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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS OF A THIRD-ORDER PERIODIC BOUNDARY VALUE PROBLEM WITH ONE-PARAMETER

DAN LI[1,2], MINGHE PEI[2,*], ZAN YANG[1]

[1]Faculty of Science, Tongji Zhejiang College, Jiaxing 314051, China

[2]School of Mathematics and Statistics, Beihua University, Jilin 132013, China

Abstract. In this paper, we study the solvability of a third-order periodic boundary value problem with one-parameter of the form

$$\begin{cases} x'''(t)+\rho^3 x(t)=\lambda a(t) f(t,x(t)), \quad 0\le t\le 2\pi, \\ x^{(i)}(0)=x^{(i)}(2\pi), \quad i=0,1,2, \end{cases}$$

where $\rho\in(0,1/\sqrt{3})$ is a constant, and $\lambda>0$ is a parameter. By applying the fixed point theorem of cone compression and expansion of norm type, we establish a series of criteria for the above one-parameter problems to have one, two, an arbitrary number, and even an infinite number of positive solutions. Criteria for the nonexistence of positive solutions are also derived.

Keywords. Positive solution; Existence; Multiplicity; Third-order periodic boundary value problem.

1. INTRODUCTION

In this paper, we consider the existence and multiplicity of positive solutions for a nonlinear third-order periodic boundary value problem with one-parameter of the form

$$\begin{cases} x'''(t)+\rho^3 x(t)=\lambda a(t) f(t,x(t)), \quad 0\le t\le 2\pi, \\ x^{(i)}(0)=x^{(i)}(2\pi), \quad i=0,1,2, \end{cases} \tag{1.1}$$

where $\lambda>0$ is a parameter, $\rho\in(0,1/\sqrt{3})$, $f:[0,2\pi]\times[0,+\infty)\to\mathbb{R}$ and $a:[0,2\pi]\to\mathbb{R}$ are continuous.

The third-order differential equations arise in many areas of applied mathematics and physics, such as the deflection of a curved beam with a constant or a varying cross-section, three layer beam, electromagnetic waves or gravity-driven flows [10]. The third-order periodic boundary

*Corresponding author.

E-mail address: peiminghe@163.com (M. Pei).

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value problems have been extensively studied by many authors, see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and reference therein. To the best of our knowledge, there is no work on the third-order periodic boundary value problems with one-parameter.

Inspired by [13] and the above literature, in this paper, we establish the existence and multiplicity of positive solutions to one-parameter third-order periodic boundary value problem (1.1). Our method is mainly based on the fixed point theorem of cone compression and expansion of norm type.

The outline of the paper as follows. In Section 2, some lemmas are given for our main results. In Section 3, by means of the fixed point theorem of cone expansion and compression of norm type, we establish the existence and multiplicity of positive solutions to one-parameter problem (1.1). As the applications of our main results, some examples are given in Section 4.

2. Preliminaries

In this section, we give some lemmas, which are needed for our main results. Throughout this paper, we always assume that $f\in C([0,2\pi]\times[0,+\infty),[0,+\infty))$ and $a\in C([0,2\pi],[0,+\infty))$ with $\int_0^{2\pi}a(s)\mathrm{d}s>0$.

First, we transform problem (1.1) into a second-order periodic boundary value problem (for short, PBVP). To do this, we define the following operator

$$(Ju)(t):=\int_0^{2\pi}g(t,s)u(s)\mathrm{d}s,\quad \forall u\in C[0,2\pi],$$

where

$$g(t,s):=\begin{cases}\frac{e^{\rho(2\pi+s-t)}}{e^{2\rho\pi}-1}, & 0\le s\le t\le 2\pi,\\ \frac{e^{\rho(s-t)}}{e^{2\rho\pi}-1}, & 0\le t\le s\le 2\pi.\end{cases}$$

Then, problem (1.1) is equivalent to the following second-order PBVP of the form

$$\begin{cases}u''-\rho u'+\rho^2u=\lambda a(t)f(t,Ju),\\ u^{(i)}(0)=u^{(i)}(2\pi),\quad i=0,1.\end{cases}\tag{2.1}$$

It is easy to verify that if u is a positive solution of second-order PBVP (2.1), then $x(t):=(Ju)(t)$ is a positive solution of third-order PBVP (1.1). Therefore, we only need to consider second-order PBVP (2.1).

Lemma 2.1. [12] *Second-order PBVP* (2.1) *is equivalent to integral equation*

$$u(t)=\lambda\int_0^{2\pi}G(t,s)a(s)f(s,(Ju)(s))\mathrm{d}s,$$

where

$$G(t,s)=\begin{cases}\frac{2e^{\frac{\rho}{2}(t-s)}[\sin\frac{\sqrt{3}}{2}\rho(2\pi+t-s)+e^{-\rho\pi}\sin\frac{\sqrt{3}}{2}\rho(t-s)]}{\sqrt{3}\rho(e^{\rho\pi}+e^{-\rho\pi}-2\cos\sqrt{3}\rho\pi)}, & s\le t,\\ \frac{2e^{\frac{\rho}{2}(2\pi+t-s)}[\sin\frac{\sqrt{3}}{2}\rho(t-s)+e^{-\rho\pi}\sin\frac{\sqrt{3}}{2}\rho(2\pi+t-s)]}{\sqrt{3}\rho(e^{\rho\pi}+e^{-\rho\pi}-2\cos\sqrt{3}\rho\pi)}, & s\ge t.\end{cases}\tag{2.2}$$

Remark 2.2. It is easy to show that $\int_0^{2\pi}g(t,s)\mathrm{d}s=1/\rho$, and $\int_0^{2\pi}G(t,s)\mathrm{d}s=1/\rho^2$.

Lemma 2.3. [12] *For the function $G(t,s)$ defined in (2.2), we have the following estimate:*

$$m := \frac{2\sin(\sqrt{3}\rho\pi)}{\sqrt{3}\rho(e^{\rho\pi}+1)^2} \le G(t,s) \le \frac{2}{\sqrt{3}\rho\sin(\sqrt{3}\rho\pi)} =: M, \quad \forall t,s \in [0,2\pi]. \tag{2.3}$$

Let

$$\alpha = \frac{m}{M}, \quad \beta = \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s)a(s)\mathrm{d}s. \tag{2.4}$$

Remark 2.4. From (2.3), we have $0 < \alpha < 1$. If $a(t) \equiv 1$, then $\beta = 1/\rho^2$ from Remark 2.2.

Let α be defined by (2.4). A cone $K \subset E = C[0,2\pi]$ is defined by

$$K = \left\{ u \in C[0,2\pi] : \min_{t\in[0,2\pi]} u(t) \ge \alpha\|u\| \right\}.$$

An operator $T : C[0,2\pi] \to C[0,2\pi]$ is defined as follows: for each $u \in C[0,2\pi]$,

$$(Tu)(t) = \int_0^{2\pi} G(t,s)a(s)f(s,(Ju)(s))\mathrm{d}s, \quad t \in [0,2\pi].$$

For $r > 0$, let

$$\Omega_r = \{u \in C[0,2\pi] : \|u\| < r\}.$$

It is easy to show the following lemma, which is directly followed from Lemma 2.3 and Arzela-Ascoli theorem.

Lemma 2.5. *$T(K) \subset K$, and T is completely continuous.*

Lemma 2.6. [11] *Let E be a Banach space, and K be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(a) $\|Tu\| \le \|u\|, \forall u \in K \cap \partial\Omega_1, \|Tu\| \ge \|u\|, \forall u \in K \cap \partial\Omega_2$; *or*
(b) $\|Tu\| \ge \|u\|, \forall u \in K \cap \partial\Omega_1, \|Tu\| \le \|u\|, \forall u \in K \cap \partial\Omega_2$.

Then T has a fixed point on $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

In this section, we first establish the existence results of positive solutions for the problem (1.1) with $\lambda = 1$ by using the fixed point theorem of cone expansion and compression of norm type.

Theorem 3.1. *Assume that there exist positive numbers r_* and r^* with $r_* < \alpha r^*$ such that*

$$f(t,x) \le \rho(\alpha\beta)^{-1}r_*, \quad \forall (t,x) \in [0,2\pi] \times [r_*, \alpha^{-1}r_*], \tag{3.1}$$

and

$$f(t,x) \ge \rho\beta^{-1}r^*, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r^*, r^*]. \tag{3.2}$$

Then problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $r_* \le \|x\| \le r^*$.

Proof. It is sufficient to show that problem (2.1) with $\lambda = 1$ has at least one positive solution $u = u(t)$ satisfying

$$\rho\alpha^{-1}r_* \leq \|u\| \leq \rho r^*.$$

To do this, we let

$$r_1 = \rho\alpha^{-1}r_*, \quad r_2 = \rho r^*.$$

Then, for every $u \in K \cap \partial\Omega_{r_1}$,

$$\|u\| = r_1, \quad \alpha r_1 \leq u(t) \leq r_1, \quad \forall t \in [0, 2\pi].$$

From Remark 2.2, it follows that, for all $u \in K \cap \partial\Omega_{r_1}$,

$$r_* \leq (Ju)(t) \leq \alpha^{-1}r_*, \quad \forall t \in [0, 2\pi].$$

Hence, from (3.1), we obtain that, for every $u \in K \cap \partial\Omega_{r_1}$,

$$\begin{aligned}(Tu)(t) &= \int_0^{2\pi} G(t,s)a(s)f(s,(Ju)(s))\mathrm{d}s \\ &\leq \rho(\alpha\beta)^{-1}r_* \int_0^{2\pi} G(t,s)a(s)\mathrm{d}s \\ &\leq \rho(\alpha\beta)^{-1}r_*\beta = \|u\|, \quad \forall t \in [0, 2\pi].\end{aligned}$$

Therefore,

$$\|Tu\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_{r_1}.$$

On the other hand, for all $u \in K \cap \partial\Omega_{r_2}$,

$$\|u\| = r_2, \quad \alpha r_2 \leq u(t) \leq r_2, \quad \forall t \in [0, 2\pi],$$

we have

$$\alpha r^* \leq (Ju)(t) \leq r^*, \quad \forall t \in [0, 2\pi].$$

Let $t_1 \in [0, 2\pi]$ be such that $\beta = \int_0^{2\pi} G(t_1, s)a(s)\mathrm{d}s$. It follows from (3.2) that

$$\begin{aligned}(Tu)(t_1) &= \int_0^{2\pi} G(t_1,s)a(s)f(s,(Ju)(s))\mathrm{d}s \\ &\geq \rho\beta^{-1}r^* \int_0^{2\pi} G(t_1,s)a(s)\mathrm{d}s \\ &= \rho\beta^{-1}r^*\beta = \|u\|, \quad \forall u \in K \cap \partial\Omega_{r_2}.\end{aligned}$$

Hence,

$$\|Tu\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_{r_2}.$$

It follows from Lemma 2.6 that there exists $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $Tu = u$. Obviously, u is a positive solution of problem (2.1) with $\lambda = 1$ satisfying $\rho\alpha^{-1}r_* \leq \|u\| \leq \rho r^*$. □

Theorem 3.2. *Assume that there exist positive numbers r_* and r^* with $r^* < \alpha r_*$ such that*

$$f(t,x) \geq \rho(\alpha\beta)^{-1} r^*, \quad \forall (t,x) \in [0,2\pi] \times [r^*, \alpha^{-1} r^*], \tag{3.3}$$

and

$$f(t,x) \leq \rho\beta^{-1} r_*, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r_*, r_*]. \tag{3.4}$$

Then problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $r^* \leq \|x\| \leq r_*$.

Proof. The proof is similar to the proof of Theorem 3.1, and hence is omitted. □

For the sake of convenience, the following notations are introduced:

$$f_0 = \liminf_{x \to 0^+} \min_{t \in [0,2\pi]} f(t,x)/x, \quad f_\infty = \liminf_{x \to +\infty} \min_{t \in [0,2\pi]} f(t,x)/x,$$

$$f^0 = \limsup_{x \to 0^+} \max_{t \in [0,2\pi]} f(t,x)/x, \quad f^\infty = \limsup_{x \to +\infty} \max_{t \in [0,2\pi]} f(t,x)/x.$$

Corollary 3.3. *If $f^0 < \rho\beta^{-1}$ and $f_\infty > \rho(\alpha\beta)^{-1}$ hold, then problem (1.1) with $\lambda = 1$ has at least one positive solution.*

Proof. Since $f^0 < \rho\beta^{-1}$, there exists $r_* > 0$ such that

$$f(t,x) < \rho\beta^{-1} x, \quad \forall (t,x) \in [0,2\pi] \times (0, \alpha^{-1} r_*].$$

Thus

$$f(t,x) < \rho(\alpha\beta)^{-1} r_*, \quad \forall (t,x) \in [0,2\pi] \times [r_*, \alpha^{-1} r_*].$$

From $f_\infty > \rho(\alpha\beta)^{-1}$, we have that there exists $r^* > \alpha^{-1} r_*$ such that

$$f(t,x) > \rho(\alpha\beta)^{-1} x, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r^*, +\infty).$$

So,

$$f(t,x) > \rho\beta^{-1} r^*, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r^*, r^*].$$

Therefore, by Theorem 3.1, problem (1.1) with $\lambda = 1$ has at least one positive solution. □

Corollary 3.4. *Assume that there exists $r_* > 0$ such that (3.1) holds and $f_\infty > \rho(\alpha\beta)^{-1}$. Then the problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $\|x\| \geq r_*$.*

Proof. From $f_\infty > \rho(\alpha\beta)^{-1}$, we have that there exists $r^* > \alpha^{-1} r_*$ such that

$$f(t,x) > \rho(\alpha\beta)^{-1} x, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r^*, +\infty),$$

and hence

$$f(t,x) > \rho\beta^{-1} r^*, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r^*, r^*].$$

From Theorem 3.1, problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $\|x\| \geq r_*$. □

Corollary 3.5. *Assume that there exists $r^* > 0$ such that (3.2) holds and $f^0 < \rho\beta^{-1}$. Then problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $\|x\| \leq r^*$.*

Proof. Since $f^0 < \rho\beta^{-1}$, there exists $r_* < \alpha r^*$ such that

$$f(t,x) < \rho\beta^{-1}x, \quad \forall (t,x) \in [0,2\pi] \times (0, \alpha^{-1}r_*],$$

and thus

$$f(t,x) < \rho(\alpha\beta)^{-1}r_*, \quad \forall (t,x) \in [0,2\pi] \times [r_*, \alpha^{-1}r_*].$$

From Theorem 3.1, problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $\|x\| \le r^*$. □

Corollary 3.6. *If $f_0 > \rho(\alpha\beta)^{-1}, f^\infty < \rho\beta^{-1}$, then problem (1.1) with $\lambda = 1$ has at least one positive solution.*

Corollary 3.7. *Assume that there exists $r^* > 0$ such that (3.3) holds and $f^\infty < \rho\beta^{-1}$. Then problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $\|x\| \ge r^*$.*

Corollary 3.8. *Assume that there exists $r_* > 0$ such that (3.4) holds and $f_0 > \rho(\alpha\beta)^{-1}$. Then problem (1.1) with $\lambda = 1$ has at least one positive solution $x = x(t)$ satisfying $\|x\| \le r_*$.*

The proofs of Corollary 3.6-3.8 are similar to the proofs of Corollary 3.3-3.5, respectively, and hence are omitted.

Theorem 3.9. *Assume that $f_0 > \rho(\alpha\beta)^{-1}$ and $f_\infty > \rho(\alpha\beta)^{-1}$ hold. Suppose also that there exists $r > 0$ such that*

$$f(t,x) < \rho\beta^{-1}r, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r, \alpha^{-1}r]. \tag{3.5}$$

Then problem (1.1) with $\lambda = 1$ has at least two positive solutions $x_1(t)$ and $x_2(t)$ satisfying $\|x_1\| < r < \|x_2\|$.

Proof. From the continuity of $f(t,x)$ on $[0,2\pi] \times [\alpha r, r]$, there exists $r_0 < r$ such that

$$f(t,x) < \rho\beta^{-1}r_0, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r, r].$$

Thus, from the finite covering theorem, there exists $\delta_1 > 0$ such that, for every $r_1 \in (r - \delta_1, r) \subset (r_0, r)$,

$$f(t,x) < \rho\beta^{-1}r_0, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r_1, r].$$

Hence, for every $r_1 \in (r - \delta_1, r) \subset (r_0, r)$,

$$f(t,x) < \rho\beta^{-1}r_1, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r_1, r_1].$$

Similarly, there exists $\delta_2 > 0$ such that, for all $r_2 \in (r, r + \delta_2) \subset (r, \alpha^{-1}r)$,

$$f(t,x) < \rho\beta^{-1}r_2 < \rho(\alpha\beta)^{-1}r_2, \quad \forall (t,x) \in [0,2\pi] \times [r_2, \alpha^{-1}r_2].$$

Therefore, from Corollary 3.8 and 3.4, the problem (1.1) with $\lambda = 1$ has two positive solutions $x_1 = x_1(t), x_2 = x_2(t)$ satisfying

$$\|x_1\| \le r_1, \quad \|x_2\| \ge r_2.$$

In view of $r_1 < r < r_2$, we have $\|x_1\| < r < \|x_2\|$. This completes the proof. □

Theorem 3.10. *Assume that* $f^0 < \rho\beta^{-1}$ *and* $f^\infty < \rho\beta^{-1}$ *hold. Suppose also that there exists* $r > 0$ *such that*

$$f(t,x) > \rho(\alpha\beta)^{-1}r, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r, \alpha^{-1}r]. \tag{3.6}$$

Then problem (1.1) with $\lambda = 1$ *has at least two positive solutions* $x_1(t)$ *and* $x_2(t)$ *satisfying* $\|x_1\| < r < \|x_2\|$.

Proof. The proof is similar to proof of the Theorem 3.9, and hence is omitted. □

Theorem 3.11. *Let* $\{r_i\}_{i=1}^{n+1} \subset (0,+\infty)$ *be such that*

$$\alpha^{-1}r_i < \alpha r_{i+1}, \quad i = 1,2,\ldots,n.$$

Assume that one of the following conditions holds:

(a) f *satisfies* (3.5) *with* $r = r_i$ *when* i *is an odd number, and* f *satisfies* (3.6) *with* $r = r_i$ *when* i *is an even number;*

(b) f *satisfies* (3.5) *with* $r = r_i$ *when* i *is an even number, and* f *satisfies* (3.6) *with* $r = r_i$ *when* i *is an odd number.*

Then problem (1.1) with $\lambda = 1$ *has at least* n *positive solutions* $x_1, x_2, \ldots, x_n$ *satisfying*

$$r_i < \|x_i\| < r_{i+1}, \quad i = 1,2,\ldots,n. \tag{3.7}$$

Proof. (a) From the proof of Theorem 3.9, we see that, for each $i = 1,2,\ldots,n$, there exist r_i', r_i'' with $r_i' < r_i < r_i''$ and $\alpha^{-1}r_i'' < \alpha r_{i+1}'$ such that if i is an odd number, then f satisfies

$$f(t,x) < \rho\beta^{-1}r_i, \quad \forall (t,x) \in ([0,2\pi] \times [\alpha r_i', r_i']) \cup ([0,2\pi] \times [r_i'', \alpha^{-1}r_i'']);$$

if i is an even number, then f satisfies

$$f(t,x) > \rho(\alpha\beta)^{-1}r_i, \quad \forall (t,x) \in ([0,2\pi] \times [\alpha r_i', r_i']) \cup ([0,2\pi] \times [r_i'', \alpha^{-1}r_i'']).$$

Therefore from Theorem 3.1 and 3.2, problem (1.1) with $\lambda = 1$ has at least n positive solutions $x_1, x_2, \ldots, x_n$ satisfying

$$r_i < r_i'' < \|x_i\| < r_{i+1}' < r_{i+1}, \quad i = 1,2,\ldots,n.$$

(b) The proof is similar to Part (a), and hence is omitted. □

Theorem 3.12. *Let* $\{r_i\}_{i=1}^{\infty} \subset (0,+\infty)$ *be such that*

$$r_i < r_{i+1}, \quad \alpha^{\frac{(-1)^i-1}{2}} r_i < \alpha^{\frac{(-1)^{i+1}-1}{2}} r_{i+1}, \quad i = 1,2,\ldots.$$

Assume that one of the following conditions holds:

(a) f *satisfies* (3.1) *with* $r_* = r_i$ *when* i *is an odd number, and* f *satisfies* (3.2) *with* $r^* = r_i$ *when* i *is an even number;*

(b) f *satisfies* (3.3) *with* $r^* = r_i$ *when* i *is an odd number, and* f *satisfies* (3.4) *with* $r_* = r_i$ *when* i *is an even number.*

Then problem (1.1) *with* $\lambda = 1$ *has an infinite number of positive solutions.*

Proof. (a) For each odd number $i = 1,3,\ldots,2k+1,\ldots,$ we let $r_* = r_i, r^* = r_{i+1}$. Then, from Theorem 3.1, the problem (1.1) with $\lambda = 1$ has solutions $x_i = x_i(t)$ satisfying

$$r_i = r_* \le \|x_i\| \le r^* = r_{i+1}, \quad i = 1,3,\ldots,2k+1,\ldots.$$

Obviously, x_i are different from each other. Hence, the problem (1.1) with $\lambda = 1$ has an infinite number of positive solutions.

Similarly, case (b) follows from Theorem 3.2. □

Corollary 3.13. *Let $\{r_i\}_{i=1}^{\infty} \subset (0,+\infty)$ be such that*

$$r_i < r_{i+1}, \quad \alpha^{\frac{(-1)^i-1}{2}} r_i < \alpha^{\frac{(-1)^{i+1}-1}{2}} r_{i+1}, \quad i = 1,2,\ldots.$$

Assume that one of the following conditions holds:

(a) $\limsup\limits_{E_1 \ni x \to \infty} \max\limits_{t\in[0,2\pi]} f(t,x)/x < \rho\beta^{-1}$ *and* $\liminf\limits_{E_2 \ni x \to \infty} \min\limits_{t\in[0,2\pi]} f(t,x)/x > \rho(\alpha\beta)^{-1}$;

(b) $\liminf\limits_{E_1 \ni x \to \infty} \min\limits_{t\in[0,2\pi]} f(t,x)/x > \rho(\alpha\beta)^{-1}$ *and* $\limsup\limits_{E_2 \ni x \to \infty} \max\limits_{t\in[0,2\pi]} f(t,x)/x < \rho\beta^{-1}$,

where

$$E_1 = \bigcup_{i=1}^{\infty} [r_{2i-1}, \alpha^{-1} r_{2i-1}], \quad E_2 = \bigcup_{i=1}^{\infty} [\alpha r_{2i}, r_{2i}].$$

Then problem (1.1) with $\lambda = 1$ has an infinite number of positive solutions.

Proof. (a) From the assumptions, we see that for sufficiently large i,

$$\frac{f(t,x)}{x} < \rho\beta^{-1}, \quad \forall (t,x) \in [0,2\pi] \times [r_{2i-1}, \alpha^{-1} r_{2i-1}]$$

and

$$\frac{f(t,x)}{x} > \rho(\alpha\beta)^{-1}, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r_{2i}, r_{2i}].$$

It follow that, for sufficiently large i,

$$f(t,x) < \rho\beta^{-1} x \le \rho(\alpha\beta)^{-1} r_{2i-1}, \quad \forall (t,x) \in [0,2\pi] \times [r_{2i-1}, \alpha^{-1} r_{2i-1}]$$

and

$$f(t,x) > \rho(\alpha\beta)^{-1} x \ge \rho\beta^{-1} r_{2i}, \quad \forall (t,x) \in [0,2\pi] \times [\alpha r_{2i}, r_{2i}].$$

Therefore, by Theorem 3.12-(a), the problem (1.1) with $\lambda = 1$ has an infinite number of positive solutions.

Similarly, case (b) follows from Theorem 3.12-(b). □

Theorem 3.14. *Assume that one of the following conditions hold*:

(a) $f(t,x)/x < \rho\beta^{-1}, \forall (t,x) \in [0,2\pi] \times (0,+\infty)$;

(b) $f(t,x)/x > \rho(\alpha\beta)^{-1}, \forall (t,x) \in [0,2\pi] \times (0,+\infty)$.

Then problem (1.1) with $\lambda = 1$ has no positive solutions.

Proof. (a) Suppose on the contrary that the problem (1.1) with $\lambda = 1$ has a positive solution $x(t)$. Set $u(t) = x'(t) + \rho y(t)$. Then u is a positive solution of the problem (2.1) with $\lambda = 1$. Thus, $u = Tu \in K$. It follows that

$$\alpha \|u\| \rho^{-1} \le x(t) = (Ju)(t) = \int_0^{2\pi} g(t,s) u(s) \mathrm{d}s \le \|u\| \rho^{-1}, \quad \forall t \in [0,2\pi].$$

Consequently,

$$\begin{aligned} u(t) &= \int_0^{2\pi} G(t,s)a(s)f(s,(Ju)(s))\mathrm{d}s \\ &< \rho\beta^{-1}\int_0^{2\pi} G(t,s)a(s)(Ju)(s)\mathrm{d}s \\ &\leq \rho\beta^{-1}\|u\|\rho^{-1}\beta = \|u\|, \quad \forall t\in[0,2\pi], \end{aligned}$$

which is a contradiction. Therefore, problem (1.1) with $\lambda = 1$ has no positive solutions.

(b) The proof is similar to Part (a), and hence is omitted. □

Next, we use the above results on the solvability of the problem (1.1) with $\lambda = 1$ to establish the existence and multiplicity of positive solutions for one-parameter problem (1.1).

Theorem 3.15. *Assume that $\lambda > 0$ satisfies one of the following conditions:*

(a) $\lambda f^0 < \rho\beta^{-1}$ *and* $\lambda f_\infty > \rho(\alpha\beta)^{-1}$;

(b) $\lambda f_\infty > \rho(\alpha\beta)^{-1}$ *and there exists* $r > 0$ *such that*

$$\lambda \leq \lambda^* := \frac{\rho(\alpha\beta)^{-1}r}{\max_{(t,x)\in[0,2\pi]\times[r,\alpha^{-1}r]} f(t,x)};$$

(c) $\lambda f^0 < \rho\beta^{-1}$ *and there exists* $r > 0$ *such that*

$$\lambda \geq \lambda_* := \frac{\rho\beta^{-1}r}{\min_{(t,x)\in[0,2\pi]\times[\alpha r,r]} f(t,x)};$$

(d) $\lambda f_0 > \rho(\alpha\beta)^{-1}$ *and* $\lambda f^\infty < \rho\beta^{-1}$;

(e) $\lambda f^\infty < \rho\beta^{-1}$ *and there exists* $r > 0$ *such that*

$$\lambda \geq \lambda_* := \frac{\rho(\alpha\beta)^{-1}r}{\min_{(t,x)\in[0,2\pi]\times[r,\alpha^{-1}r]} f(t,x)};$$

(f) $\lambda f_0 > \rho(\alpha\beta)^{-1}$ *and there exists* $r > 0$ *such that*

$$\lambda \leq \lambda^* := \frac{\rho\beta^{-1}r}{\max_{(t,x)\in[0,2\pi]\times[\alpha r,r]} f(t,x)}.$$

Then problem (1.1) has at least one positive solution.

Proof. (a) It follows from the assumptions that

$$(\lambda f)^0 = \lambda f^0 < \rho\beta^{-1}, \quad (\lambda f)_\infty = \lambda f_\infty > \rho(\alpha\beta)^{-1}.$$

Hence, by Corollary 3.3, one-parameter problem (1.1) has at least one positive solution.

The other Parts of the theorem can be proved similarly. □

Theorem 3.16. *Assume that $\lambda > 0$ satisfies one of the following conditions:*

(a) $\lambda f_0 > \rho(\alpha\beta)^{-1}$, $\lambda f_\infty > \rho(\alpha\beta)^{-1}$ *and there exists* $r > 0$ *such that*

$$\lambda < \lambda^* := \frac{\rho\beta^{-1}r}{\max\limits_{(t,x)\in[0,2\pi]\times[\alpha r,\alpha^{-1}r]} f(t,x)};$$

(b) $\lambda f^0 < \rho\beta^{-1}$, $\lambda f^\infty < \rho\beta^{-1}$ *and there exists* $r > 0$ *such that*

$$\lambda > \lambda_* := \frac{\rho(\alpha\beta)^{-1} r}{\min\limits_{(t,x)\in[0,2\pi]\times[\alpha r,\alpha^{-1}r]} f(t,x)}.$$

Then problem (1.1) *has at least two positive solutions.*

Proof. (a) It follows from the assumptions that

$$(\lambda f)_0 = \lambda f_0 > \rho(\alpha\beta)^{-1}, \quad (\lambda f)_\infty = \lambda f_\infty > \rho(\alpha\beta)^{-1},$$

and, for all $(t,x) \in [0,2\pi]\times[\alpha r, \alpha^{-1} r]$,

$$\lambda f(t,x) < \lambda^* \max_{(t,x)\in[0,2\pi]\times[\alpha r,\alpha^{-1}r]} f(t,x) = \rho\beta^{-1} r.$$

Hence, by Theorem 3.9, one-parameter problem (1.1) has at least two positive solutions.

(b) The proof is similar to Part (a), and hence is omitted. □

Theorem 3.17. *Let* $\{r_i\}_{i=1}^{n+1} \subset (0,+\infty)$ *be such that*

$$\alpha^{-1} r_i < \alpha r_{i+1}, \quad i = 1,2,\ldots,n.$$

Assume that one of the following conditions holds:

(a) $\lambda_* := \dfrac{\rho(\alpha\beta)^{-1} r_{n_2}}{\min_{(t,x)\in E_2} f(t,x)} < \dfrac{\rho\beta^{-1} r_1}{\max_{(t,x)\in E_1} f(t,x)} =: \lambda^*$;

(b) $\lambda_* := \dfrac{\rho(\alpha\beta)^{-1} r_{n_1}}{\min_{(t,x)\in E_1} f(t,x)} < \dfrac{\rho\beta^{-1} r_2}{\max_{(t,x)\in E_2} f(t,x)} =: \lambda^*$,

where $n_1 = n + (1+(-1)^n)/2$, $n_2 = n + (1-(-1)^n)/2$, *and*

$$E_1 = \bigcup_{i=1}^{n_1} [\alpha r_{2i-1}, \alpha^{-1} r_{2i-1}], \quad E_2 = \bigcup_{i=1}^{n_2} [\alpha r_{2i}, \alpha^{-1} r_{2i}].$$

Then, for any $\lambda \in (\lambda_*, \lambda^*)$, *problem* (1.1) *has at least n positive solutions.*

Proof. (a) Let $\lambda \in (\lambda_*, \lambda^*)$. For each fixed $i = 1,2,\ldots,n_1$, if $f(t,x) \equiv 0$ on $[0,2\pi]\times[\alpha r_{2i-1}, \alpha^{-1} r_{2i-1}]$, then

$$\lambda f(t,x) < \rho\beta^{-1} r_{2i-1}, \quad \forall (t,x) \in [0,2\pi]\times[\alpha r_{2i-1}, \alpha^{-1} r_{2i-1}];$$

if $f(t,x) \not\equiv 0$ on $[0,2\pi]\times[\alpha r_{2i-1}, \alpha^{-1} r_{2i-1}]$, then $\forall (t,x) \in [0,2\pi]\times[\alpha r_{2i-1}, \alpha^{-1} r_{2i-1}]$,

$$\lambda f(t,x) < \lambda^* \max_{(t,x)\in[0,2\pi]\times[\alpha r_{2i-1},\alpha^{-1}r_{2i-1}]} f(t,x) \le \rho\beta^{-1} r_1 \le \rho\beta^{-1} r_{2i-1}.$$

Similarly, we can show that, for each fixed $i = 1,2,\ldots,n_2$,

$$\lambda f(t,x) > \rho(\alpha\beta)^{-1} r_{2i}, \quad \forall (t,x) \in [0,2\pi]\times[\alpha r_{2i}, \alpha^{-1} r_{2i}].$$

Hence, by Theorem 3.11, one-parameter problem (1.1) has at least n positive solutions.

(b) The proof is similar to Part (a), and hence is omitted. □

Theorem 3.18. *Let* $\{r_i\}_{i=1}^{\infty} \subset (0,+\infty)$ *be such that*

$$r_i < r_{i+1}, \quad \alpha^{\frac{(-1)^i-1}{2}} r_i < \alpha^{\frac{(-1)^{i+1}-1}{2}} r_{i+1}, \quad i = 1,2,\ldots.$$

Assume that one of the following conditions holds:

(a) $\limsup_{E_1 \ni x \to \infty} \max_{t \in [0,2\pi]} f(t,x)/x = 0$ *and* $\liminf_{E_2 \ni x \to \infty} \min_{t \in [0,2\pi]} f(t,x)/x = +\infty$;

(b) $\liminf_{E_1 \ni x \to \infty} \min_{t \in [0,2\pi]} f(t,x)/x = +\infty$ *and* $\limsup_{E_2 \ni x \to \infty} \max_{t \in [0,2\pi]} f(t,x)/x = 0$,

where

$$E_1 = \bigcup_{i=1}^{\infty} [r_{2i-1}, \alpha^{-1} r_{2i-1}], \quad E_2 = \bigcup_{i=1}^{\infty} [\alpha r_{2i}, r_{2i}].$$

Then, for any $\lambda > 0$, problem (1.1) has an infinite number of positive solutions.

Proof. (a) From the assumptions, we have, for any fixed $\lambda > 0$,

$$\limsup_{E_1 \ni x \to \infty} \max_{t \in [0,2\pi]} \frac{\lambda f(t,x)}{x} = \lambda \limsup_{E_1 \ni x \to \infty} \max_{t \in [0,2\pi]} \frac{f(t,x)}{x} < \rho \beta^{-1}$$

and

$$\liminf_{E_2 \ni x \to \infty} \min_{t \in [0,2\pi]} \frac{\lambda f(t,x)}{x} = \lambda \liminf_{E_2 \ni x \to \infty} \min_{t \in [0,2\pi]} \frac{f(t,x)}{x} > \rho (\alpha\beta)^{-1}.$$

Therefore, the conclusion of the theorem follows directly from Corollary 3.13-(a).

(b) Similar to the proof of Part (a), the conclusion of the theorem follows directly from Corollary 3.13-(b). □

Theorem 3.19. *Assume that $\lambda > 0$ satisfies one of the following conditions:*

(a) $\lambda < \lambda^* := \frac{\rho \beta^{-1}}{\sup_{(t,x) \in [0,2\pi] \times (0,+\infty)} f(t,x)/x}$;

(b) $\lambda > \lambda_* := \frac{\rho (\alpha\beta)^{-1}}{\inf_{(t,x) \in [0,2\pi] \times (0,+\infty)} f(t,x)/x}$.

Then problem (1.1) has no positive solutions.

Proof. Without loss of generality, we assume that (a) holds. Then, for all $\lambda \in (0, \lambda^*)$,

$$\frac{\lambda f(t,x)}{x} < \lambda^* \sup_{(t,x) \in [0,2\pi] \times (0,+\infty)} \frac{f(t,x)}{x} = \rho \beta^{-1}, \quad \forall (t,x) \in [0,2\pi] \times (0,+\infty).$$

Hence, from Theorem 3.14-(a), the one-parameter problem has no positive solutions. □

4. The examples

In this section, some examples will be given to illustrate the effectiveness of the results in Section 3.

Example 4.1. Consider third-order periodic boundary value problem

$$\begin{cases} x'''(t) + \rho^3 x(t) = x^k(t), & 0 \le t \le 2\pi, \\ x^{(i)}(0) = x^{(i)}(2\pi), & i = 0,1,2, \end{cases} \tag{4.1}$$

where $\rho \in (0, 1/\sqrt{3})$ and $k > 0$ are constants.

It is noted that this periodic boundary value problem corresponds to the case when $a(t) \equiv 1, f(t,x) = x^k$ in the problem (1.1) with $\lambda = 1$.

If $k > 1$, then

$$\lim_{x\to 0^+} \frac{f(t,x)}{x} = 0, \quad \lim_{x\to +\infty} \frac{f(t,x)}{x} = +\infty.$$

So, from Corollary 3.3, problem (4.1) has at least one positive solution.

If $0 < k < 1$, then

$$\lim_{x\to 0^+} \frac{f(t,x)}{x} = +\infty, \quad \lim_{x\to +\infty} \frac{f(t,x)}{x} = 0.$$

Hence, by Corollary 3.6, problem (4.1) has at least one positive solution.

Example 4.2. Consider third-order periodic boundary value problem

$$\begin{cases} x'''(t) + \rho^3 x(t) = f(t,x(t)), & 0 \le t \le 2\pi, \\ x^{(i)}(0) = x^{(i)}(2\pi), & i = 0,1,2, \end{cases} \tag{4.2}$$

where $\rho \in (0, 1/\sqrt{3})$, and

$$f(t,x) = \begin{cases} \frac{1}{2}\rho(\alpha^{-1}+1)\beta^{-1}x(\sin(b\ln x)+1), & x > 0, \\ 0, & x = 0, \end{cases}$$

with

$$0 < b < \frac{\pi - 2\sin^{-1}\delta}{\ln(\alpha^{-1})}, \quad \delta = \frac{\alpha^{-1}-1}{\alpha^{-1}+1}.$$

Let, for each $k \in \mathbb{N} = \{1,2,\dots\}$,

$$\xi_k = \exp(b^{-1}(\sin^{-1}\delta + (k-1)\pi)), \quad \eta_k = \exp(b^{-1}(k\pi - \sin^{-1}\delta)).$$

Then

$$\xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_k < \eta_k < \cdots.$$

In addition, for each $k \in \mathbb{N}$, we have

$$\frac{\eta_k}{\xi_k} = \exp(b^{-1}(\pi - 2\sin^{-1}\delta)) > \exp(\ln(\alpha^{-1})) = \alpha^{-1}.$$

So,

$$\alpha^{-1}\xi_k < \eta_k, \quad k \in \mathbb{N}.$$

Set $k = 2i+1$, $i \in \mathbb{N}$. For all $x \in [\xi_k, \alpha^{-1}\xi_k] \subset [\xi_k, \eta_k]$, we have

$$\theta_1 := (k-1)\pi + \sin^{-1}\delta \le b\ln x \le k\pi - \sin^{-1}\delta =: \theta_2.$$

Since θ_1 belongs to the first quadrant, $\theta_2 - \theta_1 \in [0, 2\pi]$ and $\sin\theta_1 = \sin\theta_2 = \delta$, we have $\sin(b\ln x) \ge \sin(\sin^{-1}\delta) = \delta$. It follows that

$$f(t,x) \ge \frac{1}{2}\rho(\alpha^{-1}+1)\beta^{-1}\xi_k(\delta+1) = \rho(\alpha\beta)^{-1}\xi_k, \quad \forall (t,x) \in [0,2\pi] \times [\xi_k, \alpha^{-1}\xi_k],$$

that is, (3.3) with $r^* = \xi_k$ holds.

Set $k = 2i$, $i \in \mathbb{N}$. In this case, by a similar argument, we can show that (3.4) with $r_* = \eta_k$ holds.

In summary, by Theorem 3.12-(b), problem (4.2) has an infinite number of positive solutions.

Example 4.3. Consider one-parameter third-order periodic boundary value problem

$$\begin{cases} x'''(t)+\rho^3 x(t)=\lambda(x^{k_1}(t)+x^{k_2}(t)), & 0\le t\le 2\pi, \\ x^{(i)}(0)=x^{(i)}(2\pi), & i=0,1,2, \end{cases} \tag{4.3}$$

where $\rho\in(0,1/\sqrt{3})$, $0<k_1<1<k_2<+\infty$, and $\lambda>0$ is a parameter.

Obviously, $f(t,x)=x^{k_1}+x^{k_2}$ is strictly increasing with respect to x on $[0,2\pi]\times(0,+\infty)$, and

$$\lim_{x\to 0^+}\frac{f(t,x)}{x}=\lim_{x\to+\infty}\frac{f(t,x)}{x}=+\infty.$$

Let

$$r=\left(\tfrac{1-k_1}{k_2-1}\right)^{\frac{1}{k_2-k_1}}, \qquad r_1=\rho\beta^{-1}r((\alpha^{-1}r)^{k_1}+(\alpha^{-1}r)^{k_2})^{-1},$$
$$r_2=\rho\beta^{-1}r(r^{k_1}+r^{k_2})^{-1}, \quad r_3=\rho(\alpha\beta)^{-1}r(r^{k_1}+r^{k_2})^{-1}.$$

Then r is the minimum point of $f(t,x)/x$ on $[0,2\pi]\times(0,+\infty)$, and $r_1<r_2<r_3$. There are three cases to consider.

Case (1) $0<\lambda\le r_2$. In this case, since $\lambda f_0>\rho(\alpha\beta)^{-1}$ and

$$0<\lambda\le\lambda^*=\frac{\rho\beta^{-1}r}{\max_{(t,x)\in[0,2\pi]\times[\alpha r,r]}f(t,x)}=\frac{\rho\beta^{-1}r}{r^{k_1}+r^{k_2}}=r_2,$$

we have from Theorem 3.15-(f) that one-parameter problem (4.3) has at least one positive solution.

Case (2) $0<\lambda<r_1$. In this case, notice that $\lambda f_0>\rho(\alpha\beta)^{-1}$, $\lambda f_\infty>\rho(\alpha\beta)^{-1}$ and

$$0<\lambda<\lambda^*=\frac{\rho\beta^{-1}r}{\max_{(t,x)\in[0,2\pi]\times[\alpha r,\alpha^{-1}r]}f(t,x)}=\frac{\rho\beta^{-1}r}{(\alpha^{-1}r)^{k_1}+(\alpha^{-1}r)^{k_2}}=r_1.$$

Then, from Theorem 3.16-(a), one-parameter problem (4.3) has at least two positive solutions.

Case (3) $\lambda>r_3$. In this case, because of

$$\lambda>\lambda_*=\frac{\rho(\alpha\beta)^{-1}}{\inf_{(t,x)\in[0,2\pi]\times(0,+\infty)}f(t,x)/x}=\frac{\rho(\alpha\beta)^{-1}}{(r^{k_1}+r^{k_2})/r}=r_3,$$

we have from Theorem 3.19-(b) that one-parameter problem (4.3) has no positive solutions.

Example 4.4. Consider one-parameter third-order periodic boundary value problem

$$\begin{cases} x'''(t)+\rho^3 x(t)=\lambda f(t,x(t)), & 0\le t\le 2\pi, \\ x^{(i)}(0)=x^{(i)}(2\pi), & i=0,1,2, \end{cases} \tag{4.4}$$

where $\rho\in(0,1/\sqrt{3})$, $\lambda>0$ is a parameter, and

$$f(t,x)=\begin{cases} \frac{1}{x^{-k}+1}, & x>0, \\ 0, & x=0, \end{cases}$$

with $k>1$.

Clearly, $f(t,x)$ is strictly increasing with respect to x on $[0,2\pi]\times(0,+\infty)$, and

$$\lim_{x\to 0^+}\frac{f(t,x)}{x}=\lim_{x\to+\infty}\frac{f(t,x)}{x}=0.$$

Let

$$r=(k-1)^{\frac{1}{k}}, \qquad r_1=\rho\beta^{-1}(r^{1-k}+r),$$
$$r_2=\rho(\alpha\beta)^{-1}(r^{1-k}+r), \quad r_3=\rho(\alpha^2\beta)^{-1}((\alpha r)^{1-k}+\alpha r).$$

Notice that r is the maximum point of $f(t,x)/x$ on $[0,2\pi]\times(0,+\infty)$, and $r_1<r_2<r_3$. Then by a similar argument of Example 4.3, we obtain following conclusion:

(1) If $\lambda\geq r_2$, then one-parameter problem (4.4) has at least one positive solution from Theorem 3.15-(e).

(2) If $\lambda>r_3$, then one-parameter problem (4.4) has at least two positive solutions from Theorem 3.16-(b).

(3) If $0<\lambda<r_1$, then one-parameter problem (4.4) has no positive solutions from Theorem 3.19-(a).

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