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ON A NONLINEAR DIFFERENTIAL EQUATION INVOLVING THE $p(x)$-TRIHARMONIC OPERATOR

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Abstract. In this paper, we study weak solutions of a class of nonlinear elliptic Navier boundary value problems involving the $p(x)$-triharmonic operator. We determine the intervals of parameters for which the problem admits either a sequence of weak solutions converging to zero or an unbounded sequence of weak solutions.

Keywords. Weak solutions; Three critical points theorem; Navier boundary conditions; $p(x)$-triharmonic operator.

1. Introduction

Partial differential equations involving the operators with variable exponents growth conditions have been the object of increasing amount of attention in recent years. For background and recent results, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. These equations are interesting in applications and raise many problems such as the model of the motion of electroheological fluids, mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and other phenomena related to image processing, elasticity and the flow in porous media [11, 12, 13].

In [9], Yin and Liu studied the following $p(x)$-biharmonic elliptic problem with Navier boundary conditions:

$$\begin{cases} \Delta^2_{p(x)} u = \lambda a(x) f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ and μ are non-negative parameters, and $p(\cdot) \in C^0(\overline{\Omega})$ with

$$\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x).$$

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Using the three critical points theorem by Ricceri [14], they established the existence of three weak solutions to problem (1.1).

In [1], Afrouzi and Shokooh proved the existence of infinitely many weak solutions for a class of Navier boundary value problems depending on two parameters and involving the $p(x)$-biharmonic operator

$$\begin{cases} \Delta^2_{p(x)} u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$ and $p(\cdot) \in C^0(\overline{\Omega})$.

The operator $\Delta^3_{p(x)} u := div\left(\Delta(|\nabla \Delta u|^{p(x)-2} \nabla \Delta u)\right)$ is the $p(x)$-triharmonic operator, where $p(\cdot) \in C^0(\overline{\Omega})$, and is reduced to the classical p-triharmonic when p is a constant.

In [8], Rahal considered the following nonlinear Navier boundary value problem involving the $p(x)$-Kirchhoff type triharmonic operator

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)} |\nabla \Delta u|^{p(x)} dx\right) \Delta^3_{p(x)} u = \lambda \zeta(x) |u|^{\alpha(x)-2} u - \lambda \xi(x) |u|^{\beta(x)-2} u, & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary, λ is a positive parameter, $p \in C^0(\overline{\Omega})$ with $1 < p(x) < \frac{N}{3}$ for any $x \in \overline{\Omega}$ and $\zeta, \xi, \alpha, \beta \in C^0(\overline{\Omega})$.

The purpose of this paper is to establish infinitely many weak solutions for the following nonlinear elliptic Navier boundary value problem involving the $p(x)$-triharmonic operator:

$$\begin{cases} -\Delta^3_{p(x)} u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$ and $p(\cdot) \in C^0(\overline{\Omega})$ with

$$\max\{3, \frac{N}{3}\} < p^- := \inf_{x \in \overline{\Omega}} p(x) \le p^+ := \sup_{x \in \overline{\Omega}} p(x).$$

Recently, Bonanno and Bisci [15] presented a version of the infinitely many critical points theorem of Ricceri (see [16, Theorem 2.5]), and established the existence of an unbounded sequence of weak solutions for a Strum-Liouville problem, which has discontinuous nonlinearities. In such an approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used in several results in order to obtain existence results for different kinds of problems (see, for instance, [17, 18, 19, 20, 21] and references therein).

Our goal in this paper is to obtain some sufficient conditions to guarantee that problem (1.2) has infinitely many weak solutions. To this end, we require that the primitive F of f satisfies a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at zero (for finding arbitrarily small solutions), while G, the primitive of g, has an appropriate growth (see Theorems 3.1 and 3.4). Our approach is the variational method and the main tool is a general critical point theorem (see Lemma 2.3 below) contained in [15]; see also [16].

The organization of the paper is as follows. In Section 2, some known definitions and results on variable exponent Lebesgue and Sobolev spaces, which will be used in sequel, are collected.

Moreover, the abstract critical points theorem (Lemma 2.3) is recalled. Section 3, the last section, is devoted to main result.

2. Preliminaries

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$. We refer the interested reader to the papers [13, 22, 23, 24] for more details. Set

$$C_+(\Omega) := \{h \in C(\overline{\Omega}) : h(x) > 1\}, \text{ for any } x \in \overline{\Omega}.$$

For $p(\cdot) \in C_+(\Omega)$, define

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(\cdot)} = \inf\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)} dx \leq 1\}.$$

$(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ becomes a Banach space, and we call it variable exponent Lebesgue space. The Sobolev space with variable exponent $W^{m,p(\cdot)}(\Omega)$ is defined as

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq m\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u$ with $\alpha = (\alpha_1, \dots, \alpha_N)$ being a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(\cdot)} := \sum_{|\alpha| \leq m} |D^\alpha u|_{p(\cdot)},$$

becomes a separable, reflexive and uniformly convex Banach space. We denote

$$X := W^{3,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega),$$

where $W_0^{m,p(\cdot)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

Define a norm $\|\cdot\|_X$ of X by

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}.$$

It is well known that if $1 < p^- \leq p^+ < \infty$, the space $(X, \|u\|_X)$ is a separable and reflexive Banach space, $\|u\|_X$ and $|\nabla \Delta u|_{p(x)}$ are two equivalent norms on X, (see [8]).

For $u \in X$, we define

$$\|u\| = \inf\left\{\beta > 0 : \int_\Omega \left|\frac{\nabla \Delta u}{\beta}\right|^{p(x)} dx \leq 1\right\}. \tag{2.1}$$

In view of [8], it is easy to observe that $\|u\|$ is equivalent to the norms $\|u\|_X$ and $|\nabla \Delta u|_{p(x)}$ in X. In this paper, for the convenience, we will use the norm $\|\cdot\|$ on X.

Proposition 2.1 (see [13, 22]). *Let $\rho(u) = \int_\Omega |u|^{p(x)} dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < (=; >) 1 \Leftrightarrow \rho(u) < (=; >) 1$,

(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^+}$,

(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^-}$,

(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.1, for $u \in L^{p(\cdot)}(\Omega)$, the following inequalities hold:

$$\|u\|^{p^-} \leq \int_\Omega |\nabla\Delta u(x)|^{p(x)}\,dx \leq \|u\|^{p^+} \quad \text{if } \|u\| \geq 1; \tag{2.2}$$

$$\|u\|^{p^+} \leq \int_\Omega |\nabla\Delta u(x)|^{p(x)}\,dx \leq \|u\|^{p^-} \quad \text{if } \|u\| \leq 1. \tag{2.3}$$

By Theorem 1.2.26 in [25], we have following proposition:

Proposition 2.2. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N/3 < p^-$.*

From Proposition 2.2, there exists a positive constant c depending on $p(\cdot), N$ and Ω such that

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| \leq c\|u\|, \quad \forall u \in X. \tag{2.4}$$

Corresponding to f and g, we introduce the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \qquad G(x,t) := \int_0^t g(x,\xi)\,d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

For fixed $\lambda > 0$ and $\mu \geq 0$, let us define $\Phi, \Psi_{\lambda,\mu}, I_\lambda : X \to \mathbb{R}$ by putting

$$\Phi(u) := \int_\Omega \frac{1}{p(x)} |\nabla\Delta u(x)|^{p(x)} dx,$$

$$\Psi_{\lambda,\mu}(u) := \int_\Omega \Big(F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x)) \Big)\,dx,$$

$$I_\lambda(u) := \Phi(u) - \lambda\Psi_{\lambda,\mu}(u),$$

for every $u \in X$. It is simple to verify that Φ and $\Psi_{\lambda,\mu}$ satisfy the regularity assumptions of Lemma 2.3. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega |\nabla\Delta u|^{p(x)-2} \nabla\Delta u \cdot \nabla\Delta v dx$$

for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ admits a continuous inverse (see [9, Lemma 3.1]). On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional $\Psi_{\lambda,\mu}$ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'_{\lambda,\mu}(u)(v) = \int_\Omega f(x,u(x))v(x)\,dx + \frac{\mu}{\lambda}\int_\Omega g(x,u(x))v(x)\,dx,$$

for any $v \in X$.

Finally, we recall that a weak solution of problem (1.2) is a function $u \in X$ such that

$$\int_\Omega |\nabla\Delta u|^{p(x)-2} \nabla\Delta u \cdot \nabla\Delta v dx - \lambda \int_\Omega f(x,u(x))v(x)\,dx - \mu \int_\Omega g(x,u(x))v(x)\,dx = 0$$

for all $v \in X$. It is obvious that our goal is to find critical points of the functional I_λ. For achieving this aim, our main tool is the following critical point theorem of Ricceri [16, Theorem 2.5] (see also [15] for a refined version).

Lemma 2.3. *Let X be a reflexive real Banach space. Let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda \Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (b_1) *I_λ possesses a global minimum, or*
  (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
  (c_1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  (c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

3. Main results

In this section, we present our main results.

For fixed $x^0 \in \Omega$, let us pick $s > 0$ such that $B(x^0, s) \subset \Omega$, where $B(x^0, s)$ denotes the ball in $\mathbb{R}^N$ with center x^0 and radius of s. Also, put

$$\sigma := \frac{2^{(6p^+ + 1)} c^{p^-} \pi^{\frac{N}{2}} \left(s^N - \left(\frac{s}{2}\right)^N\right)}{N \Gamma\left(\frac{N}{2}\right)} \max\left\{ \left[\frac{96(N+4)^2}{s^3}\right]^{p^-}, \left[\frac{96(N+4)^2}{s^3}\right]^{p^+} \right\}, \tag{3.1}$$

where Γ denotes the Gamma function and c is defined by (2.4).

Theorem 3.1. *Assume that*

(A_1) *$F(x,t) \geq 0$ for every $(x,t) \in \Omega \times [0, +\infty[$;*

(A_2) *there exist $x^0 \in \Omega$ and $s > 0$ as considered in (3.1) such that, if we put*

$$\eta := \liminf_{\xi \to +\infty} \frac{\int_\Omega \sup_{|t| \leq \xi} F(x,t)\, dx}{\xi^{p^-}},$$

$$\theta := \limsup_{\xi\to+\infty} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\xi)\,dx}{\xi^{p^+}},$$

one has $\eta < \frac{p^-}{p^+\sigma}\theta$.

For each $\lambda \in \Lambda := (\lambda_1, \lambda_2)$, *where* $\lambda_1 := \frac{\sigma}{p^- c^{p^-}\theta}$, $\lambda_2 := \frac{1}{p^+ c^{p^-}\eta}$, *and for every* $g \in C^0(\overline{\Omega}\times\mathbb{R})$ *whose potential* $G(x,t) := \int_0^t g(x,\xi)\,d\xi$ *for all* $(x,t) \in \overline{\Omega}\times[0,+\infty[$ *is a non-negative function satisfying the condition*

$$g_\infty := \limsup_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\le\xi} G(x,t)\,dx}{\xi^{p^-}} < +\infty, \tag{3.2}$$

if we put

$$\mu_{g,\lambda} := \frac{1}{p^+ c^{p^-} g_\infty}\left(1 - \lambda p^+ c^{p^-}\eta\right),$$

where $\mu_{g,\lambda} = +\infty$ *when* $g_\infty = 0$, *then problem* (1.2) *has an unbounded sequence of weak solutions for every* $\mu \in [0, \mu_{g,\lambda})$ *in* X.

Proof. We plan to apply Lemma 2.3(b) with $X = W^{3,p(\cdot)}(\Omega)\cap W_0^{1,p(\cdot)}(\Omega)$ endowed with the norm introduced in (2.1). Fixing $\overline{\lambda} \in (\lambda_1,\lambda_2)$ and $\overline{\mu} \in (0, \mu_{g,\overline{\lambda}})$, we take $\Phi, \Psi_{\overline{\lambda},\overline{\mu}}$ as in the previous section. Similar arguments as those used in [1] and assumption (A_2) imply that

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le p^+ c^{p^-}\left(\eta + \frac{\overline{\mu}}{\overline{\lambda}} g_\infty\right) < +\infty, \tag{3.3}$$

and consequently $\overline{\lambda} < \frac{1}{\gamma}$.

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < \frac{p^- c^{p^-}}{\sigma}\theta,$$

there exist a sequence $\{\tau_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty}\tau_n = +\infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{p^- c^{p^-}}{\sigma} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\tau_n)\,dx}{\tau_n^{p^+}} \tag{3.4}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, we define $w_n \in X$ by

$$w_n(x) := \begin{cases} 0 & \text{if } \; x \in \overline{\Omega}\setminus B(x^0,s), \\ 64\frac{L^3}{s^6}(s-L)^3\tau_n & \text{if } \; x \in B(x^0,s)\setminus B(x^0,\frac{s}{2}), \\ \tau_n & \text{if } \; x \in B(x^0,\frac{s}{2}), \end{cases} \tag{3.5}$$

where $L = dist(x,x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. Then,

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0 & \text{if } \; x \in \overline{\Omega}\setminus B(x^0,s)\cup B(x^0,\frac{s}{2}), \\ \frac{64\tau_n(x_i-x_i^0)}{s^6}(3Ls^3 - 12s^2L^2 + 15sL^3 - 6L^4) & \text{if } \; x \in B(x^0,s)\setminus B(x^0,\frac{s}{2}), \end{cases}$$

$$\frac{\partial^2 w_n(x)}{\partial x_i^2}=\begin{cases} 0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,s)\cup B(x^0,\frac{s}{2}),\\ \frac{64\tau_n}{s^6}(3Ls^3-12s^2L^2+15sL^3-6L^4)+ & \\ \frac{64\tau_n(x_i-x_i^0)^2}{s^6}\left(\frac{3s^3}{L}-24s^2+45sL-24L^2\right) & \text{if } x\in B(x^0,s)\setminus B(x^0,\frac{s}{2}),\end{cases}$$

$$\sum_{i=1}^{N}\frac{\partial^2 w_n(x)}{\partial x_i^2}=\begin{cases} 0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,s)\cup B(x^0,\frac{s}{2}),\\ \frac{64\tau_n}{s^6}(3Ls^3(N+1)-12s^2L^2(N+2)+ & \\ 15sL^3(N+3)-6L^4(N+4)) & \text{if } x\in B(x^0,s)\setminus B(x^0,\frac{s}{2}),\end{cases}$$

$$\frac{\partial \Delta w_n(x)}{\partial x_i}=\begin{cases} 0 \quad \text{if } x\in\overline{\Omega}\setminus B(x^0,s)\cup B(x^0,\frac{s}{2}),\\ \frac{64\tau_n(x_i-x_i^0)}{s^6}\left(\frac{3s^3(N+1)}{L}-24s^2(N+2)+45sL(N+3)-24L^2(N+4)\right)\\ \quad \text{if } \ x\in B(x^0,s)\setminus B(x^0,\frac{s}{2}),\end{cases}$$

and

$$|\nabla\Delta w_n(x)|=\frac{64\tau_n}{s^6}|3s^3(N+1)-24s^2L(N+2)+45sL^2(N+3)-24L^3(N+4)|.$$

For any fixed $n\in\mathbb{N}$, one has

$$\Phi(w_n)=\int_{B(x^0,s)\setminus B/.?/>(x^0,\frac{s}{2})}\frac{1}{p(x)}|\nabla\Delta w_n(x)|^{p(x)}dx\leq\frac{\sigma\tau_n^{p^+}}{p^-c^{p^-}}. \tag{3.6}$$

On the other hand, bearing (A_1) in mind and since G is non-negative, from the definition of $\Psi_{\overline{\lambda},\overline{\mu}}$, we obtain

$$\Psi_{\overline{\lambda},\overline{\mu}}(w_n)=\int_{\Omega}\left[F(x,w_n(x))+\frac{\overline{\mu}}{\overline{\lambda}}G(x,w_n(x))\right]dx\geq\int_{B(x^0,\frac{s}{2})}F(x,\tau_n)\,dx. \tag{3.7}$$

By (3.4), (3.6) and (3.7), we observe that

$$I_{\overline{\lambda}}(w_n)\leq\frac{\sigma\tau_n^{p^+}}{p^-c^{p^-}}-\overline{\lambda}\int_{B(x^0,\frac{s}{2})}F(x,\tau_n)\,dx<\frac{\sigma\tau_n^{p^+}}{p^-c^{p^-}}(1-\overline{\lambda}\tau) \tag{3.8}$$

for every $n\in\mathbb{N}$ large enough. Since $\overline{\lambda}\tau>1$ and $\lim_{n\to+\infty}\tau_n=+\infty$, we have

$$\lim_{n\to+\infty}I_{\overline{\lambda}}(w_n)=-\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, by Lemma 2.3(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$ such that

$$\lim_{n\to+\infty}\|u_n\|=+\infty.$$

The conclusion is achieved. □

Now, we present the following consequence of Theorem 3.1 with $\mu=0$.

Theorem 3.2. *Let all the assumptions in Theorem 3.1 hold. Then, for each*

$$\lambda\in\left(\frac{\sigma}{p^-c^{p^-}\theta},\frac{1}{p^+c^{p^-}\eta}\right),$$

the problem

$$\begin{cases} -\Delta^3_{p(x)}u=\lambda f(x,u), & x\in\Omega,\\ u=\Delta u=\Delta^2u=0, & x\in\partial\Omega\end{cases} \tag{3.9}$$

has an unbounded sequence of weak solutions in X.

Here, we point out the following consequence of Theorem 3.1.

Corollary 3.3. *Let the assumption (A_1) in Theorem 3.1 holds. Suppose that*

$$\eta < \frac{1}{p^+ c^{p^-}}, \qquad \theta > \frac{\sigma}{p^- c^{p^-}}.$$

Then, the problem

$$\begin{cases} -\Delta^3_{p(x)} u = f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega \end{cases} \tag{3.10}$$

has an unbounded sequence of weak solutions in X.

Now, put

$$\sigma^0 := \frac{2^{(6p^+ + 1)} c^{p^+} \pi^{\frac{N}{2}} \left(s^N - \left(\frac{s}{2}\right)^N\right)}{N\Gamma(\frac{N}{2})} \max\left\{ \left[\frac{96(N+4)^2}{s^3}\right]^{p^-}, \left[\frac{96(N+4)^2}{s^3}\right]^{p^+} \right\}, \tag{3.11}$$

$$\eta^0 := \liminf_{\xi \to 0^+} \frac{\displaystyle\int_\Omega \sup_{|t| \le \xi} F(x,t)\,dx}{\xi^{p^+}},$$

$$\theta^0 := \limsup_{\xi \to 0^+} \frac{\displaystyle\int_{B(x^0, \frac{s}{2})} F(x,\xi)\,dx}{\xi^{p^-}},$$

and

$$\lambda_3 := \frac{\sigma^0}{p^- c^{p^+} \theta^0}, \qquad \lambda_4 := \frac{1}{p^+ c^{p^+} \eta^0}.$$

Using Lemma 2.3(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 3.4. *Assume that the assumption (A_1) in Theorem 3.1 holds and*

(A_5) $\eta^0 < \frac{p^-}{p^+\sigma}\theta^0$.

For every $\lambda \in (\lambda_3, \lambda_4)$ and for every $g \in C^0(\bar{\Omega} \times \mathbb{R})$, such that

(k_1) *There exists $\tau > 0$ such that $G(x,t) \ge 0$ for every $(x,t) \in \bar{\Omega} \times [0,\tau]$,*

(k_2) $g_0 := \limsup_{t \to 0^+} \frac{\int_\Omega \max_{|\xi| \le t} G(x,\xi)dx}{t^{p^+}} < +\infty$,

if we put

$$\mu'_{g,\lambda} := \frac{1}{p^+ c^{p^+} g_0}\left(1 - \lambda p^+ c^{p^+} \eta^0\right),$$

where $\mu'_{g,\lambda} = +\infty$ when $g_0 = 0$, for every $\mu \in [0, \mu'_{g,\lambda})$, problem (1.2) has a sequence of weak solutions, which strongly converges to zero in X.

Proof. Fix $\overline{\lambda} \in (\lambda_3, \lambda_4)$ and let g be a function that satisfies the condition (k_2). Since $\overline{\lambda} < \lambda_4$, we obtain

$$\mu'_{g,\overline{\lambda}} := \frac{1}{p^+ c^{p^+} g_0}\left(1 - \overline{\lambda} p^+ c^{p^+} \eta^0\right) > 0.$$

Now, we fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$J(x,t) := F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi),$$

for all $(x,t) \in \Omega \times \mathbb{R}$. We take $\Phi, \Psi_{\overline{\lambda},\overline{\mu}}$ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and $\Psi_{\overline{\lambda},\overline{\mu}}$ satisfy the regularity assumptions required in Lemma 2.3. As the first step, we will prove that $\overline{\lambda} < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\displaystyle\int_\Omega \sup_{|t|\le\xi_n} F(x,t)\,dx}{\xi_n^{p^+}} = \eta^0.$$

From the fact that $\inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$. Put $r_n = \frac{1}{p^+}\left(\frac{\xi_n}{c}\right)^{p^+}$. Following the proof in Theorem 3.1, we can prove that $\delta < +\infty$. From $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$, the following inequalities hold

$$\delta \le p^+ c^{p^+}\left(\eta^0 + \frac{\overline{\mu}}{\overline{\lambda}} g_0\right) < p^+ c^{p^+}\eta^0 + \frac{1-\overline{\lambda}p^+c^{p^+}\eta^0}{\overline{\lambda}}.$$

Therefore

$$\overline{\lambda} = \frac{1}{p^+c^{p^+}\eta^0 + \left(1-\overline{\lambda}p^+c^{p^+}\eta^0\right)/\overline{\lambda}} < \frac{1}{\delta}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\lambda}} < \frac{p^-c^{p^+}\theta^0}{\sigma^0},$$

we have that there exist a sequence $\{\tau_n\}$ of positive numbers in $]0,\tau[$ and $\zeta > 0$ such that $\lim_{n\to+\infty} \tau_n = 0^+$ and

$$\frac{1}{\overline{\lambda}} < \zeta < \frac{p^-c^{p^+}}{\sigma^0} \frac{\displaystyle\int_{B(x^0,\frac{s}{2})} F(x,\tau_n)\,dx}{\tau_n^{p^-}}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be the sequence in X defined in (3.5). From (k_1) and (A_1), one has that (3.7) holds. Note that $\overline{\lambda}\zeta > 1$. From (3.8), we can obtain that

$$I_{\overline{\lambda}}(w_n) < \frac{\tau_n^{p^-}\sigma^0}{p^-c^{p^+}}(1-\overline{\lambda}\zeta) < 0 = \Phi(0) - \overline{\lambda}\Psi(0)$$

for every $n \in \mathbb{N}$ large enough. Then, we see that zero is not a local minimum of $I_{\overline{\lambda}}$. This, together with the fact that zero is the only global minimum of Φ deduces that the energy functional $I_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ. Therefore, by Lemma 2.3(c), there exists a sequence $\{u_n\}$ of critical points of $I_{\overline{\lambda}}$, which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact, we know that the critical points converge strongly to zero. The proof is complete. □

Remark 3.5. Under the conditions $\eta^0 = 0$ and $\theta^0 = +\infty$, Theorem 3.4 ensures that, for every $\lambda > 0$ and for each $\mu \in \left[0, \frac{1}{p^+ c^{p^+} g_0}\right)$, problem (1.2) admits a sequence of weak solutions, which strongly converges to 0 in X. Moreover, if $g_0 = 0$, then the result holds for every $\lambda > 0$ and $\mu \geq 0$.

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