)-f(t_0)|;x) + \mathfrak{G}_n^{p,q}(|f(x)-f(t_0)|;x)\\
&\le M\{\mathfrak{G}_n^{p,q}(|t-t_0|^\gamma;x) + |x-t_0|^\gamma\}\\
&\le M\{\mathfrak{G}_n^{p,q}(|t-x|^\gamma + |x-t_0|^\gamma;x) + |x-t_0|^\gamma\}\\
&= M\{\mathfrak{G}_n^{p,q}(|t-x|^\gamma;x) + 2|x-t_0|^\gamma\}.
\end{aligned}$$

Choosing $p_1 = \frac{2}{\gamma}$ and $p_2 = \frac{2}{2-\gamma}$ and using the well-known Hölder inequality, we have

$$\begin{aligned}
|\mathfrak{G}_n^{p,q}(f;x) - f(x)| &\le M\{(\mathfrak{G}_n^{p,q}(|t-x|^{p_1\gamma};x))^{\frac{1}{p_1}}(\mathfrak{G}_n^{p,q}(1^{p_2};x))^{\frac{1}{p_2}} + 2\mathrm{d}^\gamma(x;E)\}\\
&\le M\{\left(\mathfrak{G}_n^{p,q}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}} + 2\mathrm{d}^\gamma(x;E)\}\\
&\le M\left((B_{p,q}(x))^{\frac{\gamma}{2}} + 2\mathrm{d}^\gamma(x;E)\right).
\end{aligned}$$

□

Next, we obtain the local direct estimate of operators $\mathfrak{G}_n^{p,q}$ by using the Lipcshitz type maximal function of the order γ introduced by Lenze [28]

$$\tilde{\omega}_\gamma(f;x) = \sup_{x,t\in\mathbb{R}_+, x\neq t} \frac{|f(t)-f(x)|}{|t-x|^\gamma},\quad \gamma\in(0,1]. \tag{3.12}$$

Theorem 3.8. *Let $\zeta \in C_B(\mathbb{R}_+)$ and $\gamma \in (0,1]$. Then, for all $x \in \mathbb{R}_+$,*

$$\left|\mathfrak{G}_n^{p,q}(f;x) - f(x)\right| \leq \tilde{\omega}_\gamma(f;x)\left(B_{p,q}(x)\right)^{\frac{\gamma}{2}}.$$

Proof. From the equation (3.12), we have

$$\left|\mathfrak{G}_n^{p,q}(f;x) - f(x)\right| \leq \tilde{\omega}_\gamma(f;x)\mathfrak{G}_n^{p,q}(|t-x|^\gamma;x).$$

From the well-known Hölder inequality, we get

$$\begin{aligned}\left|\mathfrak{G}_n^{p,q}(f;x) - f(x)\right| &\leq \tilde{\omega}_\gamma(f;x)\left(\mathfrak{G}_n^{p,q}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}} \\ &\leq \tilde{\omega}_\gamma(f;x)\left(B_{p,q}(x)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

□

3.5. **Voronovskaja type theorem.** In this subsection, we give a Voronovskaja type asymptotic formula for operators (1.2) via the second and fourth central moments.

Theorem 3.9. *Let (p_n) and (q_n) be the sequences defined in Lemma 2.3 and $f \in C_B(\mathbb{R}_+)$. Suppose that $f''(x)$ exists at $x \in \mathbb{R}_+$. Then*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(\mathfrak{G}_n^{p_n,q_n}(f;x) - f(x)\right) = (\alpha - \beta x)f'(x) + \frac{a+b}{2}\left(x^2 f''(x) - x f'(x)\right).$$

Proof. Using Taylor's expansion formula, we obtain

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + R(t,x)(t-x)^2, \tag{3.13}$$

where $R(t,x)$ is the Peano form of the remainder and $\lim_{t\to x} R(t,x) = 0$. Using $\mathfrak{G}_n^{p_n,q_n}$ to the both sides of (3.13), we have

$$\begin{aligned}&[n]_{p_n,q_n}\left(\mathfrak{G}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right) \\ &= [n]_{p_n,q_n} f'(x)\left(\mathfrak{G}_n^{p_n,q_n}(t-x;x)\right. \\ &\quad + [n]_{p_n,q_n}\frac{f''(x)}{2}\mathfrak{G}_n^{p_n,q_n}\left((t-x)^2;x\right) + [n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}\left(R(t,x)(t-x)^2;x\right)\end{aligned}$$

From the Schwarz inequality, we have

$$\mathfrak{G}_n^{p_n,q_n}\left(R(t,x)(t-x)^2;x\right) \leq \sqrt{\mathfrak{G}_n^{p_n,q_n}\left(R^2(t,x);x\right)}\sqrt{\mathfrak{G}_n^{p_n,q_n}\left((t-x)^4;x\right)}. \tag{3.14}$$

We observe that $R^2(x,x) = 0$ and $R^2(\cdot,x) \in C_B[0,\infty)$. Then, it follows from Corollary 3.2 that

$$\lim_{n\to\infty}\mathfrak{G}_n^{p_n,q_n}\left(R^2(t,x);x\right) = R^2(x,x) = 0. \tag{3.15}$$

Hence, from (2.3), (3.14), (3.15), we can obtain

$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathfrak{G}_n^{p_n,q_n}\left(R(t,x)(t-x)^2;x\right) = 0. \tag{3.16}$$

Combining (2.1), (2.2), (3.16), we obtain the required result immediately. □

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References

[1] D.D. Stancu, Approximation of functions by a new class of linear polynomials operators, Rev. Roumaine Math. Pures Appl. 13 (1968), 1173-1194.
[2] A. Aral, V. Gupta, R.P. Agarwal, Application of q-calculus in operator theory, Springer, Berlin, 2013.
[3] N. Çetin, V.A. Radu, Approximation by generalized Bernstein-Stancu operators, Turkey J. Math. 43 (2019), 2032-2048.
[4] V. Gupta, R.P. Agarwal, Convergence Estimates in Approximation Theory, Springer, New York, 2013.
[5] H. Karsli, Rate of convergence of a new Gamma type operators for functions with derivatives of bounded variation, Math. Comput. Model. 45 (2007), 617-624.
[6] Q.B. Cai, X.M. Zeng, On the convergence of a kind of q-gamma operators, J. Inequal. Appl. 2013 (2013), Article ID 105.
[7] C. Zhao, W.T. Cheng, X.M. Zeng, Some approximation properties of a kind of q-Gamma-Stancu operators, J. Inequal. Appl. 2014 (2014), Article ID 94.
[8] M. Mursaleen, F. Khan, A. Khan, On (p,q)-analogue of Bernstein operators, Appl. Math. Comput. 266 (2015), 874-882.
[9] M. Mursaleen, F. Khan, A. Khan, Erratum to "On (p,q)-analogue of Bernstein Operators" [Appl. Math. Comput. 266 (2015) 874–882], Appl. Math. Comput. 278 (2015), 70-71.
[10] M. Mursaleen, F. Khan, A. Khan, Some approximation results by (p,q)-analogue of Bernstein-Stancu operators, Appl. Math. Comput. 264 (2015), 392-402.
[11] M. Mursaleen, F. Khan, A. Khan, Corrigendum to: "Some approximation results by (p, q)-analogue of Bernstein–Stancu operators" [Appl. Math. Comput. 264 (2015) 392–402], Appl. Math. Comput. 269 (2016), 744-746.
[12] T. Acar, (p,q)-generalization of Szász-Mirakjan operators, Math. Methods Appl. Sci. 39 (2016), 2685-2695.
[13] T. Acar, A. Aral, S.A. Mohiuddine, On Kantorovich modification of (p,q)-Baskakov operators, J. Inequal. Appl. 2016 (2016), 98.
[14] T. Acar, A. Aral, S.A. Mohiuddine, On Kantorovich modification of (p,q)-Bernstein operators, Iran J. Sci. Technol. Trans. Sci. 42 (2018), 1459-1464.
[15] T. Acar, P.N. Agrawal, S. Kumar, On a modification of (p,q)-Szász-Mirakyan operators, Complex Anal. Operator Theory 12 (2018), 155-167.
[16] M. Mursaleen, M. Nasiruzzaman, A. Khan, K.J. Ansari, Some approximation results on Bleimann-Butzer-Hahn operators defined by (p,q)-integers, Filomat 30 (2016), 639-648.
[17] E.Y Özkan, N. İspir, Approximation by (p,q)-Analogue of Balázs-Szabados operators. Filomat 32 (2018), 2257-2271.
[18] W.T. Cheng, W.H. Zhang, Q.B. Cai, (p,q)-gamma operators which preserve x^2, J. Inequal. Appl. 2019 (2019), Article ID 108.
[19] T. Acar, S.A. Mohiuddine, M. Mursaleen, Approximation by (p,q)-Baskakov-Durrmeyer-Stancu operators, Complex Anal. Operator Theory 12 (2018), 1453-1468.
[20] İ. Büyükyazıcı, Ç. Atakut, On Stancu type generalization of q-Baskakov operators, Math. Comput. Model. 52 (2010), 752-759.

[21] D. Bărbosu, A.M. Acu, C.V. Muraru, On certain GBS-Durrmeyer operators based on q-integers, Turkey J. Math. 41 (2017), 368-380.
[22] Q. Liu, R.S. Qi, Stancu type generalization of the q-Phillips operators, Numer. Algo. 72 (2016), 191-193.
[23] V. Gupta, T.M. Rassias, P.N. Agarwal, A.M. Acu, Recent Advances in Constructive Approximation Theory, Springer, New York, 2018.
[24] P.N. Sadjang, On two (p,q)-analogues of the Laplace transform, J. Diff. Equ. Appl. 23 (2017), 1562-1583.
[25] A. Aral, V. Gupta, (p,q)-type beta function of second kind, Adv. Operator Theorry 1 (2016), 134-146.
[26] R.A. Devore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.
[27] A.D. Gadzhiev, Theorems of the type of P.P. Korovkin theorems, Math. Zametki 20 (1976), 781-786.
[28] B. Lenze, On Lipschitz type maximal functions and their smoothness spaces, Nederl. Akad. Indag. Math 50 (1988), 53-63.
[29] N. Ispir, On modified Baskakov operators, Turkey J. Math. 25 (2001), 355-365.