J. Nonlinear Funct. Anal. 2020 (2020), Article ID 20 https://doi.org/10.23952/jnfa.2020.20

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

DIFFERENCE INEQUALITIES FOR THE i-TH MIXED AFFINE SURFACE AREAS

XIA ZHAO[1], WEIDONG WANG[2,*]

[1]Department of Mathematics, China Three Gorges University, Yichang, China
[2]Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, China

Abstract. Lutwak introduced the notion of i-th mixed affine surface areas and Leng defined the volume difference functions. In this paper, we establish difference inequalities of Brunn-Minkowski type and cycle type for i-th mixed affine surface areas. In particular, we obtain the Minkowski type inequality for the difference of i-th mixed affine surface areas.

Keywords. i-th mixed affine surface area; Difference; Brunn-Minkowski inequality; Cycle inequality; Minkowski inequality.

1. INTRODUCTION

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. Further, let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unite sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of the body K.

The classic affine surface area is one of the important concepts in affine differential geometry. The results regarding the classic affine surface area have some surprising applications. A number of geometric inequalities obtained by associating with classic affine surface area, such as, the affine isoperimetric inequality [1] and Petty affine projection inequality [2]. In particular, Petty [2] obtained an associated result for affine surface area on the shephard type problem for projection bodies (see [3, 4]).

A body $K \in \mathscr{K}^n$ is said to have a curvature function $f(K,\cdot)$: $S^{n-1} \to \mathbb{R}$, which is defined by (see [4])

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot), \tag{1.1}$$

*Corresponding author.
E-mail addresses: 1813845707@qq.com.com (X. Zhao), wangwd722@163.com (W. Wang)
Received December 8, 2019; Accepted April 23, 2020.
©2020 Journal of Nonlinear Functional Analysis

where the surface area measure $S(K,\cdot)$ of K is absolutely continuous with respect to spherical Lebesgue measure S. For convenience, we write $f(N,\cdot)=f_N$. In addition, we write $\mathscr{F}^n$ as the subset of $\mathscr{K}^n$ that has a positive continuous curvature function.

In 1985, Petty [1], based on the notion of curvature function of convex body, gave the definition of the affine surface area. If $K\in\mathscr{F}^n$, then the affine surface area $\Omega(K)$ of K is defined by

$$\Omega(K)=\int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}}dS(u). \tag{1.2}$$

Based on the classical affine surface area, Lutwak [5] introduced the concept of the mixed affine surface area. If $K_1,\ \cdots,\ K_n\in\mathscr{F}^n$, then mixed affine surface area $\Omega(K_1,\cdots,K_n)$ is defined by

$$\Omega(K_1,\cdots,K_n)=\int_{S^{n-1}}[f(K_1,u)\cdots f(K_n,u)]^{\frac{1}{n+1}}dS(u). \tag{1.3}$$

Let $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$ $(i=0,\ \cdots,\ n)$ in (1.3), then $\Omega_i(K,L)=\Omega(\underbrace{K,\ \cdots,\ K}_{n-i},\underbrace{L,\cdots,\ L}_{i})=\Omega(K,n-i;\ L,i)$, where K appears $n-i$ times and L appears i times.

Further, if $i\in\mathbb{R}$, Lutwak [5] gave i-th mixed affine surface areas as follows. For $K,L\in\mathscr{F}^n$, i, any real number, the i-th mixed affine surface area, $\Omega_i(K,L)$ of K,L is defined by

$$\Omega_i(K,L)=\int_{S^{n-1}} f(K,u)^{\frac{n-i}{n+1}}f(L,u)^{\frac{i}{n+1}}dS(u). \tag{1.4}$$

In particular, if $L=B$ in (1.4), we write $\Omega_i(K,B)=\Omega_i(K)$ and notice that $f(B,\ \cdot)=1$. Then i-th affine surface area, $\Omega_i(K)$ of K is given by (see [5])

$$\Omega_i(K)=\int_{S^{n-1}} f(K,u)^{\frac{n-i}{n+1}}dS(u). \tag{1.5}$$

From (1.2) and (1.4), we easily get

$$\Omega_i(K,\ K)=\Omega(K),\qquad \Omega_0(K,\ L)=\Omega(K),\qquad \Omega_n(K,\ L)=\Omega(L). \tag{1.6}$$

It is easy to obtain from (1.2) and (1.5) that $\Omega_0(K)=\Omega(K)$. Regarding the studies of affine surface areas, many results were obtained recently; see, e.g., [6, 7, 8, 9, 10, 11, 12, 13] and the references therein. In 2004, Leng [14] defined the volume difference function of convex bodies D and K, where $D\subseteq K$,

$$D_v(K,D)=V(K)-V(D).$$

Meanwhile, Leng established the following Brunn-Minkowski type inequality for the volume difference function.

Theorem 1.1. *If K,L and D are compact domains, $D\subseteq K$, $D'\subseteq L$, D' is a homothetic copy of D, then*

$$(V(K+L)-V(D+D'))^{\frac{1}{n}}\geq(V(K)-V(D))^{\frac{1}{n}}+(V(L)-V(D'))^{\frac{1}{n}},$$

equality holds if and only if K and L are homothetic and $(V(K),V(D))=\mu(V(L),V(D'))$, where μ is a constant.

In 2010, Lv [15] introduced the dual volume difference function for star bodies and established the following dual cycle type inequality.

Theorem 1.2. *Suppose that $M,N,K,L \in \mathscr{S}_o^n$, if $K \subseteq M$, $L \subseteq N$ and N is a dilate of M, then for $i,j,k \in \mathbb{R}$, $0 \le i < j < k \le n$,*

$$(\widetilde{V}_j(M,\ N) - \widetilde{V}_j(K,\ L))^{k-i} \ge (\widetilde{V}_k(M,\ N) - \widetilde{V}_k(K,\ L))^{j-i}(\widetilde{V}_i(M,\ N) - \widetilde{V}_i(K,\ L))^{k-j},$$

with equality if and only if K and L are dilates and $(M,\ K) = \lambda(N,\ L)$, where λ is a constant, where $\widetilde{V}_i(M,N)$ denotes the i-th dual mixed volume of M and N (see also [16]).

From further results on the volume differences functions, we refer to [17, 18, 19] and the references therein.

2. Preliminaries

First, we introduce the Blaschke combination as follows (see [20, 21]). For $K,L \in \mathscr{K}^n$ and $\lambda,\mu \ge 0$ (not both zero), the Blaschke combination, $\lambda \odot K \mp \mu \odot L$, of K and L is defined by

$$S(\lambda \odot K \mp \mu \odot L,\ \cdot) = \lambda S(K,\cdot) + \mu S(L,\cdot),$$

where "$\mp$" denotes the Blaschke addition and "$\odot$" denotes the Blaschke scalar multiplication.

From (1.1) and the Blaschke combination, we easily know that if $K,L \in \mathscr{F}^n$, the Blaschke addition of K, L satisfies

$$f(K \mp L,\ \cdot) = f(K,\ \cdot) + f(L,\ \cdot). \tag{2.1}$$

From the Blaschke combination, Lutwak obtained the following result (see [5]).

Lemma 2.1. *For K, $L \in \mathscr{F}^n$ and $i \in \mathbb{R}$, if $i > -1$, then*

$$\Omega_i(K \mp L)^{\frac{n+1}{n-i}} \ge \Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}; \tag{2.2}$$

if $i < -1$, then

$$\Omega_i(K \mp L)^{\frac{n+1}{n-i}} \le \Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}. \tag{2.3}$$

In each case, equality holds if and only if K and L are homothetic.

Lemma 2.2. [22] *(Bellman's inequality) Let $\mathbf{a} = \{a_1,\cdots,a_n\}$ and $\mathbf{b} = \{b_1,\cdots,b_n\}$ be two series of positive real numbers, and $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. If $p < 0$ or $0 < p < 1$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \ge \left((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\right)^{\frac{1}{p}}; \tag{2.4}$$

if $p > 1$, then

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \le \left((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\right)^{\frac{1}{p}}. \tag{2.5}$$

In each case, equality holds if and only if $\mathbf{a} = v\mathbf{b}$, where v is a constant.

Lemma 2.3. [15] *Suppose that g_i, h_i $(i = 1,\ 2)$ are non-negative continuous functions on S^{n-1} such that*

$$\int_{S^{n-1}} g_1^p(\xi)d\xi \ge \int_{S^{n-1}} g_2^p(\xi)d\xi;$$

$$\int_{S^{n-1}} h_1^q(\xi)d\xi \ge \int_{S^{n-1}} h_2^q(\xi)d\xi,$$

for $p>1, \frac{1}{p}+\frac{1}{q}=1$, *and*

$$g_1^p(\xi)=\lambda h_1^q(\xi),\ \forall \xi\in S^{n-1},$$

where λ *is a constant. Then*

$$\left(\int_{S^{n-1}}(g_1^p(\xi)-g_2^p(\xi))d\xi\right)^{\frac{1}{p}}\left(\int_{S^{n-1}}(h_1^q(\xi)-h_2^q(\xi))d\xi\right)^{\frac{1}{q}}$$
$$\leq\int_{S^{n-1}}(g_1(\xi)h_1(\xi))-(g_2(\xi)h_2(\xi))d\xi,$$

and "=" holds if and only if $g_2^p(\xi)=\lambda h_2^q(\xi)$ *for any* $\xi\in S^{n-1}$, *where* λ *is as above.*

Lemma 2.4. [5] *If* $K_1,\cdots,K_n\in\mathscr{F}^n$, *and* $1<m\leq n$, *then*

$$\prod_{i=0}^{m-1}\Omega(K_1,\cdots,K_{n-m},K_{n-i},\cdots,K_{n-i})\geq\Omega^m(K_1,\cdots,K_n),$$

and "=" holds if and only if $K_{n-m+1},\cdots,K_n$ *are homothetic.*

Lemma 2.5. [23] *If* $c_i>0,\ b_i>0,\ c_i>b_i,\ i=1,\cdots,n$, *then*

$$\left(\prod_{i=1}^n(c_i-b_i)\right)^{\frac{1}{n}}\leq\left(\prod_{i=1}^n c_i\right)^{\frac{1}{n}}-\left(\prod_{i=1}^n b_i\right)^{\frac{1}{n}},$$

and "=" holds if and only if $\frac{c_1}{b_1}=\frac{c_2}{b_2}=\cdots=\frac{c_n}{b_n}$.

3. Main results

In this section, we establish the difference inequalities of Brunn-Minkowski type and cycle type for i-th mixed affine surface areas. Our main results read as follows.

Theorem 3.1. *Let* $K,L,M\in\mathscr{F}^n$, N *be a homothetic copy of* M, *and* $f_K\leq f_M$, $f_L\leq f_N$, $i\in\mathbb{R}$. *If* $i>-1$ *and* $i\neq n$, *then*

$$\left(\Omega_i(M\mp N)-\Omega_i(K\mp L)\right)^{\frac{n+1}{n-i}}\leq\left(\Omega_i(M)-\Omega_i(K)\right)^{\frac{n+1}{n-i}}+\left(\Omega_i(N)-\Omega_i(L)\right)^{\frac{n+1}{n-i}};\tag{3.1}$$

if $i<-1$, *then*

$$\left(\Omega_i(M\mp N)-\Omega_i(K\mp L)\right)^{\frac{n+1}{n-i}}\geq\left(\Omega_i(M)-\Omega_i(K)\right)^{\frac{n+1}{n-i}}+\left(\Omega_i(N)-\Omega_i(L)\right)^{\frac{n+1}{n-i}}.\tag{3.2}$$

In each case, equality holds if and only if K *and* L *are homothetic, and* $(\Omega_i(K),\Omega_i(M))=\mu(\Omega_i(L),\Omega_i(N))$, *where* μ *is a constant. Here* $M\mp N$ *denotes the Blaschke addition of* M *and* N.

Proof. Since $f_K\leq f_M$, $f_L\leq f_N$, we get from (2.1) that

$$\begin{aligned}f(M\mp N,\ \cdot)&=f(M,\ \cdot)+f(N,\ \cdot)\\&\geq f(K,\ \cdot)+f(L,\ \cdot)\\&=f(K\mp L,\ \cdot),\end{aligned}$$

which together with (1.5) entails that

$$\Omega_i(M)\geq\Omega_i(K),\qquad\Omega_i(N)\geq\Omega_i(L),\qquad\Omega_i(M\mp N)\geq\Omega_i(K\mp L).$$

If $i > -1$, we have from (2.2) that

$$\Omega_i(K \mp L)^{\frac{n+1}{n-i}} \geq \Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}, \tag{3.3}$$

and "=" holds if and only if K and L are homothetic. Since N and M are homothetic, one has

$$\Omega_i(M \mp N)^{\frac{n+1}{n-i}} = \Omega_i(M)^{\frac{n+1}{n-i}} + \Omega_i(N)^{\frac{n+1}{n-i}}. \tag{3.4}$$

Hence, if $-1 < i < n$, then $\frac{n-i}{n+1} > 0$. By (3.3) and (3.4), we obtain that

$$\begin{aligned} &\Omega_i(M \mp N) - \Omega_i(K \mp L) \\ &\leq \left(\Omega_i(M)^{\frac{n+1}{n-i}} + \Omega_i(N)^{\frac{n+1}{n-i}}\right)^{\frac{n-i}{n+1}} - \left(\Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}\right)^{\frac{n-i}{n+1}}. \end{aligned} \tag{3.5}$$

Let

$$\{a_1, \cdots, a_n\} = \{\Omega_i(M)^{\frac{n+1}{n-i}}, \Omega_i(K)^{\frac{n+1}{n-i}}, 0, \cdots, 0\}$$

and

$$\{b_1, \cdots, b_n\} = \{\Omega_i(N)^{\frac{n+1}{n-i}}, \Omega_i(L)^{\frac{n+1}{n-i}}, 0, \cdots, 0\},$$

and observe that $0 < p = \frac{n-i}{n+1} < 1$. Then inequality (3.5) and Bellman's inequality (2.4) yield

$$\begin{aligned} &\left(\Omega_i(M \mp N) - \Omega_i(K \mp L)\right)^{\frac{n+1}{n-i}} \\ &\leq \left[\left(\Omega_i(M)^{\frac{n+1}{n-i}} + \Omega_i(N)^{\frac{n+1}{n-i}}\right)^{\frac{n-i}{n+1}} - \left(\Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}\right)^{\frac{n-i}{n+1}}\right]^{\frac{n+1}{n-i}} \\ &\leq \left(\Omega_i(M) - \Omega_i(K)\right)^{\frac{n+1}{n-i}} + \left(\Omega_i(N) - \Omega_i(L)\right)^{\frac{n+1}{n-i}}. \end{aligned} \tag{3.6}$$

This is just the inequality (3.1).

If $i > n$, then $\frac{n-i}{n+1} < 0$. From this, (3.3) and (3.4), we have

$$\begin{aligned} &\Omega_i(M \mp N) - \Omega_i(K \mp L) \\ &\geq \left(\Omega_i(M)^{\frac{n+1}{n-i}} + \Omega_i(N)^{\frac{n+1}{n-i}}\right)^{\frac{n-i}{n+1}} - \left(\Omega_i(K)^{\frac{n+1}{n-i}} + \Omega_i(L)^{\frac{n+1}{n-i}}\right)^{\frac{n-i}{n+1}}. \end{aligned} \tag{3.7}$$

In view of $p = \frac{n-i}{n+1} < 0$, we have that (3.7) and (2.4) deduce inequality (3.6), i.e., inequality (3.1) is also obtained. Similarly, for $i < -1$, by use of (2.5), (2.3) and their equality conditions, we get the desired inequality (3.2). From the equality conditions of Lemma 2.1 and Lemma 2.2, we know that the equality holds in (3.1) and (3.2) if and only if K and L are homothetic, and $(\Omega_i(K), \Omega_i(M)) = \mu((\Omega_i(L), \Omega_i(N))$, where μ is a constant. □

Obviously, if $i = 0$ in Theorem 3.1, then we obtain the Brunn-Minkowski type inequality for difference of affine surface areas.

Corollary 3.2. *If $K, L, M \in \mathscr{F}^n$, N is a homothetic copy of M, $f_K \leq f_M$, $f_L \leq f_N$, then*

$$\left(\Omega(M \mp N) - \Omega(K \mp L)\right)^{\frac{n+1}{n}} \leq \left(\Omega(M) - \Omega(K)\right)^{\frac{n+1}{n}} + \left(\Omega(N) - \Omega(L)\right)^{\frac{n+1}{n}},$$

and "=" holds if and only if K and L are homothetic, and $(\Omega(K), \Omega(M)) = \mu(\Omega(L), \Omega(N))$, where μ is a constant.

Theorem 3.3. *Let $K,L,M\in\mathscr{F}^n$, N be a homothetic copy of M, $f_K\le f_M$, $f_L\le f_N$, $i,\ j,\ k\in\mathbb{R}$. If $0\le i<j<k$, then*

$$\Big(\Omega_j(M,\ N)-\Omega_j(K,\ L)\Big)^{k-i}\ge\Big(\Omega_i(M,\ N)-\Omega_i(K,\ L)\Big)^{k-j}\Big(\Omega_k(M,\ N)-\Omega_k(K,\ L)\Big)^{j-i},\tag{3.8}$$

and "=" holds if and only if K and L are homothetic and $(f_K,f_M)=\lambda(f_L,f_N)$, where λ is a constant.

Proof. For $i<j<k$, let $p=\frac{k-i}{j-i}$, $q=\frac{k-i}{k-j}$. Thus $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let

$$g_1=\Big(f(M,u)^{\frac{n-k}{n+1}}f(N,u)^{\frac{k}{n+1}}\Big)^{\frac{1}{p}},\qquad h_1=\Big(f(M,u)^{\frac{n-i}{n+1}}f(N,u)^{\frac{i}{n+1}}\Big)^{\frac{1}{q}},$$

$$g_2=\Big(f(K,u)^{\frac{n-k}{n+1}}f(L,u)^{\frac{k}{n+1}}\Big)^{\frac{1}{p}},\qquad h_2=\Big(f(K,u)^{\frac{n-i}{n+1}}f(L,u)^{\frac{i}{n+1}}\Big)^{\frac{1}{q}}.$$

From Lemma 2.3, we see that $g_1^p/h_1^q=(f_M/f_N)^{\frac{i-k}{n+1}}$, which is a constant since N is a homothetic copy of M. Now by use of (1.4) and Lemma 2.3, we obtain

$$\begin{aligned}
&\Big(\Omega_k(M,N)-\Omega_k(K,L)\Big)^{\frac{j-i}{k-i}}\Big(\Omega_i(M,N)-\Omega_i(K,L)\Big)^{\frac{k-j}{k-i}}\\
&=\left(\int_{S^{n-1}}f(M,u)^{\frac{n-k}{n+1}}f(N,u)^{\frac{k}{n+1}}-f(K,u)^{\frac{n-k}{n+1}}f(L,u)^{\frac{k}{n+1}}dS(u)\right)^{\frac{j-i}{k-i}}\\
&\quad\cdot\left(\int_{S^{n-1}}f(M,u)^{\frac{n-i}{n+1}}f(N,u)^{\frac{i}{n+1}}-f(K,u)^{\frac{n-i}{n+1}}f(L,u)^{\frac{i}{n+1}}dS(u)\right)^{\frac{k-j}{k-i}}\\
&\le\int_{S^{n-1}}\Bigg[\Big(f(M,u)^{\frac{n-k}{n+1}}f(N,u)^{\frac{k}{n+1}}\Big)^{\frac{j-i}{k-i}}\Big(f(M,u)^{\frac{n-i}{n+1}}f(N,u)^{\frac{i}{n+1}}\Big)^{\frac{k-j}{k-i}}\\
&\qquad-\Big(f(K,u)^{\frac{n-k}{n+1}}f(L,u)^{\frac{k}{n+1}}\Big)^{\frac{j-i}{k-i}}\Big(f(K,u)^{\frac{n-i}{n+1}}f(L,u)^{\frac{i}{n+1}}\Big)^{\frac{k-j}{k-i}}\Bigg]dS(u)\\
&=\int_{S^{n-1}}\Big(f(M,u)^{\frac{n-j}{n+1}}f(N,u)^{\frac{j}{n+1}}-f(K,u)^{\frac{n-j}{n+1}}f(L,u)^{\frac{j}{n+1}}\Big)dS(u)\\
&=\Omega_j(M,N)-\Omega_j(K,L).
\end{aligned}$$

This obtains (3.8). By the equality condition of Lemma 2.3, the equality in (3.8) holds if and only if $f_K^{\frac{i-k}{n+1}}=\lambda f_L^{\frac{i-k}{n+1}}$, which is equivalent to the fact that K is a homothetic copy of L, and $(f_K,f_M)=\lambda(f_L,f_N)$, where λ is a constant. This proves the Theorem 3.3 □

In particular, let $i=0$ and $k=n$ in Theorem 3.3. From (1.6), we can obtain the following Minkowski type inequality for difference of i-th mixed affine surface areas.

Corollary 3.4. *Let $K,L,M\in\mathscr{F}^n$, $j\in\mathbb{R}$, N be a homothetic copy of M, $f_K\le f_M$, $f_L\le f_N$. If $0<j<n$, then*

$$\Big(\Omega_j(M,\ N)-\Omega_j(K,\ L)\Big)^{n}\ge\Big(\Omega(M)-\Omega(K)\Big)^{n-j}\Big(\Omega(N)-\Omega(L)\Big)^{j},$$

and the equality holds if and only if K and L are homothetic and $(f_K, f_M) = \lambda(f_L, f_N)$, where λ is a constant.

In addition, we also give the following Aleksandrov-Fenchel type inequality for the difference of mixed affine surface areas.

Theorem 3.5. *Let $K_i, M_i \in \mathscr{F}^n$, and $f_{K_i} \leq f_{M_i}$ $(i = 1, \cdots, n)$. If $M_{n-r+1}, \cdots, M_n$ are homothetic, then, for all $1 \leq r \leq n$,*

$$\Big(\Omega(M_1, \cdots, M_n) - \Omega(K_1, \cdots, K_n)\Big)^r \tag{3.9}$$
$$\geq \prod_{i=1}^{r} \Big(\Omega(M_1, \cdots, M_{n-r}, M_{n-i+1}, \cdots, M_{n-i+1}) - \Omega(K_1, \cdots, K_{n-r}, K_{n-i+1}, \cdots, K_{n-i+1})\Big),$$

and the equality holds if and only if $K_{n-r+1}, \cdots, K_n$ are homothetic, and

$$\Big(\Omega(M_1, \cdots, M_{n-r}, M_n, \cdots, M_n), \cdots, \Omega(M_1, \cdots, M_{n-r}, M_{n-r+1}, \cdots, M_{n-r+1})\Big)$$
$$= \mu\Big(\Omega(K_1, \cdots, K_{n-r}, K_n, \cdots, K_n), \cdots, \Omega(K_1, \cdots, K_{n-r}, K_{n-r+1}, \cdots, K_{n-r+1})\Big),$$

where μ is a constant.

Proof. Observe that $M_{n-r+1}, \cdots, M_n$ are homothetic. From the equality condition of Lemma 2.4, we obtain

$$\prod_{i=1}^{r} \Omega(M_1, \cdots, M_{n-r}, M_{n-i+1}, \cdots, M_{n-i+1}) = \Omega^r(M_1, \cdots, M_n),$$

which together with Lemma 2.4 yields that

$$\begin{aligned} &\Omega(M_1, \cdots, M_n) - \Omega(K_1, \cdots, K_n) \\ &\geq \Big(\prod_{i=1}^{r} \Omega(M_1, \cdots, M_{n-r}, M_{n-i+1}, \cdots, M_{n-i+1})\Big)^{\frac{1}{r}} \\ &\quad - \Big(\prod_{i=1}^{r} \Omega(K_1, \cdots, K_{n-r}, K_{n-i+1}, \cdots, K_{n-i+1})\Big)^{\frac{1}{r}}. \end{aligned} \tag{3.10}$$

Since $f_{K_i} \leq f_{M_i}$ $(i = 1, \cdots, n)$, we conclude from (1.3) that

$$\Omega(M_1, \cdots, M_n) \geq \Omega(K_1, \cdots, K_n). \tag{3.11}$$

Taking $M_{n-r+1} = \cdots = M_n = M_{n-i+1}$, $K_{n-r+1} = \cdots = K_n = K_{n-i+1}$ in (3.11), we obtain

$$\Omega(M_1, \cdots, M_{n-r}, M_{n-i+1}, \cdots, M_{n-i+1}) \geq \Omega(K_1, \cdots, K_{n-r}, K_{n-i+1}, \cdots, K_{n-i+1}). \tag{3.12}$$

Therefore, from (3.10), (3.11), (3.12) and Lemma 2.5, we get

$$\Omega(M_1, \cdots, M_n) - \Omega(K_1, \cdots, K_n)$$
$$\geq \Big(\prod_{j=1}^{r} (\Omega(M_1, \cdots, M_{n-r}, M_{n-i+1}, \cdots, M_{n-i+1}) - \Omega(K_1, \cdots, K_{n-r}, K_{n-i+1}, \cdots, K_{n-i+1}))\Big)^{\frac{1}{r}}.$$

This yields (3.9).

According to the equality conditions of Lemma 2.4 and Lemma 2.5, we obtain that the equality holds in (3.9) if and only if $K_{n-r+1},\cdots,K_n$ are homothetic, and

$$\Big(\Omega(M_1,\cdots,M_{n-r},M_n,\cdots,M_n),\cdots,\Omega(M_1,\cdots,M_{n-r},M_{n-r+1},\cdots,M_{n-r+1})\Big)$$
$$=\mu\Big(\Omega(K_1,\cdots,K_{n-r},K_n,\cdots,K_n),\cdots,\Omega(K_1,\cdots,K_{n-r},K_{n-r+1},\cdots,K_{n-r+1})\Big),$$

where μ is a constant. □

Recently, affine surface area, which is an important part of convex geometry, received much attention. Many associated results were obtained. In this paper, we mainly discussed various kinds of difference inequalities of the affine surface area, which play an important role in further research, and also promote the development of the affine surface area.

Acknowledgements
The authors were supported by the innovation Foundation of Graduate Student of China Three Gorges University No. 2019SSPY144 and the National Natural Science Foundation under Grant No. 11371224, respectively.

REFERENCES

[1] C.M. Petty, Affine isoperimetric problems, Ann. N.Y. Acad. Sci. 440 (1985), 113-127.
[2] C.M. Petty, Projection bodies, Proc. Coll. Convexity, Copenhagen, 1965, Københacns Univ. Math. Inst (1967), 234-241.
[3] E. Lutwak, Centroid bodies and dual mixed volumes, Proc. London Math. Soc. 60 (1990), 365-391.
[4] E. Lutwak, Extended affine surface area, Adv. Math. 85 (1991), 39-68.
[5] E. Lutwak, Mixed affine surface area, J. Math. Anal. Appl. 125 (1987), 351-360.
[6] Y.B. Feng, W.D. Wang, Blaschke-Minkowski homomorphisms and affine surface area, Publ. Math. Debrecen, 85 (2014), 297-308.
[7] M. Ludwing, General affine surface areas, Adv. Math. 224 (2010), 2346-2360.
[8] K. Leichtweiß, On the history of the affine surface area for convex bodies, Results Math. 20 (1991), 650-656.
[9] G.S. Leng, Affine surface areas of curvature images for convex bodies, Acta Math. Sin. 45 (2002), 797-802.
[10] F.H. Lu, W.D. Wang, G.S. Leng, On inequalities for mixed affine surface area, Indian J. Pure Appl. Math. 38 (2007), 153-163.
[11] W.D. Wang, B. Wei, A new type of Brunn-Minkowski inequality for affine surface area and its applications, Acta Math. Hungar. 141 (2013), 282-290.
[12] C.J. Zhao, On quotients of i-th affine surface areas, Turkish J. Math. 37 (2013), 1022-1029.
[13] W.S. Cheung, C.J. Zhao, Width-integrals and affine surface area of convex bodies, Banach J. Math. Anal. 2 (2008), 70-77.
[14] G.S. Leng, The Brunn-Minkowski inequality for volume differences, Adv. Appl. Math. 32 (2004), 615-624.
[15] S.J. Lv, Dual Brunn-Minkowski inequality for volume differences, Geom. Dedicate 145 (2010), 169-180.
[16] E. Lutwak, Dual mixed volumes, Pacific J. Math. 58 (1975), 531-538.
[17] Y.B. Feng , W.D. Wang, J. Yuan, Differences of quermass- and dual quermassintegrals of Blaschke-Minkowski and radial Blaschke-Minkowski homomorphisms, Bull. Belg. Math. Soc. 21 (2014), 577-592.
[18] C.J. Zhao, The Aleksandrov-Fenchel type inequalities for volume differences, Balkan J. Geom. Appl. 15 (2010), 163-172.
[19] C. J. Zhao, W.S. Cheung, Radial Blaschke-Minkowski homomorphisms and volume differences, Geom. Dedicata 154 (2011), 81-91.
[20] R.J. Gardner, Geometric Tomography, Cambridge Univ. Press, Cambridge, UK., 2nd edition, 2006.

[21] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, 2nd edn, Cambridge Univ. Press, Cambridge, 2014.
[22] E.F. Beckenbach, R. Bellman, Inequalities, Berlin. Springer-Verlag., 2nd edition, 1965.
[23] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge Univ. Press, Cambridge, 1952.