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EIGENVALUE PROBLEMS FOR A CLASS OF BI-NONLOCAL EQUATIONS IN ORLICZ-SOBOLEV SPACES

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Abstract. In this paper, we consider eigenvalue problems for a class of bi-nonlocal problems with weights in Orlicz-Sobolev spaces. Under some suitable conditions on the nonlinearities, we establish the existence of a continuous family of eigenvalues in a neighborhood of the origin by using the Ekeland variational principle and the mountain pass theorem in critical point theory.
Keywords. Bi-nonlocal problems; Orlicz-Sobolev spaces; Variational methods.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Assume that $a : (0,\infty) \to \mathbb{R}$ is a function such that the odd mapping $\phi : \mathbb{R} \to \mathbb{R}$, defined by

$$\phi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. For the function ϕ above, let us define

$$\Phi(t) = \int_0^t \phi(s)ds \quad \text{for all } t \in \mathbb{R},$$

on which some suitable conditions will be imposed later.

In this paper, we are interested in the existence of weak solutions for the following bi-nonlocal problem

$$\begin{cases} -M\left(\int_\Omega \Phi(|\nabla u|)\,dx\right) \operatorname{div}\left(a(|\nabla u|)\nabla u\right) = \lambda V(x)|u|^{q(x)-2}u\left[\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx\right]^{\kappa}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\lambda > 0$ and $\kappa \geq 0$ are two real parameters, $V : \Omega \to \mathbb{R}_0^+ := [0,+\infty)$ is a weighted function, $q : \overline{\Omega} \to (1,+\infty)$ is a continuous function, and $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an increasing and continuous function satisfying

(M_0) $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}, \quad \forall t \geq 0, \quad m_2 \geq m_1 > 0, \quad \alpha > 1.$

We notice that if $\kappa = 0$, then problem (1.1) is reduced to the Kirchhoff type problem in Orlicz-Sobolev spaces which was studied in [1, 2, 3, 4, 5]. In particular, if $\varphi(t) = p|t|^{p-2}t$, $p \in (1,\infty)$, then problem (1.1) becomes the well-known p-Kirchhoff type problem, which was proposed by Kirchhoff [6] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. In the case $M(t) \equiv 1$, problem (1.1) is a version of the non-local problem

$$\begin{cases} -\Delta u = \lambda f(x,u) \left[\int_\Omega F(x,u)\,dx\right]^r, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with $F(x,t) = \int_0^t f(x,s)\,ds$, where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a regular function. This problem arises in the analytical investigation of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates. It also arises in the investigation of a fully turbulent behaviour of a real flow and in the theory of gravitational equilibrium of polytropic stars, see [7, 8, 9]. Problem (1.2) is often called nonlocal problem because it contains the integral over Ω. This causes some mathematical difficulties, which make the study of such a problem particularly interesting. In our case, problem (1.1) is even "doubly nonlocal" in the sense that the nonlocality appears both in the differential operator and in the coefficient of the source term f, so it is usually called a bi-nonlocal problem. Clearly, the form of the problems studied in the present paper is more general. In (1.1), not only does the divergence term contain a nonlocal coefficient as usual but also the external force term contains a nonlocal coefficient. Such a form can better describe several physical and biological systems. From the physical point of view, in (1.1), $M\left(\int_\Omega \Phi(|\nabla u|)\,dx\right)$ is a function depending on the average of the kinetic energy and $\left[\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx\right]^\kappa$ is a function depending on the average of the potential energy.

Although Kirchhoff type problems have been intensively studied in recent years, it is worth pointing out that the literature related to bi-nonlocal problems like (1.1), even for the case of p-Laplace operator $\Delta_p(.)$, is limited. We refer here to some interesting papers [10, 11, 12, 13, 14, 15]. In [10, 11], the authors considered bi-nonlocal problems for the p-Laplace operator with special nonlinearities by using Morse theory and variational methods. In [12], Cammaroto and Vilasi obtained at least three weak solutions for a class bi-nonlocal problems with the nonaffine dependence on the parameters. Here, the main tool for achieving the result was an interesting variational principle due to Ricceri [16]. In [13, 14, 15], the authors studied the existence and multiplicity of solutions for bi-nonlocal problems involving the $p(x)$-Laplace operator $\Delta_{p(x)}(.)$, where $p(.)$ was a continuous function on $\overline{\Omega}$. The $p(x)$-Laplace operator is an extension of the p-Laplace operator $\Delta_p(.)$ and it is not homogeneous as in the constant case. Problems with variable exponents have interesting motivation from both physical and mathematical point of view. Indeed, it appears in the so-called model of motion of electrorheological fluids, characterized by their ability to change in a drastic way the mechanical properties when influenced by an exterior electromagnetic field, see [17].

Motivated by the contribution mentioned above and recent results in [13, 14, 18, 19, 20], in this paper, we are interested in the eigenvalue problem for bi-nonlocal equation (1.1) in the

Orlicz-Sobolev space. In [18, 19, 20], the authors considered problem (1.1) in the local case. That is, $M \equiv 1$ and $\kappa = 0$. It has the following form

$$\begin{cases} -\operatorname{div}\Big(a(|\nabla u|)\nabla u\Big) = \lambda V(x)|u|^{q(x)-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

More precisely, using variational methods, Ge [18] and Mihăilescu and Rădulescu [20] considered eigenvalue problem (1.3) in the sublinear case when $q^+ < \phi_0$ and Ge and Geng [19] considered the problem in the superlinear case when $\phi^0 < q^-$ (see the definitions of q^+, q^-, ϕ_0 and ϕ^0 in Section 2). The purpose of this work is to study bi-nonlocal problem (1.1) with the weighted function V and the general subcritical growth (see conditions (V_0) and (Q_1)-(Q_2)). For this reasons, the nonlinear term appeared in our problem covers the previous cases [18, 19, 20] in the sense that it may be sublinear or superlinear at infinity. We will establish the existence of a continuous family of eigenvalues for problem (1.1) in a neighborhood of the origin. Moreover, in the above papers, it was proved the existence of a non-trivial solution for each parameter $\lambda > 0$ while we obtained here at least two non-negative non-trivial solutions in the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ (see Theorem 3.2). To this purpose, our main tools are essentially based on the mountain pass theorem [21] and the Ekeland variational principle [22]. We emphasize that our situations in this paper are different from those presented by Corrêa et al. [13, 14], in which the authors considered bi-nonlocal problems in the Sobolev space with variable exponent $W_0^{1,p(x)}(\Omega)$ and $V \equiv 1$. Finally, for more details on the topic, we refer the interested readers to some our results on Kirchhoff type problems in Orlicz-Sobolev spaces [1, 2, 3].

The remainder of the paper is organized as follows. In Section 2, we recall the definitions and some properties of generalized Lebesgue spaces and Orlicz-Sobolev spaces. We refer to [23, 24, 25, 26] for details on this class of functional spaces. In Section 3, we state and prove the main results of the paper.

2. Preliminaries

In order to study problem (1.1), let us introduce the functional spaces. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz-Sobolev spaces, which are useful for what follow. For more details, we refer the readers to the books by Adams [27], Rao and Ren [26], the papers by Clément et al. [23], and Mihăilescu and Repovš [25].

For $\phi : \mathbb{R} \to \mathbb{R}$ and Φ introduced at the beginning of the paper, we can see that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$, and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$, the function Φ is then called an N-function. Let us define the function Φ^* by the formula

$$\Phi^*(t) = \int_0^t \phi^{-1}(s)ds \text{ for all } t \in \mathbb{R},$$

which is called the complementary function of Φ and it satisfies the condition

$$\Phi^*(t) = \sup\{st - \Phi(s) : \ s \ge 0\} \quad \text{for all } t \ge 0.$$

We observe that Φ^* is also an N-function in the sense above and the following Young inequality holds

$$st \le \Phi(s) + \Phi^*(t) \quad \text{for all } s,t \ge 0.$$

The Orlicz class defined by the N-function Φ is the set

$$K_\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is measurable } : \int_\Omega \Phi(|u(x)|)\,dx < \infty \right\}$$

and the Orlicz space $L_\Phi(\Omega)$ is then defined as the linear hull of the set $K_\Phi(\Omega)$. The space $L_\Phi(\Omega)$ is a Banach space under the following Luxemburg norm

$$\|u\|_\Phi := \inf\left\{ k > 0 : \int_\Omega \Phi\left(\frac{u(x)}{k}\right) dx \le 1 \right\}$$

or the equivalent Orlicz norm

$$\|u\|_{L_\Phi} := \sup\left\{ \left| \int_\Omega u(x)v(x)\,dx \right| : v \in K_{\Phi^*}(\Omega), \int_\Omega \Phi^*(|v(x)|)\,dx \le 1 \right\}.$$

For Orlicz spaces, the Hölder inequality reads as follows (see [26]):

$$\int_\Omega uv\,dx \le 2\|u\|_{L_\Phi(\Omega)}\|u\|_{L^*_\Phi(\Omega)} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^*}(\Omega).$$

The Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ building upon $L_\Phi(\Omega)$ is the space defined by

$$W^1L_\Phi(\Omega) := \left\{ u \in L_\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, 2, \ldots, N \right\}.$$

and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

Now, we introduce the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. It turns out that the space $W_0^1L_\Phi(\Omega)$ can be renormed by using as an equivalent norm

$$\|u\| := \||\nabla u|\|_\Phi.$$

Throughout this paper, we assume that Φ and Φ^* satisfy the Δ_2-condition at infinity, namely,

$$1 < \phi_0 := \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \le \frac{t\phi(t)}{\Phi(t)} \le \phi^0 := \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} < \infty, \quad t \ge 0. \tag{2.1}$$

Furthermore, we also need the following conditions

$$\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty) \tag{2.2}$$

and

$$\lim_{t\to 0} \int_t^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds < +\infty \text{ and } \lim_{t\to+\infty} \int_1^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds = +\infty, \tag{2.3}$$

which help us to define the Orlicz-Sobolev conjugate Φ_* of Φ as follows

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\,ds. \tag{2.4}$$

We notice that Orlicz-Sobolev spaces, which are unlike the Sobolev spaces they generalize, are in general neither separable nor reflexive. A key tool to guarantee these properties is represented by the Δ_2-condition (2.1). Actually, condition (2.1) assures that both $L_\Phi(\Omega)$ and $W_0^1L_\Phi(\Omega)$ are separable, see [27]. Conditions (2.1) and (2.2) assure that $L_\Phi(\Omega)$ is a uniformly convex space. Thus, a reflexive Banach space (see [25]). Consequently, the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ is also a reflexive Banach space.

Proposition 2.1 (see [23, 25])**.** *Let* $u \in W_0^1 L_\Phi(\Omega)$. *Then*

(*i*) $\|u\|^{\phi^0} \leq \int_\Omega \Phi(|\nabla u(x)|)\,dx \leq \|u\|^{\phi_0}$ *if* $\|u\| < 1$.

(*ii*) $\|u\|^{\phi_0} \leq \int_\Omega \Phi(|\nabla u(x)|)\,dx \leq \|u\|^{\phi^0}$ *if* $\|u\| > 1$.

Next, we recall some definitions and basic properties of the generalized Lebesgue space $L^{p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [24, 17], the paper of Kováčik and Rákosník [28]. Set

$$C_+(\overline{\Omega}) := \{h;\ \ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \text{a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}$$

with respect to the following so-called *Luxemburg norm* defined by the formula

$$|u|_{p(x)} = \inf\left\{ \mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Hölder inequality holds

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)}. \tag{2.5}$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold

$$|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \tag{2.6}$$

provided that $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \tag{2.7}$$

provided that $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.8}$$

Proposition 2.2. *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq +\infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ and $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$

and

$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is a constant, then $||u|^p|_{q(x)} = |u|^p_{pq(x)}$.

3. Main results

In this section, we will state and prove the main results of the paper. Let us denote by X the Orlicz-Sobolev space $W^1_0L_\Phi(\Omega)$. Now, we assume that the function $V : \Omega \to [0,+\infty)$ belongs to $L^\infty(\Omega)$ and the continuous function $q : \overline{\Omega} \to \mathbb{R}$ satisfy the following conditions:

(V_0) there exist an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $V(x) = 0$ for $x \in \overline{B_R(x_0)\backslash B_r(x_0)}$ while $V(x) > 0$ for $x \in \Omega \backslash \overline{B_R(x_0)\backslash B_r(x_0)}$;

(Q_1) $\lim_{t\to+\infty} \frac{|t|^{q^+}}{\Phi_*(kt)} = 0$ for all $k > 0$, where Φ_* stands for the Orlicz-Sobolev conjugate of Φ, that is,

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds;$$

(Q_2) Either

$$(\kappa+1) \max_{x\in \overline{B_r(x_0)}} q(x) < \alpha\phi_0 \leq \alpha\phi^0 < \frac{m_1(\kappa+1)}{m_2} \min_{x\in \overline{\Omega\backslash B_R(x_0)}} q(x)$$

or

$$(\kappa+1) \max_{x\in \overline{\Omega\backslash B_R(x_0)}} q(x) < \alpha\phi_0 \leq \alpha\phi^0 < \frac{m_1(\kappa+1)}{m_2} \min_{x\in \overline{B_r(x_0)}} q(x).$$

We can see that there are many functions satisfying conditions (V_0) and (Q_1)-(Q_2), for example, the functions V and q defined by the formulas

$$V(x) = \begin{cases} \frac{1}{r}(r - |x-x_0|), & \text{for } x \in B_r(x_0), \\ 0, & \text{for } x \in \overline{B_R(x_0)\backslash B_r(x_0)}, \\ \frac{1}{R}(|x-x_0| - R), & \text{for } x \in \Omega \backslash \overline{B_R(x_0)\backslash B_r(x_0)} \end{cases}$$

and

$$q(x) = \begin{cases} t_1, & \text{for } x \in B_r(x_0), \\ \frac{t_1(R-|x-x_0|)}{R-r} + \frac{t_2(|x-x_0|-R)}{R-r}, & \text{for } x \in \overline{B_R(x_0)\backslash B_r(x_0)}, \\ t_2, & \text{for } x \in \overline{\Omega\backslash B_R(x_0)}. \end{cases}$$

satisfy the above conditions, where the positive constants t_1, t_2 can be chosen in a suitable manner such as $(\kappa+1)t_1 < \alpha\phi_0 \leq \alpha\phi^0 < \frac{m_1(\kappa+1)}{m_2} t_2$ for the first case in (Q_2) and $(\kappa+1)t_2 < \alpha\phi_0 \leq \alpha\phi^0 < \frac{m_1(\kappa+1)}{m_2} t_1$ for the second one. From conditions (Q_1)-(Q_2), our main results in this paper complement and improve the previous ones [18, 19, 20], in which the authors considered the local problem with sublinear or superlinear terms.

Definition 3.1. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u \in X\backslash\{0\}$ such that

$$M\left(\int_\Omega \Phi(|\nabla u|)\,dx\right)\int_\Omega a(|\nabla u|)\nabla u\nabla v\,dx-\lambda\left(\int_\Omega V(x)|u|^{q(x)}\,dx\right)^{\kappa}\int_\Omega V(x)|u|^{q(x)-2}uv\,dx=0$$

for all $v \in X$. If λ is an eigenvalue of problem (1.1), then the corresponding eigenfunction $u \in X\backslash\{0\}$ is a weak solution of (1.1).

The main results of this paper are given by the following theorem.

Theorem 3.2. *Assume that conditions (M_0), (V_0) and (Q_1)-(Q_2) are satisfied. Then there exists $\lambda^* > 0$ such that any $\lambda \in (0,\lambda^*)$ is an eigenvalue of problem (1.1). Moreover, for any $\lambda \in (0,\lambda^*)$, problem (1.1) has at least two non-negative non-trivial weak solutions.*

For any $\lambda > 0$, let us define the functional $J_\lambda : X \to \mathbb{R}$ by

$$J_\lambda(u)=\widehat{M}\left(\int_\Omega \Phi(|\nabla u|)\,dx\right)-\frac{\lambda}{\kappa+1}\left(\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx\right)^{\kappa+1}, \quad u \in X.$$

Then, by hypotheses (Q_1)-(Q_2) and the continuous embeddings, we can show that J_λ is well-defined on X and $J_\lambda \in C^1(X,\mathbb{R})$ with the derivative given by

$$\begin{aligned}J_\lambda'(u)(v)=M&\left(\int_\Omega \Phi(|\nabla u|)\,dx\right)\int_\Omega a(|\nabla u|)\nabla u\nabla v\,dx\\ -\lambda&\left(\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx\right)^{\kappa}\int_\Omega V(x)|u|^{q(x)-2}uv\,dx\end{aligned}$$

for all $u,v \in X$.

Lemma 3.3. (i) *There exists $\lambda^* > 0$ such that, for any $\lambda \in (0,\lambda^*)$, there exist $\rho > 0$ and $\beta > 0$ for which $J_\lambda(u) \geq \beta$ for any $u \in X$ with $\|u\| = \rho$.*
(ii) *There exists $\psi \in X$, $\psi \neq 0$ such that $\lim_{t\to+\infty} J_\lambda(t\psi) = -\infty$.*
(iii) *There exists $\varphi \in X$, and $\varphi \neq 0$ such that $J_\lambda(t\varphi) < 0$ for any $t > 0$ small enough.*

Proof. We will prove this lemma in details for the case

$$(\kappa+1)\max_{x\in \overline{B_r(x_0)}} q(x) < \alpha\phi_0 \leq \alpha\phi^0 < \frac{m_1(\kappa+1)}{m_2}\min_{x\in \overline{\Omega\backslash B_R(x_0)}} q(x) \tag{3.1}$$

while the remaining one can be made by similarly arguments.

(i) Let us define the functions q_1 and q_2 as follows: $q_1 : \overline{B}_r(x_1) \to (1,+\infty)$, $q_1(x) = q(x)$ for any $x \in \overline{B}_r(x_0)$ and $q_2 : \overline{\Omega\backslash B_R(x_0)} \to (1,+\infty)$, $q_2(x) = q(x)$ for any $x \in \overline{\Omega\backslash B_R(x_0)}$. We also introduce here the notations

$$q_1^- = \min_{x\in \overline{B_r(x_0)}} q_1(x), \quad q_1^+ = \max_{x\in \overline{B_r(x_0)}} q_1(x),$$

and

$$q_2^- = \min_{x\in \overline{\Omega\backslash B_R(x_0)}} q_2(x), \quad q_1^+ = \max_{x\in \overline{\Omega\backslash B_R(x_0)}} q_2(x).$$

By the conditions (Q_1) and (Q_2), we have

$$1 < (\kappa+1)q_1^- \leq (\kappa+1)q_1^+ < \alpha\phi_0 \leq \alpha\phi^0 < \frac{m_1(\kappa+1)}{m_2}q_2^- \leq \frac{m_1(\kappa+1)}{m_2}q_2^+, \tag{3.2}$$

so that X is continuously embedded in $L^{q_i^{\pm}}(\Omega)$ for $i=1,2$. Then there exists a positive constant C_1 such that

$$\int_\Omega |u|^{q_i^{\pm}}\,dx \le C_1\|u\|^{q_i^{\pm}}, \quad \forall u\in X \text{ and } i=1,2. \tag{3.3}$$

From (3.3), there exists a positive constant C_2 such that

$$\begin{aligned}\int_{B_r(x_0)} |u|^{q_1(x)}\,dx &\le \int_{B_r(x_0)} |u|^{q_1^-}\,dx + \int_{B_r(x_0)} |u|^{q_1^+}\,dx\\ &\le \int_\Omega |u|^{q_1^-}\,dx + \int_\Omega |u|^{q_1^+}\,dx\\ &\le C_2\left(\|u\|^{q_1^-}+\|u\|^{q_1^+}\right), \quad \forall u\in X,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} |u|^{q_2(x)}dx &\le \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^-}dx + \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^+}|dx\\ &\le \int_\Omega |u|^{q_2^-}dx + \int_\Omega |u|^{q_2^+}dx\\ &\le C_2\left(\|u\|^{q_2^-}+\|u\|^{q_2^+}\right), \quad \forall u\in X.\end{aligned} \tag{3.5}$$

Using relations (3.4) and (3.5) give us

$$\begin{aligned}J_\lambda(u) &= \widehat{M}\left(\int_\Omega \Phi(|\nabla u|)\,dx\right) - \frac{\lambda}{\kappa+1}\left(\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx\right)^{\kappa+1}\\ &\ge \frac{m_1}{\alpha}\|u\|^{\alpha\phi^0} - \frac{\lambda}{\kappa+1}\left(\int_{B_r(x_0)} \frac{V(x)}{q(x)}|u|^{q(x)}\,dx + \int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|u|^{q(x)}\,dx\right)^{\kappa+1}\\ &\ge \frac{m_1}{\alpha}\|u\|^{\alpha\phi^0} - \frac{\lambda 2^\kappa}{\kappa+1}\left(\frac{C_2|V|_\infty}{q^-}\right)^{\kappa+1}\left[\left(\|u\|^{q_1^-}+\|u\|^{q_1^+}\right)^{\kappa+1} + \left(\|u\|^{q_2^-}+\|u\|^{q_2^+}\right)^{\kappa+1}\right]\\ &\ge \left[\frac{m_1}{2\alpha} - \frac{\lambda 4^\kappa}{\kappa+1}\left(\frac{C_2}{q^-}|V|_\infty\right)^{\kappa+1}\left(\|u\|^{q_1^-(\kappa+1)-\alpha\phi^0} + \|u\|^{q_1^+(\kappa+1)-\alpha\phi^0}\right)\right]\|u\|^{\alpha\phi^0}\\ &\quad + \left[\frac{m_1}{2\alpha} - \frac{\lambda 4^\kappa}{\kappa+1}\left(\frac{C_2}{q^-}|V|_\infty\right)^{\kappa+1}\left(\|u\|^{q_2^-(\kappa+1)-\alpha\phi^0} + \|u\|^{q_2^+(\kappa+1)-\alpha\phi^0}\right)\right]\|u\|^{\alpha\phi^0}\end{aligned}$$

for all $u\in X$ with $\|u\|<1$. Let us define $h:[0,1]\to\mathbb{R}$ by

$$h(t) = \frac{m_1}{2\alpha} - \frac{4^\kappa}{\kappa+1}\left(\frac{C_2}{q^-}|V|_\infty\right)^{\kappa+1} t^{q_2^+(\kappa+1)-\alpha\phi^0} - \frac{4^\kappa}{\kappa+1}\left(\frac{C_2}{q^-}|V|_\infty\right)^{\kappa+1} t^{q_2^-(\kappa+1)-\alpha\phi^0}.$$

Then, for all $\lambda>0$, there exists $\rho\in(0,1)$ such that $h(\rho)>0$. Setting

$$\lambda^* = \min\left\{1, \frac{m_1(\kappa+1)}{4\alpha 4^\kappa\left(\frac{C_2|V|_\infty}{q^-}\right)^{\kappa+1}}\min\{\rho^{\alpha\phi^0-q_1^-(\kappa+1)}, \rho^{\alpha\phi^0-q_1^+(r+1)}\}\right\} > 0, \tag{3.6}$$

we have

$$J_\lambda(u) \geq \left[\frac{m_1}{2\alpha} - \frac{\lambda 4^\kappa}{\kappa+1}\left(\frac{C_2}{q^-}|V|_\infty\right)^{\kappa+1}\left(\|u\|^{q_1^-(\kappa+1)-\alpha\phi^0} + \|u\|^{q_1^+(\kappa+1)-\alpha\phi^0}\right)\right]\|u\|^{\alpha\phi^0}$$
$$\geq \frac{m_1}{4\alpha}\rho^{\alpha\phi^0},$$

for any $u \in X$ with $\|u\| = \rho$. So, we conclude that, for any $\lambda \in (0,\lambda^*)$, there exists $\beta > 0$ such that, for any $u \in X$ with $\|u\| = \rho$, $J_\lambda(u) \geq \beta$.

(ii) Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and there exist $x_1 \in \Omega\backslash B_R(x_0)$ and $\varepsilon > 0$ such that, for any $x \in B_\varepsilon(x_1) \subset (\Omega\backslash B_R(x_0))$, $\psi(x) > 0$. For any $t > 1$ large enough, we have

$$J_\lambda(t\psi) = \widehat{M}\left(\int_\Omega \Phi(|\nabla t\psi|)\,dx\right) - \frac{\lambda}{\kappa+1}\left(\int_\Omega \frac{V(x)}{q(x)}|t\psi|^{q(x)}\,dx\right)^{\kappa+1}$$
$$\leq \frac{m_2 t^{\alpha\phi^0}}{\alpha}\|\psi\|^{\alpha\phi^0} - \lambda\frac{t^{q_2^-(\kappa+1)}}{\kappa+1}\left(\int_{\Omega\backslash B_R(x_0)} \frac{V(x)}{q_2(x)}|\psi|^{q_2(x)}\,dx\right)^{\kappa+1}$$
$$\to -\infty,$$

since $\alpha\phi^0 < q_2^-(\kappa+1)$.

(iii) Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and there exist $x_2 \in B_r(x_0)$ and $\varepsilon > 0$ such that, for any $x \in B_\varepsilon(x_2) \subset B_r(x_0)$, $\varphi(x) > 0$. Letting $0 < t < 1$ small enough, then

$$J_\lambda(t\varphi) = \widehat{M}\left(\int_\Omega \Phi(|\nabla t\varphi|)\,dx\right) - \frac{\lambda}{\kappa+1}\left(\int_\Omega \frac{V(x)}{q(x)}|t\varphi|^{q(x)}\,dx\right)^{\kappa+1}$$
$$\leq \frac{m_2 t^{\alpha\phi_0}}{\alpha}\|\varphi\|^{\alpha\phi_0} - \lambda\frac{t^{q_1^+(\kappa+1)}}{\kappa+1}\left(\int_{B_r(x_0)} \frac{V(x)}{q(x)}|\varphi|^{q(x)}\,dx\right)^{\kappa+1}.$$

Obviously, we have $J_\lambda(t\varphi) < 0$ for any $0 < t < \delta^{\frac{1}{\alpha\phi_0 - q_1^+(\kappa+1)}}$, where

$$0 < \delta < \min\left\{1, \frac{\lambda\alpha\left(\int_{B_r(x_0)}\frac{V(x)}{q(x)}|\varphi|^{q(x)}\,dx\right)^{\kappa+1}}{m_2(\kappa+1)\|\varphi\|^{\alpha\phi_0}}\right\}.$$

The proof of Lemma 3.3 is complete. □

Lemma 3.4. *The functional J_λ satisfies the Palais-Smale condition in X.*

Proof. Let $\{u_n\} \subset X$ be such that

$$J_\lambda(u_n) \to \overline{c}, \quad J'_\lambda(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty, \tag{3.7}$$

where X^* is the dual space of X.

We next prove that $\{u_n\}$ is bounded in X. For this purpose, we assume by contradiction that, passing if necessary to a subsequence, denoted by $\{u_n\}$, we have $\|u_n\| \to \infty$ as $n \to \infty$. By (3.7) and Propositions 2.1 and 2.2, for n large enough and $\lambda \in (0,\lambda^*)$, we have

$$\begin{aligned}
&1+\bar{c}+\|u_n\| \\
&\geq J_\lambda(u_n) - \frac{1}{q_2^-(\kappa+1)} J'_\lambda(u_n)(u_n) \\
&= \widehat{M}\left(\int_\Omega \Phi(|\nabla u_n|)\,dx\right) - \frac{\lambda}{\kappa+1}\left(\int_\Omega \frac{V(x)}{q(x)}|u_n|^{q(x)}\,dx\right)^{\kappa+1} \\
&\quad - \frac{1}{q_2^-(\kappa+1)} M\left(\int_\Omega \Phi(|\nabla u_n|)\,dx\right)\int_\Omega a(|\nabla u_n|)|\nabla u_n|^2\,dx \\
&\quad + \frac{\lambda}{q_2^-(\kappa+1)}\left(\int_\Omega \frac{V(x)}{q(x)}|u_n|^{q(x)}\,dx\right)^{\kappa}\int_\Omega V(x)|u_n|^{q(x)}\,dx \\
&\geq \left(\frac{m_1}{\alpha} - \frac{m_2\phi^0}{q_2^-(\kappa+1)}\right)\|u_n\|^{\alpha\phi_0} \\
&\quad - \frac{\lambda}{\kappa+1}\left(\int_{B_r(x_0)} \frac{V(x)}{q_1(x)}|u_n|^{q_1(x)}\,dx\right)^{\kappa}\int_{B_r(x_0)} V(x)\left(\frac{1}{q_1(x)} - \frac{1}{q_2^-}\right)|u_n|^{q_1(x)}\,dx \\
&\geq \left(\frac{m_1}{\alpha} - \frac{m_2\phi^0}{q_2^-(\kappa+1)}\right)\|u_n\|^{\alpha\phi_0} - \frac{C_2 2^\kappa |V|_\infty^{\kappa+1}\lambda^*}{q_1^-(\kappa+1)}\left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right)\left(\|u_n\|^{q_1^-(\kappa+1)} + \|u_n\|^{q_1^+(\kappa+1)}\right),
\end{aligned}$$

where λ^* is given by (3.6).

Dividing the above inequality by $\|u_n\|^{\alpha\phi_0}$, and passing to the limit as $n\to\infty$, we obtain a contradiction. By (3.2), it follows that $\{u_n\}$ is bounded in X. Since X is a reflexive Banach space, there exists $u_1 \in X$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u_1 in X. Then $\{\|u_n-u\|\}$ is bounded. By (Q_1), the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is compact. Then, using the Hölder inequality, we have

$$\begin{aligned}
\left|\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)\,dx\right| &\leq |V|_\infty \int_\Omega |u_n|^{q(x)-1}|u_n-u|\,dx \\
&\leq 2|V|_\infty \left||u_n|^{q(x)-1}\right|_{\frac{q(x)}{q(x)-1}} |u_n-u|_{q(x)} \\
&\to 0 \text{ as } n\to\infty
\end{aligned}$$

and thus,

$$\lim_{n\to\infty}\left(\int_\Omega \frac{V(x)}{q(x)}|u_n|^{q(x)}\,dx\right)^{\kappa}\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)\,dx = 0. \tag{3.8}$$

From (3.7) and the boundedness of $\{u_n-u\}$ in X, we get $J'_\lambda(u_n)(u_n-u)\to 0$ as $n\to\infty$. Combining this with (3.8), it follows that

$$\lim_{n\to\infty} M\left(\int_\Omega \Phi(|\nabla u_n|)\,dx\right)\int_\Omega a(|\nabla u_n|)\nabla u_n(\nabla u_n - \nabla u)\,dx = 0. \tag{3.9}$$

Using Proposition 2.1, since $\{u_n\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \Phi(|\nabla u_n|)\,dx \to t_0 \geq 0 \text{ as } n\to\infty.$$

If $t_0 = 0$, then we find from Proposition 2.1 that $\{u_n\}$ converges strongly to $u_0 = 0$ in X and the proof is finished. If $t_0 > 0$, it follows from the continuity of the function M that

$$M\left(\int_\Omega \Phi(|\nabla u_n|)\,dx\right) \to M(t_0) \text{ as } n \to \infty.$$

Thus, by (M_0), for sufficiently large n, we have

$$M\left(\int_\Omega \Phi(|\nabla u_n|)\,dx\right) \geq C_4 > 0. \tag{3.10}$$

From (3.9), (3.10), it follows that

$$\lim_{m\to\infty} \int_\Omega a(|\nabla u_n|)\nabla u_n \cdot (\nabla u_n - \nabla u)\,dx = 0. \tag{3.11}$$

From (3.11) and the fact that $\{u_n\}$ converges weakly to u in X, we can apply Lemma 5 of [25] in order to deduce that $\{u_n\}$ converges strongly to u in X and the functional J_λ satisfies the Palais-Smale condition. □

Proof of Theorem 3.2. From Lemma 3.3, for any $u \in X$ with $\|u\| = \rho$, we have $J_\lambda(u) \geq \beta > 0$ and there exists $e \in X$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$. We set

$$\Gamma := \{\gamma \in C([0,1], X) : \; \gamma(0) = 0, \quad \gamma(1) = e\}$$

and define

$$\overline{c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

It should be noticed that since $\|e\| > \rho$, every path $\gamma \in \Gamma$ intersects the sphere $\|u\| = \rho$. So, we have $\overline{c} \geq \inf_{\|u\|=\rho} J_\lambda(u) \geq \beta > 0$.

By Lemma 3.3 and Lemma 3.4, all assumptions of the mountain pass theorem in [21] are satisfied. Then, we deduce u_1 as a non-trivial critical point of the functional J_λ with $J_\lambda(u_1) = \overline{c}$ and thus a non-trivial weak solution of problem (1.1).

We now prove that there exists a second weak solution $u_2 \in X$ such that $u_2 \neq u_1$. Indeed, let λ^* be as in the proof of Lemma 3.3(i) and assume that $\lambda \in (0, \lambda^*)$. By Lemma 3.3(i), it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0) = \{u \in X : \; \|u\| < \rho\}$, we have

$$\inf_{u\in\partial B_\rho(0)} J_\lambda(u) > 0. \tag{3.12}$$

On the other hand, by Lemma 3.3(ii), there exists $\varphi \in X$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, from the proof of Lemma 3.3, functional J_λ is bounded from below on $B_\rho(0)$. It follows that

$$-\infty < \underline{c} := \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u) < 0. \tag{3.13}$$

From (3.12)-(3.13), let us choose $\varepsilon > 0$ such that $0 < \varepsilon < \inf_{u\in\partial B_\rho(0)} J_\lambda(u) - \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u)$. Applying the Ekeland variational principle in [22] to functional $J_\lambda : \overline{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\varepsilon \in \overline{B}_\rho(0)$ such that

$$J_\lambda(u_\varepsilon) < \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u) + \varepsilon,$$
$$J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

We then deduce that $J_\lambda(u_\varepsilon) < \inf_{u\in\partial B_\rho(0)} J_\lambda(u)$. Thus, $u_\varepsilon \in B_\rho(0)$.

Now, we define the functional $\overline{J}_\lambda : \overline{B}_\rho(0) \to \mathbb{R}$ by $\overline{J}_\lambda(u) = J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of $\overline{J}_\lambda$ and thus

$$\frac{\overline{J}_\lambda(u_\varepsilon + tv) - \overline{J}_\lambda(u_\varepsilon)}{t} \geq 0$$

for all $t > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\langle J'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| \geq 0,$$

we then infer that $\|J'_\lambda(u_\varepsilon)\|_{X^*} \leq \varepsilon$. Therefore, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \to \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J(u) < 0 \text{ and } J'_\lambda(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{3.14}$$

From Lemma 3.4, the sequence $\{u_n\}$ converges strongly to u_2 as $n \to \infty$. Moreover, since $J_\lambda \in C^1(X, \mathbb{R})$, by (3.14) it follows that $J_\lambda(u_2) = \underline{c}$ and $J'_\lambda(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1).

Finally, we point out the fact that $u_1 \neq u_2$ since $J_\lambda(u_1) = \overline{c} > 0 > \underline{c} = J_\lambda(u_2)$. Moreover, since $J_\lambda(u) = J_\lambda(|u|)$, problem (1.1) has at least two non-negative non-trivial weak solutions. The proof of Theorem 3.2 is complete. □

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