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SOME ESTIMATES ON THE WEIGHTED SIMPSON-LIKE TYPE INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. A weighted integral identity of Simpson-like type is investigated. Based on this identity, some estimation-type results related to the weighted Simpson-like type integral inequalities for the first-order differentiable functions are obtained. These results are then applied to some error estimates for random variable, special means of real numbers, and two special functions including modified Bessel function and q-digamma function, respectively.

Keywords. Simpson's inequality; s-convex function; Special means; Modified Bessel function; q-digamma function.

1. INTRODUCTION-PRELIMINARIES

Throughout this paper, we let $K \subseteq \mathbb{R}$ be an interval and K° be the interior of K.

In 1978, Breckner [1] proposed a class of s-convex mappings, which is a generalization of convexity. Recall that a mapping $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense, for certain fixed $s \in (0,1]$, if the following inequality

$$f\big(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2\big) \le \lambda^s f(\varepsilon_1) + (1-\lambda)^s f(\varepsilon_2)$$

holds for all $\varepsilon_1, \varepsilon_2 \in [0,\infty)$ and $\lambda \in [0,1]$.

Clearly, if $s = 1$, then an s-convex mapping is reduced to a convex mapping. For the related results involving such kind of mappings, we refer to Dragomir and Fitzpatrick [2], Khan et al. [3], Kórus [4], Krtinić and Mikić [5], Latif [6], Özcan and Işcan [7] etc.

Recently, Sarikaya and Bardak [8] established the following result.

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Lemma 1.1. *Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous on K° such that $f' \in L_1([\varepsilon_1, \varepsilon_2])$ with $\varepsilon_1, \varepsilon_2 \in K$, $\varepsilon_1 < \varepsilon_2$. Then the following equality holds:*

$$\begin{aligned}&\frac{(\omega-\varepsilon_1)^2}{2(\varepsilon_2-\varepsilon_1)}\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}\varepsilon_1\right)\mathrm{d}t\\&\quad+\frac{(\varepsilon_2-\omega)^2}{2(\varepsilon_2-\varepsilon_1)}\int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}\varepsilon_2\right)\mathrm{d}t\\&=\frac{1}{6}f(\omega)+\frac{1}{3(\varepsilon_2-\varepsilon_1)}\left[(\omega-\varepsilon_1)f\left(\frac{\varepsilon_1+\omega}{2}\right)+(\varepsilon_2-\omega)f\left(\frac{\omega+\varepsilon_2}{2}\right)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\frac{\varepsilon_1+\omega}{2}}^{\frac{\omega+\varepsilon_2}{2}}f(x)\mathrm{d}x,\end{aligned}$$

where $\omega = \mu\varepsilon_1 + (1-\mu)\varepsilon_2$, $\forall \mu \in [0,1]$.

They also based on the convexity gave the following generalized Simpson type inequality by using the above identity.

Theorem 1.2. *Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K° such that $f' \in L_1([\varepsilon_1, \varepsilon_2])$ where $\varepsilon_1, \varepsilon_2 \in K$, $\varepsilon_1 < \varepsilon_2$. If $|f'|^q$ is a convex mapping on $[\varepsilon_1, \varepsilon_2]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}f(\omega)+\frac{1}{3(\varepsilon_2-\varepsilon_1)}\left[(\omega-\varepsilon_1)f\left(\frac{\varepsilon_1+\omega}{2}\right)+(\varepsilon_2-\omega)f\left(\frac{\omega+\varepsilon_2}{2}\right)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\frac{\varepsilon_1+\omega}{2}}^{\frac{\omega+\varepsilon_2}{2}}f(x)\mathrm{d}x\right|\\&\leq\frac{1}{2(\varepsilon_2-\varepsilon_1)}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[(\omega-\varepsilon_1)^2\left(\frac{29|f'(\varepsilon_1)|^q+61|f'(\omega)|^q}{3^42^3}\right)^{\frac{1}{q}}\right.\\&\quad\left.+(\varepsilon_2-\omega)^2\left(\frac{61|f'(\omega)|^q+29|f'(\varepsilon_2)|^q}{3^42^3}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where $\omega = \mu\varepsilon_1 + (1-\mu)\varepsilon_2$, $\forall \mu \in [0,1]$.

In [9], Shuang, Wang and Qi used the following identity to obtain Simpson type inequalities and some applications.

Lemma 1.3. *Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K° and $\varepsilon_1, \varepsilon_2 \in K^\circ$ with $\varepsilon_1 < \varepsilon_2$. If $f' \in L_1([\varepsilon_1, \varepsilon_2])$, then the following equality holds:*

$$\begin{aligned}&\frac{1}{8}\left[f(\varepsilon_1)+6f\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)+f(\varepsilon_2)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}f(x)\mathrm{d}x\\&=\frac{\varepsilon_2-\varepsilon_1}{4}\int_0^1\left[\left(\frac{3}{4}-t\right)f'\left(t\varepsilon_1+(1-t)\frac{\varepsilon_1+\varepsilon_2}{2}\right)+\left(\frac{1}{4}-t\right)f'\left(t\frac{\varepsilon_1+\varepsilon_2}{2}+(1-t)\varepsilon_2\right)\right]\mathrm{d}t.\end{aligned}$$

Simpson's inequality in the integral inequality area has been attracted much attention due to the close coupling to special means and special functions, differential geometry and linear differential equations. Simpson's integral inequality reads as follows:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(\varepsilon_1)+4f\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)+f(\varepsilon_2)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}f(t)\mathrm{d}t\right|\\&\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(\varepsilon_2-\varepsilon_1)^4,\end{aligned}\tag{1.1}$$

where $f:[\varepsilon_1,\varepsilon_2]\to\mathbb{R}$ is a four-order differentiable mapping on $(\varepsilon_1,\varepsilon_2)$, and

$$\left\|f^{(4)}\right\|_\infty=\sup_{x\in(\varepsilon_1,\varepsilon_2)}\left|f^{(4)}(x)\right|<\infty.$$

For the Simpson type inequalities, many researchers generalized and extended them. For example, Hsu, Hwang and Tseng [10], Du et al. [11], Noor, Noor and Awan [12] and Tunç et al. [13] obtained certain Simpson type inequalities for differentiable mappings, which are convex, and extended the (s,m)-convex, the geometrically relative convex and the h-convex, respectively.

Recently, many authors studied the weighted Simpson type inequality for differentiable mappings, for example, Luo et al. [14] studied the weighted Simpson-like type inequality for the (α,m,h)-convex mappings, and Matloka [15] investigated the weighted Simpson type inequality for the h-convex mappings. Further results involving the Simpson type inequality in question with applications to fractional integrals were also explored recently, for example, Set, Akdemir and Özdemir [16] considered the Simpson type inequalities using the Riemann-Liouville fractional integrals, and Ertŭgral and Sarikaya [17] investigated Simpson type inequalities using the generalized fractional integral. For more results related to the Simpson type inequalities, we refer to [18, 19, 20, 21, 22, 23] and the references cited therein.

Motivated by the results mentioned above, especially the results developed in [15] [8] and [24], this paper aims to obtain some new bounds related to the weighted Simpson-like type integral inequalities. To this end, we consider the following three cases: (i) the mapping whose absolute value of the first derivative is s-convex; (ii) the first derivative of the considered mapping is bounded; (iii) the first derivative of the considered mapping satisfies the Lipschitz condition.

2. Main results

Supposed that $f,g:K\subseteq\mathbb{R}\to\mathbb{R}$ are differentiable mappings on K°, $a,b\in K$ with $a<b$. Before presenting the results, we denote the following symbols:

$$\begin{aligned}\mathscr{T}_{f,g}(\omega;a,b):=&\frac{f(\omega)}{2(b-a)}\int_{\frac{\omega+a}{2}}^{\frac{\omega+b}{2}}g(x)\mathrm{d}x+\frac{f\left(\frac{\omega+b}{2}\right)}{2(b-a)}\int_{\omega}^{\frac{\omega+b}{2}}g(x)\mathrm{d}x\\&+\frac{f\left(\frac{\omega+a}{2}\right)}{2(b-a)}\int_{\frac{\omega+a}{2}}^{\omega}g(x)\mathrm{d}x-\frac{1}{b-a}\int_{\frac{\omega+a}{2}}^{\frac{\omega+b}{2}}f(x)g(x)\mathrm{d}x,\end{aligned}\tag{2.1}$$

where $\omega=\mu a+(1-\mu)b$, for all $\mu\in[0,1]$.
(i) If $\mu=0$, then equation (2.1) is reduced to

$$\mathscr{T}_{f,g}(b;a,b):=\frac{1}{2(b-a)}\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\int_{\frac{a+b}{2}}^{b}g(x)\mathrm{d}x-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)g(x)\mathrm{d}x.$$

(ii) If $\mu=\frac{1}{2}$ and $g:\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\to\mathbb{R}$ is symmetric about $\frac{a+b}{2}$, i.e., $g(x)=g(a+b-x)$, $\forall x\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]$, then equation (2.1) is reduced to

$$\begin{aligned}&\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)\\&:=\frac{\left[f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+3b}{4}\right)\right]}{2(b-a)}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}g(x)\mathrm{d}x-\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}f(x)g(x)\mathrm{d}x.\end{aligned}$$

(iii) If $\mu = 1$, then equation (2.1) is reduced to

$$\mathscr{T}_{f,g}(a;a,b) := \frac{1}{2(b-a)}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]\int_a^{\frac{a+b}{2}} g(x)\mathrm{d}x - \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)g(x)\mathrm{d}x.$$

(iv) If $g(x) = 1$, then equation (2.1) is reduced to

$$\begin{aligned}&\mathscr{T}_f(\omega;a,b)\\ &:= \frac{1}{4}f(\omega) + \frac{1}{4(b-a)}\left[(\omega-a)f\left(\frac{a+\omega}{2}\right)+(b-\omega)f\left(\frac{\omega+b}{2}\right)\right] - \frac{1}{b-a}\int_{\frac{a+\omega}{2}}^{\frac{\omega+b}{2}} f(x)\mathrm{d}x.\end{aligned}$$

We first prove the following lemma for our main results.

Lemma 2.1. *Let $f,g : K \to \mathbb{R}$ be differentiable mappings on K°, $a,b \in K$ with $a < b$. If $f', g \in L^1([a,b])$, then, for $\omega \in [a,b]$,*

$$\begin{aligned}\mathscr{T}_{f,g}(\omega;a,b) = &\frac{(\omega-a)^2}{2(b-a)}\int_0^1 h_1(t) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\mathrm{d}t\\ &+ \frac{(b-\omega)^2}{2(b-a)}\int_0^1 h_2(t) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}b\right)\mathrm{d}t,\end{aligned} \tag{2.2}$$

where

$$h_1(t) = \frac{1}{2}\int_0^t g\left(\frac{1+\nu}{2}\omega + \frac{1-\nu}{2}a\right)\mathrm{d}\nu - \frac{1}{4}\int_0^1 g\left(\frac{1+\nu}{2}\omega + \frac{1-\nu}{2}a\right)\mathrm{d}\nu$$

and

$$h_2(t) = \frac{1}{4}\int_0^1 g\left(\frac{1+\nu}{2}\omega + \frac{1-\nu}{2}b\right)\mathrm{d}\nu - \frac{1}{2}\int_0^t g\left(\frac{1+\nu}{2}\omega + \frac{1-\nu}{2}b\right)\mathrm{d}\nu.$$

In particular,

$$\begin{aligned}\left|\mathscr{T}_{f,g}(\omega;a,b)\right| \le &\frac{(\omega-a)^2}{2(b-a)}||g||_{[a,b],\infty}\int_0^1 \left|\frac{t}{2}-\frac{1}{4}\right|\left|f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\right|\mathrm{d}t\\ &+ \frac{(b-\omega)^2}{2(b-a)}||g||_{[a,b],\infty}\int_0^1 \left|\frac{1}{4}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}b\right)\right|\mathrm{d}t,\end{aligned} \tag{2.3}$$

where $||g||_{[a,b],\infty} = \sup_{x\in[a,b]}|g(x)|$.

Proof. Integrating by parts and changing the variables, for $\omega \in (a,b)$, we have that

$$\begin{aligned}
\mathscr{I}_1 &= \int_0^1 h_1(t) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\mathrm{d}t \\
&= \frac{2}{\omega - a}\left[\frac{1}{2}\int_0^t g\left(\frac{1+s}{2}\omega + \frac{1-s}{2}a\right)\mathrm{d}s \right.\\
&\quad \left. - \frac{1}{4}\int_0^1 g\left(\frac{1+s}{2}\omega + \frac{1-s}{2}a\right)\mathrm{d}s\right] f\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\Bigg|_0^1 \\
&\quad - \frac{1}{\omega - a}\int_0^1 f\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right) g\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\mathrm{d}t \\
&= \frac{1}{(\omega - a)^2}\left[f(\omega) + f\left(\frac{\omega + a}{2}\right)\right]\int_{\frac{\omega+a}{2}}^{\omega} g(x)\mathrm{d}x - \frac{2}{(\omega - a)^2}\int_{\frac{\omega+a}{2}}^{\omega} f(x)g(x)\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{I}_2 &= \int_0^1 h_2(t) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}b\right)\mathrm{d}t \\
&= \frac{1}{(b-\omega)^2}\left[f(\omega) + f\left(\frac{\omega + b}{2}\right)\right]\int_{\omega}^{\frac{\omega+b}{2}} g(x)\mathrm{d}x - \frac{2}{(b-\omega)^2}\int_{\omega}^{\frac{\omega+b}{2}} f(x)g(x)\mathrm{d}x.
\end{aligned}$$

Multiplying $\mathscr{I}_1$ and $\mathscr{I}_2$ with $\frac{(\omega-a)^2}{2(b-a)}$ and $\frac{(b-\omega)^2}{2(b-a)}$, respectively, and summing over these two results, we obtain equation (2.2). For $\omega = a$ and $\omega = b$, the identities

$$\mathscr{T}_{f,g}(a;a,b) = \frac{b-a}{2}\int_0^1 h_2(t) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}b\right)\mathrm{d}t$$

and

$$\mathscr{T}_{f,g}(b;a,b) = \frac{b-a}{2}\int_0^1 h_1(t) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\mathrm{d}t$$

can be proved by performing an integration by parts once in the integrals on the right-hand side and changing the variable. Inequality (2.3) follows immediately from (2.2) by using the trigonometric inequality. This completes the proof. □

Remark 2.2. Consider Lemma 2.1.
(i) If $\omega = a$, then

$$\mathscr{T}_{f,g}(a;a,b) = \frac{b-a}{2}\int_0^1 h_2(t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t. \tag{2.4}$$

(ii) If $\omega = \frac{a+b}{2}$ and $g : \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \to \mathbb{R}$ is symmetric about $\frac{a+b}{2}$, then

$$\begin{aligned}
\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right) &= \frac{b-a}{8}\int_0^1 h_1(t) f'\left(\frac{3-t}{4}a + \frac{1+t}{2}b\right)\mathrm{d}t \\
&\quad + \frac{b-a}{8}\int_0^1 h_2(t) f'\left(\frac{1+t}{4}a + \frac{3-t}{4}b\right)\mathrm{d}t.
\end{aligned} \tag{2.5}$$

(iii) If $\omega = b$, then

$$\mathscr{T}_{f,g}(b;a,b) = \frac{b-a}{2}\int_0^1 h_1(t) f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\mathrm{d}t. \tag{2.6}$$

(iv) If $g(x) = 1$, then

$$\begin{aligned}\mathscr{T}_f(\omega;a,b) = &\frac{(\omega-a)^2}{2(b-a)}\int_0^1\left(\frac{t}{2}-\frac{1}{4}\right) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}a\right)\mathrm{d}t \\ &+ \frac{(b-\omega)^2}{2(b-a)}\int_0^1\left(\frac{1}{4}-\frac{t}{2}\right) f'\left(\frac{1+t}{2}\omega + \frac{1-t}{2}b\right)\mathrm{d}t.\end{aligned} \tag{2.7}$$

It is worth mentioning that, to the best of our knowledge, the identities (2.4)-(2.7) obtained here are new in the literature.

Theorem 2.3. *Let $f,g:\mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be differentiable and continuous mappings on $\mathbb{R}_0$, $a,b \in \mathbb{R}_0$, $a < b$, and let $f', g \in L^1([a,b])$. If $|f'|^q$ for $q > 1$ is an s-convex mapping with $p^{-1}+q^{-1} = 1$ for certain fixed $s \in (0,1]$, then, for $\omega \in [a,b]$, the following inequality holds:*

$$\begin{aligned}\left|\mathscr{T}_{f,g}(\omega;a,b)\right| \le &\frac{||g||_{[a,b],\infty}}{(b-a)2^{3+\frac{s}{q}}(s+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left\{(\omega-a)^2\left[\left|f'(a)\right|^q + \left(2^{s+1}-1\right)\left|f'(\omega)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+ (b-\omega)^2\left[\left(2^{s+1}-1\right)\left|f'(\omega)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{2.8}$$

Proof. Observe that $|f'|^q$ is s-convex on $[a,b]$. By using Lemma 2.1 and Hölder's inequality, we have that

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b)\right| \\ \le &\frac{(\omega-a)^2||g||_{[a,b],\infty}}{8(b-a)}\left(\int_0^1 |1-2t|^p \mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\ &+ \frac{(b-\omega)^2||g||_{[a,b],\infty}}{8(b-a)}\left(\int_0^1 |1-2t|^p \mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\ \le &\frac{(\omega-a)^2||g||_{[a,b],\infty}}{8(b-a)}\left(\int_0^1 |1-2t|^p \mathrm{d}t\right)^{\frac{1}{p}}\left[\left|f'(a)\right|^q\int_0^1\left(\frac{1-t}{2}\right)^s\mathrm{d}t + \left|f'(\omega)\right|^q\int_0^1\left(\frac{1+t}{2}\right)^s\mathrm{d}t\right]^{\frac{1}{q}} \\ &+ \frac{(b-\omega)^2||g||_{[a,b],\infty}}{8(b-a)}\left(\int_0^1 |1-2t|^p \mathrm{d}t\right)^{\frac{1}{p}}\left[\left|f'(\omega)\right|^q\int_0^1\left(\frac{1+t}{2}\right)^s\mathrm{d}t + \left|f'(b)\right|^q\int_0^1\left(\frac{1-t}{2}\right)^s\mathrm{d}t\right]^{\frac{1}{q}}.\end{aligned} \tag{2.9}$$

The desired inequality (2.8) follows from the above by noting that

$$\int_0^1\left(\frac{1-t}{2}\right)^s\mathrm{d}t = \frac{1}{2^s(s+1)}, \quad \int_0^1\left(\frac{1+t}{2}\right)^s\mathrm{d}t = \frac{2^{s+1}-1}{2^s(s+1)}$$

and

$$\int_0^1 |2t-1|^p\mathrm{d}t = \int_0^1 |1-2t|^p\mathrm{d}t = \frac{1}{p+1}. \tag{2.10}$$

Thus, the proof is completed. □

Corollary 2.4. *Consider Theorem 2.3.*
(i) If $\omega = a$, then

$$\left|\mathscr{T}_{f,g}(a;a,b)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{2^{3+\frac{s}{q}}(s+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}. \tag{2.11}$$

Specially, if $s=1$, then

$$\left|\mathscr{T}_{f,g}(a;a,b)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{8(1+p)^{\frac{1}{p}}}\left(\frac{3\left|f'(a)\right|^{q}}{4}+\frac{\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}.$$

(ii) If $\omega = \frac{a+b}{2}$ and $g:\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\to\mathbb{R}$ is symmetric about $\frac{a+b}{2}$, then

$$\begin{aligned}\left|\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{2^{5+\frac{s}{q}}(s+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}&\left\{\left[\left|f'(a)\right|^{q}+\left(2^{s+1}-1\right)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\left(2^{s+1}-1\right)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{2.12}$$

Specially, if $s=1$, then

$$\begin{aligned}\left|\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{32(1+p)^{\frac{1}{p}}}&\left[\left(\frac{\left|f'(a)\right|^{q}}{4}+\frac{3\left|f'(\frac{a+b}{2})\right|^{q}}{4}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{3\left|f'(\frac{a+b}{2})\right|^{q}}{4}+\frac{\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

(iii) If $\omega = b$, then

$$\left|\mathscr{T}_{f,g}(b;a,b)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{2^{3+\frac{s}{q}}(s+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left[\left|f'(a)\right|^{q}+\left(2^{s+1}-1\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}. \tag{2.13}$$

Specially, if $s=1$, then

$$\left|\mathscr{T}_{f,g}(b;a,b)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{8(1+p)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^{q}}{4}+\frac{3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}.$$

Remark 2.5. Combining the inequalities (2.11) and (2.13) with $g(x)=1$, we have that

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\leq \frac{b-a}{2^{3+\frac{s}{q}}(s+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left\{\left[\left|f'(a)\right|^{q}+\left(2^{s+1}-1\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right.\\ &\quad\left.+\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Specially, if $s=1$ and $|f'(x)|\le \mathscr{M}$, $x\in[a,b]$, then

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le \frac{\mathscr{M}}{4(1+p)^{\frac{1}{p}}}(b-a).$$

Theorem 2.6. *Let $f,g:\mathbb{R}_0\to\mathbb{R}$ be differentiable and continuous mappings on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$, $a<b$, and let $f',g\in L^1([a,b])$. If $|f'|^q$ for $q\ge 1$ is an s-convex mapping with certain fixed $s\in(0,1]$, then, for $\omega\in[a,b]$, the following inequality holds:*

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b)\right|\\&\le \frac{||g||_{[a,b],\infty}}{2^{4+\frac{2s-1}{q}}(b-a)(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}}\\&\quad\times\left\{(\omega-a)^2\left[(s2^s+1)\left|f'(a)\right|^q+\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'(\omega)\right|^q\right]^{\frac{1}{q}}\right.\\&\quad\left.+(b-\omega)^2\left[\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'(\omega)\right|^q+(s2^s+1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.14}$$

Proof. From the inequality (2.3) in Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b)\right|\\&\le \frac{(\omega-a)^2||g||_{[a,b],\infty}}{8(b-a)}\left(\int_0^1|1-2t|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\&\quad+\frac{(b-\omega)^2||g||_{[a,b],\infty}}{8(b-a)}\left(\int_0^1|1-2t|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}\tag{2.15}$$

Utilizing the s-convexity of $\left|f'\right|^q$, we have

$$\begin{aligned}&\int_0^1|1-2t|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)\right|^q\mathrm{d}t\\&\le\frac{\left|f'(a)\right|^q}{2^s}\int_0^1\left|1-2t\right|(1-t)^s\mathrm{d}t+\frac{\left|f'(\omega)\right|^q}{2^s}\int_0^1\left|1-2t\right|(1+t)^s\mathrm{d}t\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}&\int_0^1|1-2t|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\\&\le\frac{\left|f'(\omega)\right|^q}{2^s}\int_0^1\left|1-2t\right|(1+t)^s\mathrm{d}t+\frac{\left|f'(b)\right|^q}{2^s}\int_0^1\left|1-2t\right|(1-t)^s\mathrm{d}t.\end{aligned}\tag{2.17}$$

The desired inequality (2.14) follows from inequalities (2.15)-(2.17) by noting that

$$\int_0^1|2t-1|\mathrm{d}t=\frac{1}{2},$$

$$\int_0^1\left|1-2t\right|(1-t)^s\mathrm{d}t=\frac{s2^s+1}{2^s(s+1)(s+2)},$$

and

$$\int_0^1 \left|1-2t\right|(1+t)^s \mathrm{d}t = \frac{3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s}{2^s(s+1)(s+2)}.$$

Thus, the proof is completed. □

Corollary 2.7. *Consider Theorem 2.6.*
(i) If $\omega = a$, then

$$\begin{aligned}\left|\mathscr{T}_{f,g}(a;a,b)\right| \leq & \frac{(b-a)||g||_{[a,b],\infty}}{2^{4+\frac{2s-1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}} \\ & \times \left(\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'(a)\right|^q+(s2^s+1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.18}$$

Specially, if $s=1$, then

$$\left|\mathscr{T}_{f,g}(a;a,b)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{16}\left(\frac{3\left|f'(a)\right|^q}{4}+\frac{\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}.$$

(ii) If $\omega = \frac{a+b}{2}$, and $g : \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \to \mathbb{R}$ is symmetric about $\frac{a+b}{2}$, then

$$\begin{aligned}&\left|\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)\right| \\ &\leq \frac{(b-a)||g||_{[a,b],\infty}}{2^{6+\frac{2s-1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}} \\ &\times \left\{\left[(s2^s+1)\left|f'(a)\right|^q+\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\ &\left.+\left[\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+(s2^s+1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{2.19}$$

Specially, if $s=1$, then

$$\begin{aligned}&\left|\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)\right| \\ &\leq \frac{(b-a)||g||_{[a,b],\infty}}{64}\left[\left(\frac{\left|f'(a)\right|^q}{4}+\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^q}{4}+\frac{\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

(iii) If $\omega = b$, then

$$\begin{aligned}\left|\mathscr{T}_{f,g}(b;a,b)\right| \leq & \frac{(b-a)||g||_{[a,b],\infty}}{2^{4+\frac{2s-1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}}\left((s2^s+1)\left|f'(a)\right|^q\right. \\ & \left.+\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.20}$$

Specially, if $s=1$, then

$$\left|\mathscr{T}_{f,g}(b;a,b)\right| \leq \frac{(b-a)||g||_{[a,b],\infty}}{16}\left(\frac{\left|f'(a)\right|^q}{4}+\frac{3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}.$$

Remark 2.8. Combining the inequalities (2.18) and (2.20) with $g(x) = 1$, we have that

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{b-a}{2^{4+\frac{2s-1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}} \\
&\quad\times\left\{\left[(s2^s+1)\left|f'(a)\right|^q+\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)\left|f'(a)\right|^q+(s2^s+1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Specially, if $q = 1$, then

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{3^{s+2}+(s-2)2^{2s+1}-2^{s+2}+1}{2^{3+2s}(s+1)(s+2)}(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).
\end{aligned}$$

Theorem 2.9. *Let $f,g:\mathbb{R}_0\to\mathbb{R}$ be differentiable and continuous mappings on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$, $a<b$, and let $f',g\in L^1([a,b])$. If $|f'|$ is an s-convex mapping for certain fixed $s\in(0,1]$ and $p^{-1}+q^{-1}=1$ with $q>1$, then, for $\omega\in[a,b]$, the following inequality holds:*

$$\begin{aligned}
\left|\mathscr{T}_{f,g}(\omega;a,b)\right| \le{}& \frac{||g||_{[a,b],\infty}}{2^{\frac{1}{p}+s+3}(b-a)(sq+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\Big\{(\omega-a)^2|f'(a)| \\
&+\left(2^{sq+1}-1\right)^{\frac{1}{q}}\left[(\omega-a)^2+(b-\omega)^2\right]|f'(\omega)|+(b-\omega)^2|f'(b)|\Big\}.
\end{aligned} \tag{2.21}$$

Proof. Since $|f'|$ is s-convex on $[a,b]$, by using the inequality 2.3 in Lemma 2.1, we have that

$$\begin{aligned}
&\left|\mathscr{T}_{f,g}(\omega;a,b)\right| \\
&\le \frac{(\omega-a)^2}{2(b-a)}\int_0^1|h_1(t)|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)\right|\mathrm{d}t \\
&\quad+\frac{(b-\omega)^2}{2(b-a)}\int_0^1|h_2(t)|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)\right|\mathrm{d}t \\
&\le \frac{(\omega-a)^2}{2(b-a)}||g||_{[a,b],\infty}\int_0^1\left|\frac{t}{2}-\frac{1}{4}\right|\left[\left(\frac{1-t}{2}\right)^s|f'(a)|+\left(\frac{1+t}{2}\right)^s|f'(\omega)|\right]\mathrm{d}t \\
&\quad+\frac{(b-\omega)^2}{2(b-a)}||g||_{[a,b],\infty}\int_0^1\left|\frac{1}{4}-\frac{t}{2}\right|\left[\left(\frac{1+t}{2}\right)^s|f'(\omega)|+\left(\frac{1-t}{2}\right)^s|f'(b)|\right]\mathrm{d}t \\
&= \frac{||g||_{[a,b],\infty}}{2^{s+3}(b-a)}\left\{(\omega-a)^2\int_0^1|2t-1|\left[\left(1-t\right)^s|f'(a)|+\left(1+t\right)^s|f'(\omega)|\right]\mathrm{d}t\right. \\
&\quad\left.+(b-\omega)^2\int_0^1|1-2t|\left[\left(1+t\right)^s|f'(\omega)|+\left(1-t\right)^s|f'(b)|\right]\mathrm{d}t\right\}.
\end{aligned}$$

Using the Hölder's inequality, the above inequality can be written as

$$\begin{aligned}
&\left|\mathscr{T}_{f,g}(\omega;a,b)\right| \\
&\le \frac{||g||_{[a,b],\infty}}{2^{s+3}(b-a)(p+1)^{\frac{1}{p}}}\Bigg\{(\omega-a)^2\Bigg[\left(\int_0^1(1-t)^{sq}\mathrm{d}t\right)^{\frac{1}{q}}|f'(a)| \\
&\quad+\left(\int_0^1(1+t)^{sq}\mathrm{d}t\right)^{\frac{1}{q}}|f'(\omega)|\Bigg] \\
&\quad+(b-\omega)^2\Bigg[\left(\int_0^1(1+t)^{sq}\mathrm{d}t\right)^{\frac{1}{q}}|f'(\omega)|+\left(\int_0^1(1-t)^{sq}\mathrm{d}t\right)^{\frac{1}{q}}|f'(b)|\Bigg]\Bigg\}.
\end{aligned}$$

The desired inequality (2.21) follows from the above by noting that

$$\int_0^1(1+t)^{sq}\mathrm{d}t=\frac{2^{sq+1}-1}{sq+1}$$

and

$$\int_0^1(1-t)^{sq}\mathrm{d}t=\frac{1}{sq+1}.$$

Thus, the proof is completed. □

Corollary 2.10. *Consider Theorem 2.9.*
(i) If $\omega=a$, then

$$\left|\mathscr{T}_{f,g}(a;a,b)\right|\le\frac{(b-a)||g||_{[a,b],\infty}}{2^{\frac{1}{p}+s+3}(sq+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(2^{sq+1}-1\right)^{\frac{1}{q}}|f'(a)|+|f'(b)|\right]. \tag{2.22}$$

(ii) If $\omega=\frac{a+b}{2}$ and $g:\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\to\mathbb{R}$ is symmetric about $\frac{a+b}{2}$, then

$$\left|\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)\right|\le\frac{(b-a)||g||_{[a,b],\infty}}{2^{\frac{1}{p}+s+5}(sq+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[|f'(a)|+2\left(2^{sq+1}-1\right)^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right]. \tag{2.23}$$

(iii) If $\omega=b$, then

$$\left|\mathscr{T}_{f,g}(b;a,b)\right|\le\frac{(b-a)||g||_{[a,b],\infty}}{2^{\frac{1}{p}+s+3}(sq+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[|f'(a)|+\left(2^{sq+1}-1\right)^{\frac{1}{q}}|f'(b)|\right]. \tag{2.24}$$

Remark 2.11. Combining the inequalities (2.22) and (2.24) with $g(x)=1$, we have that

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le\frac{\left(2^{sq+1}-1\right)^{\frac{1}{q}}+1}{2^{\frac{1}{p}+s+3}(sq+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}(b-a)\left(|f'(a)|+|f'(b)|\right).
\end{aligned}$$

3. Further estimation results

If the function f' is bounded from below and above, then we have the following result.

Theorem 3.1. *Let $f,g : K \to \mathbb{R}$ be differentiable and continuous mappings on K°, $a,b \in K$ with $a < b$, and let $f',g \in L^1([a,b])$. Assume that there exist constants $m < M$ such that $-\infty < m \leq f' \leq M < +\infty$. Then, for $\omega \in [a,b]$,*

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b) - \frac{M+m}{4(b-a)}\left\{(\omega-a)^2\int_0^1 h_1(t)\mathrm{d}t + (b-\omega)^2\int_0^1 h_2(t)\mathrm{d}t\right\}\right|\\ &\leq \frac{(M-m)||g||_{[a,b],\infty}}{32(b-a)}\left[(\omega-a)^2+(b-\omega)^2\right],\end{aligned} \tag{3.1}$$

where $h_1(t)$ and $h_2(t)$ are defined in Lemma 2.1.

Proof. From Lemma 2.1, we have that

$$\begin{aligned}\mathscr{T}_{f,g}(\omega;a,b) &= \frac{(\omega-a)^2}{2(b-a)}\int_0^1 h_1(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-\frac{M+m}{2}+\frac{M+m}{2}\right]\mathrm{d}t\\ &\quad+\frac{(b-\omega)^2}{2(b-a)}\int_0^1 h_2(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-\frac{M+m}{2}+\frac{M+m}{2}\right]\mathrm{d}t\\ &=\frac{1}{2(b-a)}\left\{(\omega-a)^2\int_0^1 h_1(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-\frac{M+m}{2}\right]\mathrm{d}t\right.\\ &\quad\left.+(b-\omega)^2\int_0^1 h_2(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-\frac{M+m}{2}\right]\mathrm{d}t\right\}\\ &\quad+\frac{M+m}{4(b-a)}\left[(\omega-a)^2\int_0^1 h_1(t)\mathrm{d}t+(b-\omega)^2\int_0^1 h_2(t)\mathrm{d}t\right].\end{aligned}$$

So

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b)-\frac{M+m}{4(b-a)}\left[(\omega-a)^2\int_0^1 h_1(t)\mathrm{d}t+(b-\omega)^2\int_0^1 h_2(t)\mathrm{d}t\right]\right|\\ &\leq\frac{(\omega-a)^2}{2(b-a)}\int_0^1 |h_1(t)|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-\frac{M+m}{2}\right|\mathrm{d}t\\ &\quad+\frac{(b-\omega)^2}{2(b-a)}\int_0^1 |h_2(t)|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-\frac{M+m}{2}\right|\mathrm{d}t.\end{aligned}$$

Since f' satisfies $-\infty < m \leq f' \leq M < +\infty$, we have that

$$m-\frac{M+m}{2}\leq f'(x)-\frac{M+m}{2}\leq M-\frac{M+m}{2},$$

which implies that

$$\left|f'(x)-\frac{M+m}{2}\right|\leq\frac{M-m}{2}.$$

Hence,

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b)-\frac{M+m}{4(b-a)}\left[(\omega-a)^2\int_0^1 h_1(t)\mathrm{d}t+(b-\omega)^2\int_0^1 h_2(t)\mathrm{d}t\right]\right|\\&\le\frac{M-m}{4(b-a)}\left[(\omega-a)^2\int_0^1|h_1(t)|\mathrm{d}t+(b-\omega)^2\int_0^1|h_2(t)|\mathrm{d}t\right]\\&\le\frac{(M-m)||g||_{[a,b],\infty}}{16(b-a)}\left[(\omega-a)^2\int_0^1|2t-1|\mathrm{d}t+(b-\omega)^2\int_0^1|1-2t|\mathrm{d}t\right]\\&=\frac{(M-m)||g||_{[a,b],\infty}}{32(b-a)}\left[(\omega-a)^2+(b-\omega)^2\right].\end{aligned}$$

This ends the proof. □

Corollary 3.2. If we take $\omega=\frac{a+b}{2}$ and $g:\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\to\mathbb{R}$ is symmetric about $\frac{a+b}{2}$ in Theorem 3.1, then we have that

$$\begin{aligned}&\left|\mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right)-\frac{(M+m)(b-a)}{16}\left(\int_0^1\bar{h}_1(t)\mathrm{d}t+\int_0^1\bar{h}_2(t)\mathrm{d}t\right)\right|\\&\le\frac{(M-m)(b-a)||g||_{[a,b],\infty}}{64},\end{aligned}\tag{3.2}$$

where

$$\bar{h}_1(t)=\frac{1}{2}\int_0^t g\left(\frac{b-a}{2}\nu+\frac{3a+b}{4}\right)\mathrm{d}\nu-\frac{1}{4}\int_0^1 g\left(\frac{b-a}{2}\nu+\frac{3a+b}{4}\right)\mathrm{d}\nu$$

and

$$\bar{h}_2(t)=\frac{1}{4}\int_0^1 g\left(\frac{a+3b}{4}-\frac{b-a}{2}\nu\right)\mathrm{d}\nu-\frac{1}{2}\int_0^t g\left(\frac{a+3b}{4}-\frac{b-a}{2}\nu\right)\mathrm{d}\nu.$$

Corollary 3.3. If we take $g(x)=1$ in Theorem 3.1, then we get that

$$\left|\mathscr{T}_f(\omega;a,b)\right|\le\frac{M-m}{32(b-a)}\left[(\omega-a)^2+(b-\omega)^2\right].$$

Our next aim is an estimation-type result considering the weighted Simpson-like type inequality when f' satisfies a Lipschitz condition.

Theorem 3.4. *Let $f,g:K\to\mathbb{R}$ be differentiable and continuous mappings on K°, $a,b\in K$ with $a<b$, and let $f',g\in L^1([a,b])$. Assume that f' satisfies the Lipschitz condition for some $L>0$. Then, for $\omega\in[a,b]$,*

$$\begin{aligned}&\left|\mathscr{T}_{f,g}(\omega;a,b)-\frac{(\omega-a)^2}{2(b-a)}f'\left(\frac{\omega+a}{2}\right)\int_0^1 h_1(t)\mathrm{d}t-\frac{(b-\omega)^2}{2(b-a)}f'\left(\frac{\omega+b}{2}\right)\int_0^1 h_2(t)\mathrm{d}t\right|\\&\le\frac{L||g||_{[a,b],\infty}}{64(b-a)}\left[(\omega-a)^3+(b-\omega)^3\right],\end{aligned}\tag{3.3}$$

where $h_1(t)$ and $h_2(t)$ are defined in Lemma 2.1.

Proof. From Lemma 2.1, we have that

$$
\begin{aligned}
&\mathscr{T}_{f,g}(\omega;a,b)\\
&=\frac{(\omega-a)^2}{2(b-a)}\int_0^1 h_1(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-f'\left(\frac{\omega+a}{2}\right)+f'\left(\frac{\omega+a}{2}\right)\right]\mathrm{d}t\\
&\quad+\frac{(b-\omega)^2}{2(b-a)}\int_0^1 h_2(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-f'\left(\frac{\omega+b}{2}\right)+f'\left(\frac{\omega+b}{2}\right)\right]\mathrm{d}t\\
&=\frac{1}{2(b-a)}\Bigg\{(\omega-a)^2\int_0^1 h_1(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-f'\left(\frac{\omega+a}{2}\right)\right]\mathrm{d}t\\
&\quad+(b-\omega)^2\int_0^1 h_2(t)\left[f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-f'\left(\frac{\omega+b}{2}\right)\right]\mathrm{d}t\Bigg\}\\
&\quad+\frac{1}{2(b-a)}\left[(\omega-a)^2 f'\left(\frac{\omega+a}{2}\right)\int_0^1 h_1(t)\mathrm{d}t+(b-\omega)^2 f'\left(\frac{\omega+b}{2}\right)\int_0^1 h_2(t)\mathrm{d}t\right].
\end{aligned}
$$

So

$$
\begin{aligned}
&\left|\mathscr{T}_{f,g}(\omega;a,b)-\frac{(\omega-a)^2}{2(b-a)}f'\left(\frac{\omega+a}{2}\right)\int_0^1 h_1(t)\mathrm{d}t-\frac{(b-\omega)^2}{2(b-a)}f'\left(\frac{\omega+b}{2}\right)\int_0^1 h_2(t)\mathrm{d}t\right|\\
&\le\frac{(\omega-a)^2}{2(b-a)}\int_0^1|h_1(t)|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-f'\left(\frac{\omega+a}{2}\right)\right|\mathrm{d}t\\
&\quad+\frac{(b-\omega)^2}{2(b-a)}\int_0^1|h_2(t)|\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-f'\left(\frac{\omega+b}{2}\right)\right|\mathrm{d}t.
\end{aligned}
$$

Since f' satisfies Lipschitz conditions for some $L>0$, we have that

$$
\begin{aligned}
\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}a\right)-f'\left(\frac{\omega+a}{2}\right)\right|&\le L\left|\frac{1+t}{2}\omega+\frac{1-t}{2}a-\frac{\omega+a}{2}\right|\\
&=\frac{\omega-a}{2}L|t|
\end{aligned}
$$

and

$$
\begin{aligned}
\left|f'\left(\frac{1+t}{2}\omega+\frac{1-t}{2}b\right)-f'\left(\frac{\omega+b}{2}\right)\right|&\le L\left|\frac{1+t}{2}\omega+\frac{1-t}{2}b-\frac{\omega+b}{2}\right|\\
&=\frac{b-\omega}{2}L|t|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|\mathscr{T}_{f,g}(\omega;a,b)-\frac{(\omega-a)^2}{2(b-a)}f'\left(\frac{\omega+a}{2}\right)\int_0^1 h_1(t)\mathrm{d}t-\frac{(b-\omega)^2}{2(b-a)}f'\left(\frac{\omega+b}{2}\right)\int_0^1 h_2(t)\mathrm{d}t\right|\\
&\le\frac{(\omega-a)^3L}{4(b-a)}\int_0^1|h_1(t)|t\mathrm{d}t+\frac{(b-\omega)^3L}{4(b-a)}\int_0^1|h_2(t)|t\mathrm{d}t\\
&\le\frac{L||g||_{[a,b],\infty}}{16(b-a)}\left[(\omega-a)^3+(b-\omega)^3\right]\int_0^1|t-2t^2|\mathrm{d}t\\
&=\frac{L||g||_{[a,b],\infty}}{64(b-a)}\left[(\omega-a)^3+(b-\omega)^3\right].
\end{aligned}
$$

This ends the proof. □

Corollary 3.5. In Theorem 3.4, if we take $\omega = \frac{a+b}{2}$ and $g : \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \to \mathbb{R}$ is symmetric about $\frac{a+b}{2}$, then we have that

$$\begin{aligned} &\left| \mathscr{T}_{f,g}\left(\frac{a+b}{2};a,b\right) - \frac{b-a}{8} f'\left(\frac{3a+b}{4}\right) \int_0^1 \bar{h}_1(t)\mathrm{d}t - \frac{b-a}{8} f'\left(\frac{a+3b}{4}\right) \int_0^1 \bar{h}_2(t)\mathrm{d}t \right| \\ &\leq \frac{L(b-a)^2 ||g||_{[a,b],\infty}}{256}, \end{aligned} \tag{3.4}$$

where $\bar{h}_1(t)$ and $\bar{h}_2(t)$ are defined in Corollary 3.2.

Corollary 3.6. In Theorem 3.4, if we take $g(x) = 1$, then we get that

$$\left| \mathscr{T}_f(\omega;a,b) \right| \leq \frac{L}{64(b-a)} \left[(\omega - a)^3 + (b-\omega)^3 \right].$$

4. The comparison of the results

In this section, we compare the bounds of some results obtained.

Let the bounds in the inequality of (2.13) and (2.24) established in Corollary 2.4 and 2.10 be denoted by $E_1(s,q)$, $E_2(s,q)$ respectively, that is,

$$E_1(s,q) = \frac{1}{2^{3+\frac{s}{q}}(s+1)^{\frac{1}{q}}} \left[\left|f'(a)\right|^q + \left(2^{s+1}-1\right)\left|f'(b)\right|^q \right]^{\frac{1}{q}}$$

and

$$E_2(s,q) = \frac{1}{2^{3+s+\frac{1}{p}}(sq+1)^{\frac{1}{q}}} \left[\left|f'(a)\right| + \left(2^{sq+1}-1\right)^{\frac{1}{q}}\left|f'(b)\right| \right],$$

where we omit $(b-a)||g||_{[a,b],\infty}$ and $(1+p)^{\frac{1}{p}}$ since they are fixed in the two error bounds above.

If $f(x) = \frac{e^{sx}}{s}, x \in [2,3]$ and $s \in (0,1]$, then $|f'(x)|$ and $|f'(x)|^q$ are s-convex. Let $q \in (1,100)$. Then it is obvious from Fig 1(a) that $E_2(s,q)$ is a better error bound than $E_1(s,q)$ in the vast majority of situations; see also Fig 1(b). Hence, in most instances, it reveals that the result of Corollary 2.10 is better than the result given in Corollary 2.4.

5. Applications

5.1. **Random variable.** Suppose that, for $0<a<b$, $g : \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \to [0,+\infty)$ is a continuous probability density function related to a continuous random variable X, which is symmetric about $\frac{a+b}{2}$. Also, for $r \in \mathbb{R}$, we suppose that the r-moment

$$E_r(X) = \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} x^r g(x)\mathrm{d}x \tag{5.1}$$

is finite.

From the fact that w is symmetric and $\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} g(x)\mathrm{d}x = 1$, we have that

$$E(X) = \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} xg(x)\mathrm{d}x = \frac{a+b}{2}, \tag{5.2}$$

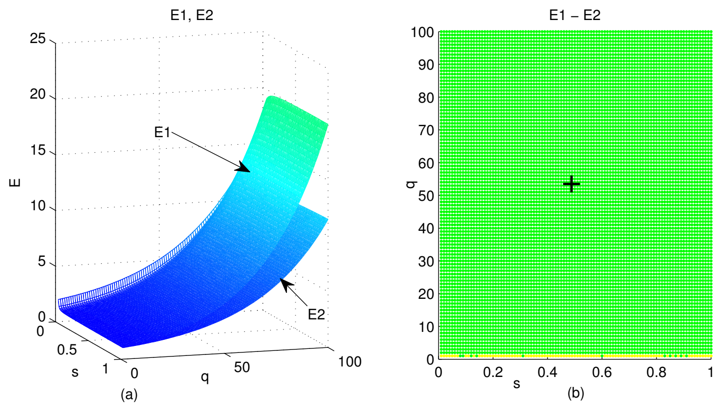

FIGURE 1. (**a**) Error surface of E_1 and E_2 on the variables s and q; (**b**) The green part describes $E_1 - E_2 > 0$ and the yellow part describes $E_1 - E_2 \leq 0$.

as a result of

$$\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} xg(x)\mathrm{d}x = \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} (b+a-x)g(b+a-x)\mathrm{d}x = \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} (b+a-x)g(x)\mathrm{d}x.$$

Based on the above-mentioned derivations, we obtain the following estimations of the r-moment.

(a) If we consider $f(x) = x^r$ for $r \geq 2$ and $x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \subseteq [a,b]$, then the function $\left|f'(x)\right|^q = r^q x^{q(r-1)}$ for $q > 1$ with $p^{-1} + q^{-1} = 1$, which is convex. Using this function and from the inequality (2.12) in Corollary 2.4 with $s = 1$, we have that

$$\begin{aligned}
&\left|\frac{1}{4}\left[\left(\frac{3a+b}{4}\right)^r + 2\big(E(X)\big)^r + \left(\frac{a+3b}{4}\right)^r\right] - E_r(X)\right| \\
&\leq \frac{(b-a)^2 r||g||_{[a,b],\infty}}{2^{5+\frac{2}{q}}(1+p)^{\frac{1}{p}}}\left\{\left[a^{q(r-1)} + 3\left(\frac{a+b}{2}\right)^{q(r-1)}\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left[3\left(\frac{a+b}{2}\right)^{q(r-1)} + b^{q(r-1)}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

(b) If we consider $f(x) = x^r$ and $x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \subseteq [a,b]$ for $r \in \mathbb{R}$, then

$$m = ra^{r-1} \leq f'(x) = rx^{r-1} \leq rb^{r-1} = M.$$

From the (3.4) in Corollary 3.2, we have that

$$\begin{aligned}
&\left|\frac{1}{4}\left[\left(\frac{3a+b}{4}\right)^r + 2\big(E(X)\big)^r + \left(\frac{a+3b}{4}\right)^r\right] - E_r(X)\right. \\
&\quad \left. - \frac{(ra^{r-1} + rb^{r-1})(b-a)^2}{16}\left(\int_0^1 \bar{h}_1(t)\mathrm{d}t + \int_0^1 \bar{h}_2(t)\mathrm{d}t\right)\right| \\
&\leq \frac{(rb^{r-1} - ra^{r-1})(b-a)^2||g||_{[a,b],\infty}}{64},
\end{aligned}$$

where $\bar{h}_1(t)$ and $\bar{h}_2(t)$ are defined in Corollary 3.2.

(c) If we consider $f(x)=x^r$, $x\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\subseteq[a,b]$ for $r\in\mathbb{R}$, then the Lipschitz constant $L=\sup_{x\in[a,b]}|f'(x)|=\sup_{x\in[a,b]} rx^{r-1}$ is equivalent to

$$L=\begin{cases} rb^{r-1}, & r\geq 1,\\ ra^{r-1}, & r<1.\end{cases}$$

From (3.4) in Corollary 3.5, we have that

$$\begin{aligned}&\left|\frac{1}{4}\left[\left(\frac{3a+b}{4}\right)^r+2\big(E(X)\big)^r+\left(\frac{a+3b}{4}\right)^r\right]-E_r(X)\right.\\&\quad\left.-\frac{r(b-a)}{8}\left(\frac{3a+b}{4}\right)^{r-1}\int_0^1\bar{h}_1(t)\mathrm{d}t-\frac{r(b-a)}{8}\left(\frac{a+3b}{4}\right)^{r-1}\int_0^1\bar{h}_2(t)\mathrm{d}t\right|\\&\leq\frac{L(b-a)^2||g||_{[a,b],\infty}}{256},\end{aligned}$$

which yields that

$$\begin{aligned}&\left|\frac{1}{4}\left[\left(\frac{3a+b}{4}\right)^r+2\big(E(X)\big)^r+\left(\frac{a+3b}{4}\right)^r\right]-E_r(X)\right.\\&\quad\left.-\frac{r(b-a)}{8}\left(\frac{3a+b}{4}\right)^{r-1}\int_0^1\bar{h}_1(t)\mathrm{d}t-\frac{r(b-a)}{8}\left(\frac{a+3b}{4}\right)^{r-1}\int_0^1\bar{h}_2(t)\mathrm{d}t\right|\\&\leq\begin{cases}\frac{rb^{r-1}(b-a)^2||g||_{[a,b],\infty}}{256}, & r\geq 1,\\[2ex] \frac{ra^{r-1}(b-a)^2||g||_{[a,b],\infty}}{256}, & r<1,\end{cases}\end{aligned}$$

where $\bar{h}_1(t)$ and $\bar{h}_2(t)$ are defined in Corollary 3.2.

5.2. **Special means.** For $0\leq a<b$, we consider the following special means:

(a) The arithmetic mean: $A(a,b)=\frac{a+b}{2}$.

(b) The geometric mean: $G(a,b)=\sqrt{ab}$.

(c) The logarithmic mean: $L(a,b)=\frac{b-a}{\ln b-\ln a}$, $ab\neq 0$.

(d) The γ-logarithmic mean: $L_\gamma(a,b)=\left(\frac{b^{\gamma+1}-a^{\gamma+1}}{(\gamma+1)(a-b)}\right)^{\frac{1}{\gamma}}$, $\gamma\in\mathbb{Z}\backslash\{0,-1\}$.

Next, using the main results obtained in Section 2, we give some applications to special means of real numbers.

Proposition 5.1. *Let $0<a<b$, $s\in(0,1)$, $q>1$ and $p^{-1}+q^{-1}=1$. Then*

$$\begin{aligned}&\left|\frac{1}{2}\left[A\left(a^{\frac{s}{q}+1},b^{\frac{s}{q}+1}\right)+A^{\frac{s}{q}+1}(a,b)\right]-L^{\frac{s}{q}+1}_{\frac{s}{q}+1}(a,b)\right|\\&\leq\frac{(b-a)\left(\frac{s}{q}+1\right)}{2^{3+\frac{s}{q}}(s+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left\{\left[a^s+\left(2^{s+1}-1\right)b^s\right]^{\frac{1}{q}}+\left[\left(2^{s+1}-1\right)a^s+b^s\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. Applying Remark 2.5 to the mapping $f(x)=\frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$, $x>0$, $q>1$ and $s\in(0,1)$ since $|f'(x)|^q=x^s$ is an s-convex function, we conclude the desired conclusion immediately. □

Proposition 5.2. *Let* $0<a<b$, $s\in(0,1)$ *and* $q\geq 1$. *Then*

$$\begin{aligned}&\left|\frac{1}{2}\left[A\left(a^{1-\frac{s}{q}},b^{1-\frac{s}{q}}\right)+A^{1-\frac{s}{q}}(a,b)\right]-L_{1-\frac{s}{q}}^{1-\frac{s}{q}}(a,b)\right|\\&\leq\frac{(b-a)(1-\frac{s}{q})}{2^{4+\frac{2s-1}{q}}(s+1)^{\frac{1}{q}}(s+2)^{\frac{1}{q}}}\\&\quad\times\left\{\left[\left(s2^s+1\right)a^{-s}+\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)b^{-s}\right]^{\frac{1}{q}}\right.\\&\quad\left.+\left[\left(3^{s+2}+(s-2)2^{2s+1}-(s+4)2^s\right)a^{-s}+\left(s2^s+1\right)b^{-s}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Specially, if $q=1$, *then*

$$\begin{aligned}&\left|\frac{1}{2}\left[A\left(a^{1-s},b^{1-s}\right)+A^{1-s}(a,b)\right]-L_{1-s}^{1-s}(a,b)\right|\\&\leq\frac{3^{s+2}+(s-2)2^{2s+1}-2^{s+2}+1}{2^{2+2s}(s+1)(s+2)}(1-s)(b-a)A\left(a^{-s},b^{-s}\right).\end{aligned}$$

Proof. The proof is obvious from the Remark 2.8 applied to the mapping $f(x)=\frac{x^{(1-\frac{s}{q})}}{1-\frac{s}{q}}$, $x>0$, $q\geq 1$ and $s\in(0,1)$ since $|f'(x)|^q=\frac{1}{x^s}$ is an s-convex function. □

Proposition 5.3. *Let* $0<a<b$, $s\in(0,1)$, $q>1$ *and* $p^{-1}+q^{-1}=1$. *Then*

$$\begin{aligned}&\left|A\left(\alpha^{\frac{3s}{4}}\beta^{\frac{s}{4}},\alpha^{\frac{s}{4}}\beta^{\frac{3s}{4}}\right)+G^s(\alpha,\beta)-L\left(\alpha^{\frac{3s}{4}}\beta^{\frac{s}{4}},\alpha^{\frac{s}{4}}\beta^{\frac{3s}{4}}\right)\right|\\&\leq\frac{(\ln\beta-\ln\alpha)s}{2^{3+s+\frac{1}{p}}(sq+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left[A(\alpha^s,\beta^s)+\left(2^{sq+1}-1\right)^{\frac{1}{q}}G^s(\alpha,\beta)\right].\end{aligned}$$

Proof. By putting $g(x)=1$ and applying inequality (2.23) to the mapping $f(x)=e^{sx}$, $x>0$, $s\in(0,1)$ and $\alpha=e^a$, $\beta=e^b$, we obtain the desired conclusion immediately. □

Proposition 5.4. *Let* $0<a<b$, $s\in(0,1)$, $q>1$ *and* $p^{-1}+q^{-1}=1$. *Then*

$$\begin{aligned}&\left|A(\alpha^s,\beta^s)+G^s(\alpha,\beta)-L(\alpha^s,\beta^s)\right|\\&\leq\frac{\left(2^{sq+1}-1\right)^{\frac{1}{q}}+1}{2^{1+s+\frac{1}{p}}(sq+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}s(\ln\beta-\ln\alpha)A(\alpha^s,\beta^s).\end{aligned}$$

Proof. The proof is obvious from the Remark 2.11 applied $f(x)=e^{sx}$, $x>0$ with $s\in(0,1)$ and $g(x)=1$ for which $\alpha=e^a$, $\beta=e^b$. □

Remark 5.5. Note that the left-hand side of the inequality above does not depend on q. So,

$$\begin{aligned}&\left|A(\alpha^s,\beta^s)+G^s(\alpha,\beta)-L(\alpha^s,\beta^s)\right|\\&\leq \inf_{q\in(1,+\infty)}\frac{\left(2^{sq+1}-1\right)^{\frac{1}{q}}+1}{2^{1+s+\frac{1}{p}}(sq+1)^{\frac{1}{q}}(1+p)^{\frac{1}{p}}}s(\ln\beta-\ln\alpha)A(\alpha^s,\beta^s),\end{aligned}$$

for any $q>1$, $a<b$.

Consider $M(q)=\ln\left[\left(2^{sq+1}-1\right)^{\frac{1}{q}}+1\right]-\ln\left[2^{2+s-\frac{1}{q}}(sq+1)^{\frac{1}{q}}(1+\frac{q}{q-1})^{1-\frac{1}{q}}\right]$ for $q>1$ and $s\in(0,1)$. Then $\lim_{q\to1^+}M'(q)<0$ and $\lim_{q\to+\infty}M'(q)=0$. From Figs 2 and 3, which reflect the positive and negative characteristic of $M''(q)$, we know that there exists an s_0 resulting in the following two cases:

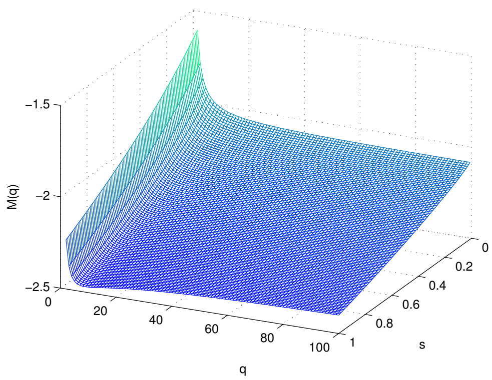

FIGURE 2. The surface of $M(q)$ on the variables s and q.

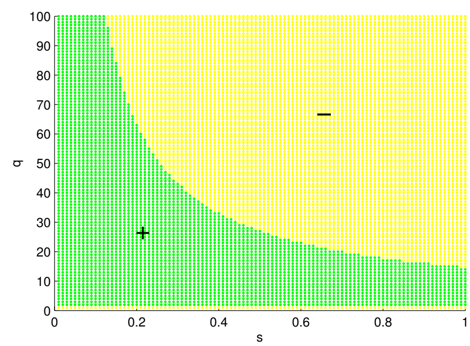

FIGURE 3. The positive and negative distribution corresponding to $M''(q)$

Case 1: If $0<s\leq s_0$, then $M''(q)\geq 0$ for $q\in(1,+\infty)$. So $M'(q)<0$. It follows that $\inf_{q\in(1,+\infty)}M(q)=\lim_{q\to+\infty}M(q)=-(s+3)\ln2$. Hence, $q>1$, $s\in(0,s_0]$,

$$\left|A(\alpha^s,\beta^s)+G^s(\alpha,\beta)-L(\alpha^s,\beta^s)\right|\leq\frac{s(\ln\beta-\ln\alpha)}{2^{s+2}}A(\alpha^s,\beta^s).$$

Case 2: If $s_0<s<1$, then $M'(q)\leq 0$ for $q\in(1,q_0(s)]$ and $M'(q)>0$ for $q\in(q_0(s),+\infty)$, where $q_0(s)$ is an unique solution of equation $M'(q)=0$. It follows that $M(q)$ decreases on $(1,q_0(s))$ and increases on $(q_0(s),+\infty)$. Hence,

$$\begin{aligned}&\left|A(\alpha^s,\beta^s)+G^s(\alpha,\beta)-L(\alpha^s,\beta^s)\right|\\&\leq\frac{\left(2^{sq_0(s)+1}-1\right)^{\frac{1}{q_0(s)}}+1}{2^{2+s-\frac{1}{q_0(s)}}(sq_0(s)+1)^{\frac{1}{q_0(s)}}\left(1+\frac{q_0(s)}{q_0(s)-1}\right)^{1-\frac{1}{q_0(s)}}}s(\ln\beta-\ln\alpha)A(\alpha^s,\beta^s).\end{aligned}$$

5.3. **Inequalities for some special functions.**

5.3.1. *Modified Bessel functions.* Recall the first kind modified Bessel function I_ρ, which has the series representation [[25], p.77]

$$I_\rho(x)=\sum_{n\geq 0}\frac{(\frac{x}{2})^{\rho+2n}}{n!\Gamma(\rho+n+1)},$$

where $x\in\mathbb{R}$ and $\rho>-1$, while the second kind modified Bessel function K_ρ [[25], p.78] is usually defined as

$$K_\rho(x)=\frac{\pi}{2}\frac{I_{-\rho}(x)-I_\rho(x)}{\sin\rho\pi}.$$

Here, we consider the function $\mathscr{I}_\rho\colon\mathbb{R}\to[1,\infty)$ defined by

$$\mathscr{I}_\rho(x)=2^\rho\Gamma(\rho+1)x^{-\rho}I_\rho(x),$$

where $\Gamma(\cdot)$ is the gamma function.

Proposition 5.6. *Let* $\rho>-1$ *and* $0<a<b$. *Then*

$$\begin{aligned}&\left|\frac{1}{4}\left[\mathscr{I}_\rho\Big(\frac{3a+b}{4}\Big)+2\mathscr{I}_\rho\Big(\frac{a+b}{2}\Big)+\mathscr{I}_\rho\Big(\frac{a+3b}{4}\Big)\right]-\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\mathscr{I}_\rho(x)\mathrm{d}x\right|\\&\leq\frac{b-a}{256(\rho+1)}\left(a\big|\mathscr{I}_{\rho+1}(a)\big|+3\big(a+b\big)\Big|\mathscr{I}_{\rho+1}\Big(\frac{a+b}{2}\Big)\Big|+b\big|\mathscr{I}_{\rho+1}(b)\big|\right).\end{aligned}\tag{5.3}$$

Specially, if $\rho=-\frac{1}{2}$, *then*

$$\begin{aligned}&\left|\frac{1}{4}\left[\cosh\Big(\frac{3a+b}{4}\Big)+2\cosh\Big(\frac{a+b}{2}\Big)+\cosh\Big(\frac{a+3b}{4}\Big)\right]-\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\cosh(x)\mathrm{d}x\right|\\&\leq\frac{b-a}{128}\left(\big|\sinh(a)\big|+6\Big|\sinh\Big(\frac{a+b}{2}\Big)\Big|+\big|\sinh(b)\big|\right).\end{aligned}\tag{5.4}$$

Proof. Put $g(x)=1$, $q=1=s$ and apply inequality (2.19) to the mapping $f(x)=\mathscr{I}_\rho(x)$, $x>0$ and $\mathscr{I}_\rho'(x)=\frac{x}{\rho+1}\mathscr{I}_{\rho+1}(x)$. Now taking into account the relations $\mathscr{I}_{-\frac{1}{2}}(x)=\cosh(x)$ and $\mathscr{I}_{\frac{1}{2}}(x)=\frac{\sinh(x)}{x}$, we have that inequality (5.3) is reduced to inequality (5.4). □

5.3.2. *q-digamma function.* Let $0<q<1$. The q-digamma function ψ_q is the q-analogue of the ψ or digamma function ψ defined by

$$\begin{aligned}\psi_q(\tau)&=-\ln(1-q)+\ln q\sum_{k=0}^{\infty}\frac{q^{k+\tau}}{1-q^{k+\tau}}\\&=-\ln(1-q)+\ln q\sum_{k=1}^{\infty}\frac{q^{k\tau}}{1-q^{k}}.\end{aligned}$$

For $q>1$ and $\tau>0$, the q-digamma function ψ_q is defined by

$$\begin{aligned}\psi_q(\tau)&=-\ln(q-1)+\ln q\Big(\tau-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{q^{-(k+\tau)}}{1-q^{-(k+\tau)}}\Big)\\&=-\ln(q-1)+\ln q\Big(\tau-\frac{1}{2}-\sum_{k=1}^{\infty}\frac{q^{-k\tau}}{1-q^{-k\tau}}\Big).\end{aligned}$$

Proposition 5.7. *Let* $0 < a < b$, $q > 1$ *and* $p^{-1} + q^{-1} = 1$. *Then*

$$\left|\frac{1}{4}\left[\psi_q\left(\frac{3a+b}{4}\right) + 2\psi_q\left(\frac{a+b}{2}\right) + \psi_q\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \psi_q(x)\mathrm{d}x\right|$$
$$\leq \frac{b-a}{2^{\frac{1}{p}+6}(q+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[|\psi_q'(a)| + 2\left(2^{q+1}-1\right)^{\frac{1}{q}}\left|\psi_q'\left(\frac{a+b}{2}\right)\right| + |\psi_q'(b)|\right].$$

Proof. The assertion can be obtained immediately by putting $g(x) = 1$, $s = 1$ and using inequality (2.23) to $f(x) = \psi_q(x)$, and $x > 0$, since $f'(x) = \psi_q'(x)$ is convex on $(0, +\infty)$. □

6. Conclusion

In this paper, with a new weighted identity of Simpson-like type, we proposed a set of generalized Simpson-like type integral inequalities for s-convex mappings. Furthermore, some interesting applications were considered, for example, we applied the investigated results to certain error estimates for random variable and special means of real numbers, and we also applied them to two special functions including modified Bessel function and q-digamma function. With these techniques and ideas developed in this paper, it is of interest to further explore the weighted inequalities of Simpson-like type.

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References

[1] W. W. Breckner, Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, Publ. Inst. Math. 23 (1978), 13-20.
[2] S. S. Dragomir, S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense, Demonstr. Math. 32 (1999), 687-696.
[3] M. A. Khan, Y. Chu, T. U. Khan, J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math. 15 (2017), 1414-1430.
[4] P. Kórus, An extension of the Hermite–Hadamard inequality for convex and s-convex functions, Aequat. Math. 93 (2019), 527-534.
[5] D. Krtinić, M. Mikić, Refinement of Hermite-Hadamard type inequalities for s-convex functions, Miskolc Math. Notes 19 (2018), 997-1005.
[6] M. A. Latif, On some new inequalities of Hermite–Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value, Ukrainian Math. J. 67 (2016), 1552-1571.
[7] S. Özcan, I. Işcan, Some new Hermite-Hadamard type inequalities for s-convex functions and their applications, J. Inequal. Appl. 2019 (2019), Article ID 201.
[8] M. Z. Sarikaya, S. Bardak, Generalized Simpson type integral inequalities, Konuralp J. Math. 7 (2019), 186-191.
[9] Y. Shuang, Y. Wang, F. Qi, Integral inequalities of Simpson's type for (α, m)-convex functions, J. Nonlinear Sci. Appl. 9 (2016), 6364-6370.
[10] K. C. Hsu, S. R. Hwang, K. L. Tseng, Some extended Simpson-type inequalities and applications, Bull. Iranian Math. Soc. 43 (2017), 409-425.
[11] T. S. Du, Y. J. Li, Z. Q. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions, Appl. Math. Comput. 293 (2017), 358-369.
[12] M. A. Noor, K. I. Noor, M. U. Awan, Simpson-type inequalities for geometrically relative convex functions, Ukrainian Math. J. 70 (2018), 1145-1154.

[13] M. Tunç, Ç. Yildiz, A. Ekinci, On some inequalities of Simpson's type via h-convex functions, Hacet. J. Math. Stat. 42 (2013), 309-317.
[14] C. Y. Luo, T. S. Du, M. Kunt, Y. Zhang, Certain new bounds considering the weighted Simpson-like type inequality and applications, J. Inequal. Appl. 2018 (2018), Article ID 332.
[15] M. Matłoka, Weighted Simpson type inequalities for h-convex functions, J. Nonlinear Sci. Appl. 10 (2017), 5770-5780.
[16] E. Set, A. O. Akdemir, M. E. Özdemir, Simpson type integral inequalities for convex functions via Riemann-Liouville integrals, Filomat 31 (2017), 4415-4420.
[17] F. Ertŭgral, M. Z. Sarikaya, Simpson type integral inequalities for generalized fractional integral, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. 113 (2019), 3115-3124.
[18] S. Hussain, S. Qaisar, More results on Simpson's type inequality through convexity for twice differentiable continuous mappings, SpringerPlus 5 (2016), 77.
[19] M. Iqbal, S. Qaisar, S. Hussain, On Simpson's type inequalities utilizing fractional integrals, J. Comput. Anal. Appl. 23 (2017), 1137-1145.
[20] T.S. Du, J.G. Liao, Y.J. Li, Properties and integral inequalities of Hadamard–Simpson type for the generalized (s,m)-preinvex functions, J. Nonlinear Sci. Appl. 9 (2016), 3112-3126.
[21] İ. İşcan, S. Turhan, S. Maden, Hermite-Hadamard and Simpson-like type inequalities for differentiable p-quasi-convex functions, Filomat 31 (2017), 5945-5953.
[22] M. Tunç, E. Göv, S. Balgeçti, Simpson type quantum integral inequalities for convex functions, Miskolc Math. Notes 19 (2018), 649-664.
[23] S. Qaisar, C.J. He, S. Hussain, A generalizations of Simpson's type inequality for differentiable functions using (α,m)-convex functions and applications, J. Inequal. Appl. 2013 (2013), Article ID 158.
[24] Y. Shuang, F. Qi, Some integral inequalities for s-convex functions, Gazi Univ. J. Sci. 31 (2018), 1192-1200.
[25] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1944.