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# POSITIVE SOLUTIONS FOR HADAMARD FRACTIONAL BOUNDARY VALUE PROBLEMS ON AN INFINITE INTERVAL

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**Abstract.** In this paper, we consider a boundary value problem of Hadamard fractional differential equations on an infinite interval

$$\begin{cases} {}^{H}D^{\alpha}u(t)+\lambda a(t)F(t,u(t))=0,\ 1<\alpha<2,\ t\in(1,\infty), \\ u(1)=0,\ {}^{H}D^{\alpha-1}u(\infty)=\sum_{i=1}^{m}\gamma_i {}^{H}I^{\beta_i}u(\eta), \end{cases}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative of order $\alpha$, $\eta\in(1,\infty)$, ${}^{H}I(\cdot)$ denotes the Hadamard fractional integral, $\lambda>0$ is a parameter, $\beta_i,\gamma_i\geq 0 (i=1,2,\ldots,\ m)$ are constants and $\Gamma(\alpha)>\sum_{i=1}^{m}\frac{\gamma_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1}$. The existence and uniqueness of positive solutions is given for each fixed $\lambda>0$. The relations between the positive solution and the parameter $\lambda$ are presented. The results obtained in this paper show that the unique positive solution $u_{\lambda}^{*}$ has good properties: continuity, monotonicity, iteration and approximation. The method of this paper is based upon different fixed point theorems and properties for two types of operators: monotone operators and mixed monotone operators. Finally, two examples are also provided.

**Keywords.** Hadamard fractional derivative; Positive solution; Infinite interval; Monotone operator; Mixed monotone operator.

## 1. INTRODUCTION

We are concerned with a boundary value problem of Hadamard fractional differential equations on an infinite interval

$$\begin{cases} {}^{H}D^{\alpha}u(t)+\lambda a(t)F(t,u(t))=0,\ 1<\alpha<2,\ t\in(1,\infty), \\ u(1)=0,\ {}^{H}D^{\alpha-1}u(\infty)=\sum\limits_{i=1}^{m}\gamma_i {}^{H}I^{\beta_i}u(\eta), \end{cases} \tag{1.1}$$

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where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative of order $\alpha$, $\eta \in (1,\infty)$ and ${}^{H}I(\cdot)$ denotes the Hadamard fractional integral, $\lambda > 0$ is a parameter, $\beta_i, \gamma_i \geq 0 (i = 1, 2, \ldots, m)$ are constants and

$$\Gamma(\alpha) > \sum_{i=1}^{m} \frac{\gamma_i \Gamma(\alpha)}{\Gamma(\alpha + \beta_i)} (\log \eta)^{\alpha + \beta_i - 1} > 0.$$

Fractional differential equations, which received much attention, have real applications in many fields: physics, chemistry, engineering and biological science, etc; see, e.g., [1, 2, 3, 4, 5, 6] and references therein. We can find that most of these results are Riemann-Liouville and Caputo-type fractional equations. As we know, there is another kind of fractional derivatives which can be found in the literature due to Hadamard [7]. Recently, More studies on boundary value problems of Hadamard fractional differential equations were obtained; see, e.g., [8, 9, 10, 11, 12]. For example, Ahmad and Ntouyas [8, 9] studied some fractional integral boundary value problems involving Hadamard fractional differential equations/systems, and gave the existence and uniqueness of solutions to these problems by applying the Banach fixed point theorem, the Leray-Schauder alternative, respectively.

In [10], Ahmad, Ntouyas and Alsaedi investigated the boundary value problem of Hadamard fractional differential inclusions

$$\begin{cases} {}^{H}D^{\alpha}x(t) \in F(t, x(t)), \ 1 < t < e, \ 1 < \alpha \leq 2, \\ x(1) = 0, \ x(e) = {}^{H}I^{\beta}x(\eta), \ 1 < \eta < e, \end{cases}$$

where $F : [1, e] \times (-\infty, +\infty) \to \rho(-\infty, +\infty)$ is a multivalued map, $\rho(-\infty, +\infty)$ is the family of all nonempty subsets of $(-\infty, +\infty)$. By using fixed point theorems for multivalued maps, they obtained the existence of solutions.

In [11], by using Leggett-Williams and Guo-Krasnoselkii's fixed point theorems, Thiramanus, Ntouyas and Tariboon studied multiple positive solutions for Hadamard fractional differential equations on an infinite interval

$$\begin{cases} {}^{H}D^{\alpha}u(t) + a(t)f(u(t)) = 0, \ 1 < \alpha \leq 2, \ t \in (1, \infty), \\ u(1) = 0, \ D^{\alpha-1}u(\infty) = \sum\limits_{i=1}^{m} \lambda_i {}^{H}I^{\beta_i}u(\eta), \end{cases}$$

where $\eta \in (1, \infty), \lambda_i \geq 0, \beta_i > 0 \ (i = 1, 2, \ldots, m)$ are constants.

Very recently, Pei, Wang and Sun [12] discussed a Hadamard fractional integro-differential equation on an infinite domain

$$\begin{cases} {}^{H}D^{\alpha}u(t) + f(t, u(t), {}^{H}I^{r}u(t), {}^{H}D^{\alpha-1}u(t)) = 0, \ 1 < \alpha < 2, \ t \in (1, \infty), \\ u(1) = 0, \ {}^{H}D^{\alpha-1}u(\infty) = \sum\limits_{i=1}^{m} \gamma_i {}^{H}I^{\beta_i}u(\eta), \end{cases}$$

where $\eta \in (1, \infty), r, \beta_i, \gamma_i \geq 0 \ (i = 1, 2, \ldots, m)$ are constants and

$$\Gamma(\alpha) > \sum_{i=1}^{m} \frac{\gamma_i \Gamma(\alpha)}{\Gamma(\alpha + \beta_i)} (\log \eta)^{\alpha + \beta_i - 1}.$$

By using monotone iterative techniques, they obtained the existence of positive solutions and proved that two monotone iterative sequences converge to the extremal solutions.

In this paper, we use new methods to study problem (1.1) under some different conditions. The existence and uniqueness of positive solutions are obtained for every fixed parameter $\lambda > 0$. We also give some properties of positive solutions which depend on the parameters. Compared with [12], we do not need to find two special functions for constructing an iterative sequence. We can construct iterative sequences for any point of a set $P_h$ and our sequence can find the unique solution. Moreover, our nonlinear term contains two monotone forms which guarantee that we can use two methods to discuss problem (1.1).

## 2. Preliminaries

**Definition 2.1.** [13] For a function $g:[1,\infty)\to\mathbf{R}$, $g\in L^1[1,\infty)$, the Hadamard fractional integral of order $\alpha$ is defined by

$$^{H}I^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds,\ \alpha>0,$$

where the integral exists.

**Definition 2.2.** [13] For a function $g:[1,\infty)\to\mathbf{R}$, the Hadamard fractional derivative of fractional order $\alpha$ is defined by

$$^{H}D^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{g(s)}{s}ds,\ n-1<\alpha<n,\ n=[\alpha]+1,$$

where $[\alpha]$ is the integer part of the real number $\alpha$ and $\log(\cdot)=\log_e(\cdot)$.

In the sequel, we let

$$\Omega=\Gamma(\alpha)-\sum_{i=1}^m\frac{\gamma_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1}.$$

From the initial condition, we know $\Omega>0$.

**Lemma 2.3.** [11] *If $h\in C[1,\infty)$ with $0<\int_1^\infty h(s)\frac{ds}{s}<\infty$, then the unique solution of the following equation with integral boundary value conditions*

$$\begin{cases} {}^{H}D^{\alpha}u(t)+h(t)=0,\ 1<\alpha<2,\ t\in(1,\infty),\\ u(1)=0,\ {}^{H}D^{\alpha-1}u(\infty)=\sum\limits_{i=1}^m\gamma_i{}^{H}I^{\beta_i}u(\eta),\end{cases}\tag{2.1}$$

*is*

$$u(t)=\int_1^\infty G(t,s)h(s)\frac{ds}{s},$$

*where*

$$G(t,s)=g(t,s)+\sum_{i=1}^m\frac{\gamma_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}g_i(\eta,s),\tag{2.2}$$

*and*

$$g(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\log t)^{\alpha-1}-\left(\log(\frac{t}{s})\right)^{\alpha-1},\ 1\le s\le t<\infty,\\ (\log t)^{\alpha-1},\ 1\le t\le s<\infty,\end{cases}\tag{2.3}$$

$$g_i(\eta,s)=\begin{cases}(\log\eta)^{\alpha+\beta_i-1}-\left(\log(\frac{\eta}{s})\right)^{\alpha+\beta_i-1},\ 1\le s\le\eta<\infty,\\ (\log\eta)^{\alpha+\beta_i-1},\ 1\le\eta\le s<\infty.\end{cases}\tag{2.4}$$

**Lemma 2.4.** [11] *The Green's function $G(t,s)$ defined by (2.2) has the following properties:*
*(i) $G(t,s)$ is continuous and $G(t,s) \geq 0$ for $(t,s) \in [1,\infty) \times [1,\infty)$;*
*(ii) $\frac{G(t,s)}{1+(\log t)^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\gamma_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}$ for $(t,s) \in [1,\infty) \times [1,\infty)$;*
*(iii) $G(t,s) \leq \left( \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\gamma_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)} \right) (\log t)^{\alpha-1}$ for $(t,s) \in [1,\infty) \times [1,\infty)$.*

## 3. THE MONOTONE OPERATOR METHOD FOR PROBLEM (1.1) WITH $F(t,u) := f(t,u)$

In this section, we consider (1.1) with $F(t,u) := f(t,u)$. Let $(E, \|\cdot\|)$ be a real Banach space, and $P \subset E$ a cone. Then $E$ is partially ordered by $P$, i.e., $x \leq y$ if and only if $y - x \in P$. $\theta$ is the zero element in $E$. $P$ is said to be normal if there is a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. An operator $A : E \to E$ is said to be increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay (Ax \geq Ay)$. For $x, y \in E$, $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), if we define a set $P_h = \{x \in E | x \sim h\}$, then $P_h \subset P$.

**Lemma 3.1.** [14] *Let $E$ be a real Banach space, $P \subset E$ a normal cone, and $h > \theta$. Let $A : P \to P$ be an increasing operator. In addition,*
*(i) there exists $h_0 \in P_h$ such that $Ah_0 \in P_h$;*
*(ii) for $x \in P$ and $r \in (0,1)$, there is $\varphi(r) \in (r,1)$ such that $A(ru) \geq \varphi(r)Au$.*
*Then:*
*(1) $Ax = x$ has a unique solution $x^*$ in $P_h$;*
*(2) for $x_0 \in P_h$, letting $x_n = Ax_{n-1}$, $n = 1,2,\ldots$, we get $x_n \to x^*$ as $n \to \infty$.*

**Lemma 3.2.** [14] *Assume that all the conditions of Lemma 3.1 hold, and $x_\lambda (\lambda > 0)$ is the unique solution of $Ax = \lambda x$. Then,*
*(i) if $0 < \lambda_1 < \lambda_2$, then $x_{\lambda_1} > x_{\lambda_2}$ that is, $x_\lambda$ is strictly decreasing in $\lambda$;*
*(ii) if there is $\gamma \in (0,1)$ such that $\varphi(t) \geq t^\gamma$ for $t \in (0,1)$, then $x_\lambda$ is continuous in $\lambda$, that is, if $\lambda \to \lambda_0$ $(\lambda_0 > 0)$, then $\|x_\lambda - x_{\lambda_0}\| \to 0$;*
*(iii) $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.*

We are next concerned with problem (1.1) in the following space $E$ defined by

$$E = \{u \in C([1,\infty)) : \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\alpha-1}} < \infty\}.$$

From [12], we know that $E$ is a Banach apace with the norm

$$\|u\| = \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\alpha-1}}.$$

We define a cone $P \subset E$ by

$$P = \{u \in E : \ u(t) \geq 0, \ t \in [1,\infty)\}.$$

For $u, \ v \in P$ with $u \leq v$, we have $0 \leq u(t) \leq v(t)$ and thus

$$\sup_{1\leq t<\infty} \frac{u(t)}{1+(\log t)^{\alpha-1}} \leq \sup_{1\leq t<\infty} \frac{v(t)}{1+(\log t)^{\alpha-1}}.$$

So, $\|u\| \leq \|v\|$. Hence $P$ is a normal cone.

**Theorem 3.3.** *Let $\Gamma(\alpha) > \sum_{i=1}^{m} \frac{\gamma_i \Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} (\log \eta)^{\alpha+\beta_i-1}$. Suppose that*
*$(H_1)$ $f : [1,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $f(t,0) \not\equiv 0$, $t \in [1,\infty)$;*
*$(H_2)$ $f(t,u)$ is increasing in $u \in [0,\infty)$ for each $t \in [1,\infty)$;*
*$(H_3)$ if $u$ is bounded, then $f(t,(1+(\log t)^{\alpha-1})u)$ is bounded on $[1,\infty)$;*
*$(H_4)$ for $\tau \in (0,1)$, there exists $\varphi(\tau) \in (\tau,1)$ such that $f(t,\tau u) \geq \varphi(\tau) f(t,u)$, $t \in [1,\infty)$, $u \in [0,\infty)$;*
*$(H_5)$ $a(t)$ is continuous with $0 < \int_1^\infty a(s)\frac{ds}{s} < \infty$.*
*Then:*
*(a) for each $\lambda > 0$, problem (1.1) has a unique solution $u_\lambda^*$ in $P_h$, where $h(t) = (\log t)^{\alpha-1}$, $t \in [1,\infty)$. For $u_0 \in P_h$, defining the sequence*

$$u_n(t) = \lambda \int_1^\infty G(t,s)a(s)f(s,u_{n-1}(s))\frac{ds}{s},\ n = 1,2,\ldots$$

*one has $u_n(t) \to u_\lambda^*(t)$ as $n \to \infty$, where $G(t,s)$ is given (2.2);*
*(b) $u_\lambda^*$ is strictly increasing in $\lambda$, namely, $0 < \lambda_1 < \lambda_2$ ensures $u_{\lambda_1}^* < u_{\lambda_2}^*$;*
*(c) if there exist $\gamma \in (0,1)$ and a nonnegative function $\psi$ such that $\varphi(t) = t^\gamma[1+\psi(t)]$ for $t \in (0,1)$, then $u_\lambda^*$ is continuous in $\lambda$, namely, $\lambda \to \lambda_0 (\lambda_0 > 0)$ ensures $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$;*
*(d) $\lim_{\lambda \to 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda \to +\infty} \|u_\lambda^*\| = \infty$.*

*Proof.* Define an operator $T : E \to E$ by

$$Tu(t) = \int_1^{+\infty} G(t,s)a(s)f(s,u(s))\frac{ds}{s},$$

where $G(t,s)$ is given in (2.2). For $u \in P$, we have $\frac{u(t)}{1+(\log t)^{\alpha-1}} < \infty$, $t \in [1,\infty)$. By $(H_3)$, there exists $M_u > 0$ such that

$$f\left(s,(1+(\log s)^{\alpha-1})\frac{u(s)}{1+(\log s)^{\alpha-1}}\right) \leq M_u.$$

Further, from Lemma 2.4, $(H_3)$ and $(H_5)$, we have

$$\begin{aligned}
& \frac{Tu(t)}{1+(\log t)^{\alpha-1}} \\
= & \int_1^\infty \frac{G(t,s)}{1+(\log t)^{\alpha-1}} a(s)f(s,u(s),u(s))\frac{ds}{s} \\
\leq & \int_1^\infty \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\gamma_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}\right) a(s) f\left(s,(1+(\log s)^{\alpha-1})\frac{u(s)}{1+(\log s)^{\alpha-1}}\right)\frac{ds}{s} \\
\leq & \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\gamma_i (\log \eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha+\beta_i)}\right) M_u \int_1^\infty a(s)\frac{ds}{s} < \infty.
\end{aligned}$$

Also, by $(H_1)$ and Lemma 2.4, $Tu \in C[1,\infty)$. So we get $T : P \to P$. From $(H_2)$, it is easy to know that $T : P \to P$ is increasing.

In the sequel, we check that $T$ satisfies other conditions of Lemma 3.1. Take $h(t) = (\log t)^{\alpha-1}$, $t \in [1,\infty)$. It is clear that

$$\sup_{1 \leq t < \infty} \frac{h(t)}{1+(\log t)^{\alpha-1}} = 1 < \infty,$$

that is, $h \in P$.

Next we mainly prove that $Th \in P_h$. Since $\frac{h(t)}{1+(\log t)^{\alpha-1}} \leq 1$ for $t \in [1,\infty)$, and from $(H_3)$, we find that there exists $M_h > 0$ such that

$$f\left(t,(1+(\log t)^{\alpha-1})\frac{h(t)}{1+(\log t)^{\alpha-1}}\right) \leq M_h. \tag{3.1}$$

Let

$$l_1 = \sum_{i=1}^{m} \frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)} \int_1^m a(s)g_i(\eta,s)f(s,0)\frac{ds}{s},$$

and

$$l_2 = M_h\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Omega}\sum_{i=1}^{m}\frac{\gamma_i(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}\right)\cdot\int_1^\infty a(s)\frac{ds}{s}.$$

In view of

$$\Gamma(\alpha) > \sum_{i=1}^{m}\frac{\gamma_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1} > 0,$$

there exist nonzero elements in $\gamma_1,\ \gamma_2,\cdots,\ \gamma_m$ such that $\sum_{i=1}^{m}\frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)} > 0$. From $(H_1)$ and $(H_5)$, $a(s)f(s,0)$ is continuous with $a(s)f(s,0) \not\equiv 0$ for $s \in [1,\infty)$. Hence, $\int_1^m a(s)f(s,0)\frac{ds}{s} > 0$ and thus $l_1 > 0$. Further, from $(H_2)$, we obtain $M_h \geq f(t,0)$ for $t \in [1,\infty)$. Note that $g_i(\eta,s) \leq (\log\eta)^{\alpha+\beta_i-1}$, we easily get $l_2 \geq l_1$. From $(H_2)$, we have

$$\begin{aligned}
Th(t) &= \int_1^\infty G(t,s)a(s)f(s,(\log s)^{\alpha-1})\frac{ds}{s}\\
&\geq \int_1^\infty G(t,s)a(s)f(s,0)\frac{ds}{s}\\
&\geq \int_1^\infty \sum_{i=1}^{m}\frac{\gamma_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}g_i(\eta,s)a(s)f(s,0)\frac{ds}{s}\\
&\geq \sum_{i=1}^{m}\frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)}\int_1^m a(s)g_i(\eta,s)f(s,0)\frac{ds}{s}\cdot(\log t)^{\alpha-1}\\
&= l_1(\log t)^{\alpha-1} = l_1h(t).
\end{aligned}$$

Using Lemma 2.4, $(H_2)$ and (3.1), we have

$$\begin{aligned}
Th(t) &= \int_1^\infty G(t,s)a(s)f(s,(\log s)^{\alpha-1})\frac{ds}{s}\\
&= \int_1^\infty G(t,s)a(s)f\left(s,(1+(\log s)^{\alpha-1})\frac{(\log s)^{\alpha-1}}{1+(\log s)^{\alpha-1}}\right)\frac{ds}{s}\\
&\leq \int_1^\infty G(t,s)a(s)M_h\frac{ds}{s}\\
&\leq \int_1^\infty \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}a(s)M_h\frac{ds}{s}+\int_1^{+\infty}\sum_{i=1}^{m}\frac{\gamma_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}g_i(\eta,s)a(s)M_h\frac{ds}{s}\\
&\leq M_h\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Omega}\sum_{i=1}^{m}\frac{\gamma_i(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}\right)\cdot\int_1^\infty a(s)\frac{ds}{s}\cdot(\log t)^{\alpha-1}\\
&= l_2(\log t)^{\alpha-1} = l_2h(t).
\end{aligned}$$

Hence, $l_1 h(t) \le Th(t) \le l_2 h(t)$, $t \in [1,\infty)$. Therefore, $l_1 h \le Th \le l_2 h$. That is, $Th \in P_h$.

Next we claim that the condition (ii) of Lemma 3.1 is also satisfied. For $\tau \in (0,1)$, $u \in P$, from $(H_4)$, we have

$$\begin{aligned} T(\tau u)(t) &= \int_1^\infty G(t,s)a(s)f(s,\tau u(s))\frac{ds}{s} \\ &\ge \varphi(\tau)\int_1^\infty G(t,s)a(s)f(s,u(s))\frac{ds}{s} \\ &= \varphi(\tau)Tu(t), \end{aligned}$$

that is, $T(\tau u) \ge \varphi(\tau)Tu$, $\tau \in (0,1)$, $u \in P$. So, $T$ satisfies the conditions of Lemma 3.1. Consequently, by Lemma 3.2, for $\lambda > 0$, there exists a unique $u_\lambda^* \in P_h$ such that $Tu_\lambda^* = \frac{1}{\lambda}u_\lambda^*$. It follows that $\lambda Tu_\lambda^* = u_\lambda^*$ and

$$u_\lambda^*(t) = \lambda\int_1^\infty G(t,s)a(s)f(s,u_\lambda^*(s))\frac{ds}{s}.$$

From Lemma 2.3, $u_\lambda^*$ is a unique positive solution of problem (1.1) in $P_h$. Further, by Lemma 3.2 (i), it can easily check that $u_\lambda^*$ is strictly increasing in $\lambda$, that is, $u_{\lambda_1}^* < u_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$. On the other hand, Lemma 3.2 (iii) ensures $\lim_{\lambda\to\infty}\|u_\lambda^*\| = \infty$, and $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = 0$. Moreover, if $\varphi(t) = t^\gamma[1+\psi(t)]$ for $t \in (0,1)$, then $\varphi(t) \ge t^\gamma$ for $t \in (0,1)$. So, Lemma 3.2 (ii) shows that $u_\lambda^*$ is continuous in $\lambda$, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ means $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$.

Moreover, let $T_\lambda = \lambda T$. Then $T_\lambda$ also satisfies the conditions of Lemma 3.1. Thus, for $u_0 \in P_h$, by defining the sequence $u_n = T_\lambda u_{n-1}$, $n = 1,2,\ldots$, one has $u_n \to u_\lambda^*$ as $n \to \infty$. Namely,

$$u_n(t) = \lambda\int_1^\infty G(t,s)a(s)f(s,u_{n-1}(s))\frac{ds}{s},$$

and $u_n(t) \to u_\lambda^*(t)$ as $n \to \infty$. □

**Corollary 3.4.** *Let $\Gamma(\alpha) > \sum_{i=1}^m \frac{\gamma_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1}$. Assume $(H_1)-(H_5)$ hold. Then the following boundary value problem*

$$\begin{cases} {}^H D^\alpha u(t) + a(t)f(t,u(t)) = 0, \ 1<\alpha<2, \ t \in (1,\infty), \\ u(1) = 0, \ {}^H D^{\alpha-1}u(\infty) = \sum\limits_{i=1}^m \gamma_i {}^H I^{\beta_i}u(\eta), \end{cases}$$

*has a unique positive solution $u^*$ in $P_h$, where $h(t) = (\log t)^{\alpha-1}$, $t \in [1,\infty)$. And, for $u_0 \in P_h$, setting a sequence*

$$u_n(t) = \int_1^\infty G(t,s)a(s)f(s,u_{n-1}(s))\frac{ds}{s}, n = 1,2,\ldots,$$

*we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t,s)$ is given in (2.2).*

## 4. The mixed monotone operator method for problem (1.1) with $F(t,u) := f(t,u,u)$

In this section, we consider (1.1) with $F(t,u) := f(t,u,u)$.

**Definition 4.1.** [15, 16] $A : P\times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in $x$ and decreasing in $y$, i.e., $u_i, v_i (i = 1,2) \in P$, $u_1 \le u_2$, $v_1 \ge v_2$ implies $A(u_1,v_1) \le A(u_2,v_2)$. Element $x \in P$ is called a fixed point of $A$ if $A(x,x) = x$.

**Lemma 4.2.** [16] *Let $P$ be a normal cone of $E$, and let $A: P\times P\to P$ be a mixed monotone operator satisfying:*
*$(A_1)$ there exists $h\in P$ with $h\neq\theta$ such that $A(h,h)\in P_h$;*
*$(A_2)$ for $u,\ v\in P$ and $t\in(0,1)$, there exists $\varphi(t)\in(t,1)$ such that $A(tu,t^{-1}v)\geq\varphi(t)A(u,v)$.*
*Then $A(x,x)=x$ has a unique solution $x^*$ in $P_h$. For $x_0,\ y_0\in P_h$, define the sequences*

$$x_n=A(x_{n-1},\ y_{n-1}),\ y_n=A(y_{n-1},\ x_{n-1}),\ n=1,2,\ldots.$$

*Then $\|x_n-x^*\|\to 0$ and $\|y_n-x^*\|\to 0$ as $n\to\infty$.*

**Lemma 4.3.** [16] *Assume that $A$ satisfies the conditions of Lemma 4.2. Let $x_\lambda$ denote the unique solution of $A(x,x)=\lambda x$ in $P_h$. Then,*
*$(R_1)$ if $\varphi(t)>t^{\frac{1}{2}}$ for $t\in(0,1)$, then $x_\lambda$ is strictly decreasing in $\lambda$, that is, $0<\lambda_1<\lambda_2$ ensures $x_{\lambda_1}>x_{\lambda_2}$;*
*$(R_2)$ if there exists $\beta\in(0,1)$ such that $\varphi(t)\geq t^\beta$ for $t\in(0,1)$, then $x_\lambda$ is continuous in $\lambda$, that is, $\lambda\to\lambda_0(\lambda_0>0)$ ensures $\|x_\lambda-x_{\lambda_0}\|\to 0$;*
*$(R_3)$ if there exists $\beta\in(0,\frac{1}{2})$ such that $\varphi(t)\geq t^\beta$ for $t\in(0,1)$, then*

$$\lim_{\lambda\to\infty}\|x_\lambda\|=0,\ \lim_{\lambda\to 0^+}\|x_\lambda\|=\infty.$$

**Theorem 4.4.** *Let $\Gamma(\alpha)>\sum_{i=1}^m\frac{\gamma_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1}$. Suppose that $(H_3)$, $(H_5)$ hold and*
*$(H_1)'$ $f:[1,\infty)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ is continuous, $f(t,0,(\log m)^{\alpha-1})\not\equiv 0$, $t\in[1,\infty)$;*
*$(H_6)$ $f(t,u,w)$ is increasing in $u$ for $t\in[1,\infty)$ and $w\in[0,\infty)$, decreasing in $w$ for each $t\in[1,\infty)$ and $u\in[0,\infty)$;*
*$(H_7)$ for $\tau\in(0,1)$, there exist $\varphi_1(\tau),\ \varphi_2(\tau)\in(0,1)$ with $\varphi_1(\tau)\varphi_2(\tau)>\tau$ such that*

$$f(t,\tau u,w)\geq\varphi_1(\tau)f(t,u,w),\ f(t,u,\tau w)\leq\frac{1}{\varphi_2(\tau)}f(t,u,w),$$

*for $t\in[1,\infty)$, $u,w\in[0,\infty)$.*
*Then:*
*(a) for fixed $\lambda>0$, problem (1.1) has a unique positive solution $u_\lambda^{**}$ in $P_h$, where $h(t)=(\log t)^{\alpha-1}$, $t\in[1,\infty)$. For $u_0,\ v_0\in P_h$, define two sequences*

$$u_n(t)=\lambda\int_1^\infty G(t,s)a(s)f(s,u_{n-1}(s),v_{n-1}(s))\frac{ds}{s},\ n=1,2,\ldots,$$

$$v_n(t)=\lambda\int_1^\infty G(t,s)a(s)f(s,v_{n-1}(s),u_{n-1}(s))\frac{ds}{s},\ n=1,2,\ldots,$$

*we have $u_n(t)\to u_\lambda^{**}(t)$, $v_n(t)\to u_\lambda^{**}(t)$ as $n\to\infty$, where $G(t,s)$ is given in (2.2);*
*(b) if $\varphi_1(t)\varphi_2(t)>t^{\frac{1}{2}}$ for $t\in(0,1)$, then $u_\lambda^{**}$ is strictly increasing in $\lambda$, that is, $0<\lambda_1<\lambda_2$ can ensure $u_{\lambda_1}^{**}<u_{\lambda_2}^{**}$. If there exists $\beta\in(0,1)$ such that $\varphi_1(t)\varphi_2(t)\geq t^\beta$ for $t\in(0,1)$, then $u_\lambda^{**}$ is continuous in $\lambda$, that is, $\lambda\to\lambda_0(\lambda_0>0)$ ensures $\|u_\lambda^{**}-u_{\lambda_0}^{**}\|\to 0$. If there exists $\beta\in(0,\frac{1}{2})$ such that $\varphi_1(t)\varphi_2(t)\geq t^\beta$ for $t\in(0,1)$, then $\lim_{\lambda\to\infty}\|u_\lambda^{**}\|=\infty$, and $\lim\_\lambda\to 0^+\|u_\lambda^{**}\|=0$.*

*Proof.* Define an operator

$$T(u,v)(t)=\int_1^\infty G(t,s)a(s)f(s,u(s),v(s))\frac{ds}{s},$$

where $G(t,s)$ is given in (2.2). From Theorem 3.3, it can easily prove that $T : P \times P \to P$. Now we prove that $T$ is a mixed monotone operator. Indeed, for $u_i,\ v_i \in P (i = 1,2)$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we have $u_1(t) \geq u_2(t)$ and $v_1(t) \leq v_2(t)$, $t \in [1,\infty)$. By $(H_6)$ and Lemma 2.3, we have

$$\begin{aligned} T(u_1,v_1)(t) &= \int_1^\infty G(t,s)a(s)f(s,u_1(s),v_1(s))\frac{ds}{s} \\ &\geq \int_1^\infty G(t,s)a(s)f(s,u_2(s),v_2(s))\frac{ds}{s} = T(u_2,v_2)(t), \end{aligned}$$

for $t \in [1,\infty)$. That is, $T(u_1,v_1) \geq T(u_2,v_2)$.

In the sequel, we check that $T$ satisfies other conditions of Lemma 4.2. Take $h(t) = (\log t)^{\alpha-1}$, $t \in [1,\infty)$. It is clear that $h \in P$. Next we mainly show that $T(h,h) \in P_h$. Let

$$l_1 = \sum_{i=1}^m \frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)} \int_1^m a(s)g_i(\eta,s)f(s,0,(\log m)^{\alpha-1})\frac{ds}{s},$$

and

$$l_2 = M_h\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Omega}\sum_{i=1}^m \frac{\gamma_i(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}\right)\cdot\int_1^\infty a(s)\frac{ds}{s},$$

where $M_h$ is given as in (3.1). In view of

$$\Gamma(\alpha) > \sum_{i=1}^m \frac{\gamma_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1} > 0,$$

we obtain that

$$\sum_{i=1}^m \frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)} > 0.$$

From $(H_1)'$ and $(H_5)$, we have that $a(s)f(s,0,(\log m)^{\alpha-1})$ is continuous with

$$a(s)f(s,0,(\log m)^{\alpha-1}) \not\equiv 0$$

for $s \in [1,\infty)$. Hence, $\int_1^m a(s)f(s,0,(\log m)^{\alpha-1})\frac{ds}{s} > 0$ and $l_1 > 0$. Further, from $(H_3),(H_6)$, we obtain $M_h \geq f(t,0,(\log m)^{\alpha-1})$ for $t \in [1,\infty)$. Note that $g_i(\eta,s) \leq (\log\eta)^{\alpha+\beta_i-1}$. It follows that $l_2 \geq l_1$. From $(H_6)$, we have

$$\begin{aligned} T(h,h)(t) &= \int_1^\infty G(t,s)a(s)f(s,(\log s)^{\alpha-1},(\log s)^{\alpha-1})\frac{ds}{s} \\ &\geq \int_1^\infty G(t,s)a(s)f(s,0,(\log s)^{\alpha-1})\frac{ds}{s} \\ &\geq \int_1^\infty \sum_{i=1}^m \frac{\gamma_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)} g_i(\eta,s)a(s)f(s,0,(\log s)^{\alpha-1})\frac{ds}{s} \\ &\geq \sum_{i=1}^m \frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)}\int_1^m a(s)g_i(\eta,s)f(s,0,(\log s)^{\alpha-1})\frac{ds}{s}\cdot(\log t)^{\alpha-1} \\ &\geq \sum_{i=1}^m \frac{\gamma_i}{\Omega\Gamma(\alpha+\beta_i)}\int_1^m a(s)g_i(\eta,s)f(s,0,(\log m)^{\alpha-1})\frac{ds}{s}\cdot(\log t)^{\alpha-1} \\ &= l_1(\log t)^{\alpha-1} = l_1 h(t). \end{aligned}$$

Also, from Lemma 2.4 and (3.1), we have

$$\begin{aligned}
T(h,h)(t) &= \int_1^\infty G(t,s)a(s)f(s,(\log s)^{\alpha-1},(\log s)^{\alpha-1})\frac{ds}{s} \\
&= \int_1^\infty G(t,s)a(s)f(s,(1+(\log s)^{\alpha-1})\frac{(\log s)^{\alpha-1}}{1+(\log s)^{\alpha-1}}, \\
&\quad (1+(\log s)^{\alpha-1})\frac{(\log s)^{\alpha-1}}{1+(\log s)^{\alpha-1}})\frac{ds}{s} \\
&\le \int_1^\infty G(t,s)a(s)M_h\frac{ds}{s} \\
&\le \int_1^\infty \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}a(s)M_h\frac{ds}{s} + \int_1^{+\infty}\sum_{i=1}^m \frac{\gamma_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}g_i(\eta,s)a(s)M_h\frac{ds}{s} \\
&\le M_h\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Omega}\sum_{i=1}^m \frac{\gamma_i(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}\right)\cdot\int_1^\infty a(s)\frac{ds}{s}\cdot(\log t)^{\alpha-1} \\
&= l_2(\log t)^{\alpha-1} = l_2h(t).
\end{aligned}$$

Hence, $l_1h(t)\le T(h,h)(t)\le l_2h(t)$, $t\in[1,\infty)$. Therefore, $T(h,h)\in P_h$.

Next, we prove that the condition (ii) of Lemma 4.2 is also satisfied. From $(H_7)$, for $\tau\in(0,1)$, we can get $f(t,u,\tau^{-1}v)\ge\varphi_2(\tau)f(t,u,v)$ for any $t\in[1,\infty)$, $u,v\in[0,\infty)$, and then

$$\begin{aligned}
T(\tau u,\tau^{-1}v)(t) &= \int_1^\infty G(t,s)a(s)f(s,\tau u(s),\tau^{-1}v(s))\frac{ds}{s} \\
&\ge \varphi_1(\tau)\int_1^\infty G(t,s)a(s)f(s,u(s),\tau^{-1}v(s))\frac{ds}{s} \\
&\ge \varphi_1(\tau)\varphi_2(\tau)\int_1^\infty G(t,s)a(s)f(s,u(s),v(s))\frac{ds}{s} \\
&= \varphi_1(\tau)\varphi_2(\tau)T(u,v)(t).
\end{aligned}$$

Let $\varphi(t)=\varphi_1(t)\varphi_2(t)$, $t\in(0,1)$. From $(H_7)$, $\varphi(t)\in(t,1)$ for $t\in(0,1)$. Hence, $T(\tau u,\tau^{-1}v)\ge\varphi(\tau)T(u,v)$, for any $\tau\in(0,1)$, $u,v\in P$. So, $T$ satisfies the conditions of Lemma 4.2. Therefore, Lemma 4.3 ensures that there exists a unique $u_\lambda^{**}\in P_h$ such that $T(u_\lambda^{**},u_\lambda^{**})=\frac{1}{\lambda}u_\lambda^{**}$. So, $\lambda T(u_\lambda^{**},u_\lambda^{**})=u_\lambda^{**}$. From Lemma 2.3, we have that $u_\lambda^{**}$ is a unique positive solution of problem (1.1) for given $\lambda>0$. Further, if $\varphi_1(t)\varphi_2(t)>t^{\frac{1}{2}}$ for $t\in(0,1)$, then Lemma 4.3 $(R_1)$ ensures that $u_\lambda^{**}$ is strictly increasing in $\lambda$, that is, $0<\lambda_1<\lambda_2$ can guarantee $u_{\lambda_1}^{**}<u_{\lambda_2}^{**}$. If there exists $\beta\in(0,1)$ such that $\varphi_1(t)\varphi_2(t)\ge t^\beta$ for $t\in(0,1)$, then Lemma 4.3 $(R_2)$ tells us that $u_\lambda^{**}$ is continuous in $\lambda$, that is, $\lambda\to\lambda_0(\lambda_0>0)$ ensures $\|u_\lambda^{**}-u_{\lambda_0}^{**}\|\to0$. If there exists $\beta\in(0,\frac{1}{2})$ such that $\varphi_1(t)\varphi_2(t)\ge t^\beta$ for $t\in(0,1)$, then Lemma 4.3 $(R_3)$ tells us $\lim_{\lambda\to\infty}\|u_\lambda^{**}\|=\infty$, and $\lim_{\lambda\to0^+}\|u_\lambda^{**}\|=0$.

Obviously, $\lambda T$ also satisfies the conditions of Lemma 4.2. By Lemma 4.2, for $u_0,\ v_0\in P_h$, two sequences $u_{n+1}=\lambda T(u_n,v_n)$, $v_{n+1}=\lambda T(v_n,u_n)$, $n=0,1,2,\dots$, satisfy $u_n\to u_\lambda^{**}$ and $v_n\to u_\lambda^{**}$ as $n\to\infty$. That is,

$$u_n(t)=\lambda\int_1^\infty G(t,s)a(s)f(s,u_{n-1}(s),v_{n-1}(s))\frac{ds}{s},\ n=1,2,\dots,$$

and

$$v_n(t) = \lambda \int_1^\infty G(t,s)a(s)f(s, v_{n-1}(s), u_{n-1}(s))\frac{ds}{s},\ n = 1,2,\ldots,$$

satisfy $u_n(t) \to u^{**}(t)$, and $v_n(t) \to v^{**}(t)$ as $n \to \infty$. $\square$

**Corollary 4.5.** *Let $\Gamma(\alpha) > \sum_{i=1}^{m} \frac{\gamma_i \Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log \eta)^{\alpha+\beta_i-1}$. Assume $(H_1)'$, $(H_3)$, $(H_5)$, $(H_6)$, $(H_7)$ hold. Then the following boundary value problem*

$$\begin{cases} {}^H D^{\alpha} u(t) + a(t) f(t, u(t), u(t)) = 0,\ 1 < \alpha < 2,\ t \in (1, \infty), \\ u(1) = 0,\ {}^H D^{\alpha-1} u(\infty) = \sum\limits_{i=1}^{m} \gamma_i {}^H I^{\beta_i} u(\eta), \end{cases}$$

*has a unique positive solution $u^{**}$ in $P_h$, where $h(t) = (\log t)^{\alpha-1}$, $t \in [1, \infty)$. For $u_0,\ v_0 \in P_h$, defining two sequences*

$$u_n(t) = \int_1^\infty G(t,s)a(s)f(s, u_{n-1}(s), v_{n-1}(s))\frac{ds}{s},\ n = 1,2,\ldots,$$

*and*

$$v_n(t) = \int_1^\infty G(t,s)a(s)f(s, v_{n-1}(s), u_{n-1}(s))\frac{ds}{s},\ n = 1,2,\ldots,$$

*we have $u_n(t) \to u^{**}(t)$, and $v_n(t) \to u^{**}(t)$ as $n \to \infty$, where $G(t,s)$ is given in (2.2).*

## 5. Examples

**Example 5.1.** We consider the following Hadamard fractional problem

$$\begin{cases} {}^H D^{\frac{3}{2}} u(t) + \lambda t^{-2} \left( \frac{u^{\frac{1}{3}}(t)}{1+(\log t)^{\frac{1}{2}}} + \frac{u^{\frac{1}{5}}(t)}{1+(\log t)^{\frac{1}{2}}} + 1 \right) = 0,\ t \in (1, \infty), \\ u(1) = 0,\ {}^H D^{\frac{1}{2}} u(\infty) = {}^H I^{\frac{3}{2}} u(e) + 2 {}^H I^{\frac{5}{2}} u(e), \end{cases} \tag{5.1}$$

where $\alpha = \frac{3}{2}$, $m = 2$, $\eta = e$, $\gamma_1 = 1$, $\gamma_2 = 2$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{5}{2}$, $a(t) = t^{-2}$ and

$$f(t,u,v) = \frac{u^{\frac{1}{3}}}{1+(\log t)^{\frac{1}{2}}} + \frac{v^{\frac{1}{5}}}{1+(\log t)^{\frac{1}{2}}} + 1.$$

Then

$$\Omega = \Gamma(\frac{3}{2}) - \sum_{i=1}^{2} \frac{\gamma_i \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\beta_i)} (\log e)^{\frac{3}{2}+\beta_i-1} \approx 0.1477 > 0.$$

Evidently, $f(t,u,v)$ satisfies $(H_1)$, $(H_2)$. If $u$ is bounded, then

$$f(t, (1+(\log t)^{\frac{1}{2}})u, (1+(\log t)^{\frac{1}{2}})u) = u^{\frac{1}{3}} + u^{\frac{1}{5}} + 1 < \infty$$

for $t \in [1, \infty)$, so $(H_3)$ is also satisfied with $f(t,0,0) = 1 > 0$. Let $\varphi(\tau) = \tau^{\frac{1}{3}}$. Then $\varphi(\tau) \in (\tau, 1)$ for $\tau \in (0,1)$. For $\tau \in (0,1)$, $u, v \geq 0$, we have

$$\begin{aligned} f(t, \tau u, \tau v) &= \frac{\tau^{\frac{1}{3}} u^{\frac{1}{3}}}{1+(\log t)^{\frac{1}{2}}} + \frac{\tau^{\frac{1}{5}} v^{\frac{1}{5}}}{1+(\log t)^{\frac{1}{2}}} + 1 \\ &\geq \tau^{\frac{1}{3}} \left( \frac{u^{\frac{1}{3}}}{1+(\log t)^{\frac{1}{2}}} + \frac{v^{\frac{1}{5}}}{1+(\log t)^{\frac{1}{2}}} + 1 \right) \\ &= \varphi(\tau) f(t,u,v). \end{aligned}$$

Moreover,

$$\int_1^\infty a(t)\frac{dt}{t} = \int_1^\infty t^{-2}\frac{dt}{t} = \frac{1}{2} < \infty.$$

So conditions $(H_4)$, $(H_5)$ are satisfied. In addition, $\varphi(t) = t^{\frac{1}{3}}[1+\psi(t)]$, where $\psi(t) \equiv 0$. From Theorem 3.3, we can claim that:
(a) for each $\lambda > 0$, problem (5.1) has a unique solution $u_\lambda^*$ in $P_h$, where $h(t) = (\log t)^{\frac{1}{2}}, t \in [1,\infty)$. For $u_0 \in P_h$, defining a sequence

$$u_n(t) = \lambda \int_1^\infty G(t,s)s^{-2}\left(\frac{u_{n-1}^{\frac{1}{3}}(s)}{1+(\log s)^{\frac{1}{2}}} + \frac{u_{n-1}^{\frac{1}{5}}(s)}{1+(\log s)^{\frac{1}{2}}} + 1\right)\frac{ds}{s},\ n = 1,2,\ldots,$$

we have $u_n(t) \to u_\lambda^*(t)$ as $n \to \infty$, where $G(t,s)$ is given (2.2);
(b) $u_\lambda^*$ is strictly increasing in $\lambda$, namely, $0 < \lambda_1 < \lambda_2$ ensures $u_{\lambda_1}^* < u_{\lambda_2}^*$;
(c) $u_\lambda^*$ is continuous in $\lambda$, namely, $\lambda \to \lambda_0 (\lambda_0 > 0)$ ensures $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$;
(d) $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = 0$, and $\lim_{\lambda\to+\infty}\|u_\lambda^*\| = \infty$.

**Example 5.2.** We consider the following Hadamard fractional problem

$$\begin{cases} {}^H D^{\frac{3}{2}}u(t) + \lambda t^{-2}\left(\frac{u^{\frac{1}{6}}(t)}{1+(\log t)^{\frac{1}{2}}} + \frac{[u(t)+1]^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}}\right) = 0,\ t \in (1,\infty), \\ u(1) = 0,\ {}^H D^{\frac{1}{2}}u(\infty) = {}^H I^{\frac{3}{2}}u(e) + 2{}^H I^{\frac{5}{2}}u(e), \end{cases} \tag{5.2}$$

where $\alpha = \frac{3}{2}$, $m = 2$, $\eta = e$, $\gamma_1 = 1$, $\gamma_2 = 2$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{5}{2}$, $a(t) = t^{-2}$ and

$$f(t,u,v) = \frac{u^{\frac{1}{6}}}{1+(\log t)^{\frac{1}{2}}} + \frac{(v+1)^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}}.$$

Then

$$\Omega = \Gamma(\frac{3}{2}) - \sum_{i=1}^{2}\frac{\gamma_i\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\beta_i)}(\log e)^{\frac{3}{2}+\beta_i-1} \approx 0.1477 > 0.$$

Evidently, $f(t,u,v)$ satisfies $(H_1)'$, $(H_3)$, $(H_5)$ and $(H_6)$ with

$$f(t,0,(\log m)^{\alpha-1}) = \frac{[(\log 2)^{\frac{1}{2}}+1]^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}} > 0.$$

Let $\varphi_1(\tau) = \tau^{\frac{1}{6}}$, and $\varphi_2(\tau) = \tau^{\frac{1}{4}}$. Then $\varphi_1(\tau), \varphi_2(\tau) \in (\tau,1)$ for $\tau \in (0,1)$. For $\tau \in (0,1)$, we have $\varphi_1(\tau)\varphi_2(\tau) = \tau^{\frac{5}{12}} > \tau$,

$$\begin{aligned} f(t,\tau u,v) &= \frac{\tau^{\frac{1}{6}}u^{\frac{1}{6}}}{1+(\log t)^{\frac{1}{2}}} + \frac{(v+1)^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}} \\ &\geq \tau^{\frac{1}{6}}\left(\frac{u^{\frac{1}{6}}}{1+(\log t)^{\frac{1}{2}}} + \frac{(v+1)^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}}\right) \\ &= \varphi_1(\tau)f(t,u,v), \end{aligned}$$

and

$$\begin{aligned}f(t,u,\tau v) &= \frac{u^{\frac{1}{6}}}{1+(\log t)^{\frac{1}{2}}} + \frac{(\tau v+1)^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}}\\ &\le \tau^{-\frac{1}{4}}\left(\frac{u^{\frac{1}{6}}}{1+(\log t)^{\frac{1}{2}}} + \frac{(v+1)^{-\frac{1}{4}}}{1+(\log t)^{\frac{1}{2}}}\right)\\ &= \frac{1}{\varphi_2(\tau)} f(t,u,v),\end{aligned}$$

for $t \in [1,\infty)$, $u,v \in [0,\infty)$. Hence, all the conditions of Theorem 4.4 are satisfied. So we can claim that:

(a) for each $\lambda > 0$, problem (5.2) has a unique solution $u_\lambda^{**}$ in $P_h$, where $h(t) = (\log t)^{\frac{1}{2}}$, $t \in [1,\infty)$. For $u_0, v_0 \in P_h$, constructing the sequences

$$u_n(t) = \lambda \int_1^\infty G(t,s)s^{-2}\left(\frac{u_{n-1}^{\frac{1}{6}}(s)}{1+(\log s)^{\frac{1}{2}}} + \frac{(v_{n-1}(s)+1)^{-\frac{1}{4}}}{1+(\log s)^{\frac{1}{2}}}\right)\frac{ds}{s},\ n = 1,2,\ldots,$$

and

$$v_n(t) = \lambda \int_1^\infty G(t,s)s^{-2}\left(\frac{v_{n-1}^{\frac{1}{6}}(s)}{1+(\log s)^{\frac{1}{2}}} + \frac{(u_{n-1}(s)+1)^{-\frac{1}{4}}}{1+(\log s)^{\frac{1}{2}}}\right)\frac{ds}{s},\ n = 1,2,\ldots.$$

we have $u_n(t) \to u_\lambda^{**}(t)$ as $n \to \infty$, where $G(t,s)$ is given (2.2);

(b) Since $\varphi_1(\tau)\varphi_2(\tau) = \tau^{\frac{5}{12}} > \tau^{\frac{1}{2}}$ for $\tau \in (0,1)$, we find from Theorem 4.4 that $u_\lambda^{**}$ is strictly increasing in $\lambda$, that is, $0 < \lambda_1 < \lambda_2$ ensures $u_{\lambda_1}^{**} < u_{\lambda_2}^{**}$. Taking $\beta \in [\frac{5}{12}, \frac{1}{2})$ and using Theorem 4.4, we know that $u_\lambda^{**}$ is continuous in $\lambda$ and $\lim_{\lambda\to\infty}\|u_\lambda^{**}\| = \infty$, and $\lim_{\lambda\to 0^+}\|u_\lambda^{**}\| = 0$.

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