J. Nonlinear Funct. Anal. 2020 (2020), Article ID 21 https://doi.org/10.23952/jnfa.2020.21

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

OPTIMALITY CONDITIONS AND DUALITY FOR E-DIFFERENTIABLE SEMI-INFINITE PROGRAMMING WITH MULTIPLE INTERVAL-VALUED OBJECTIVE FUNCTIONS UNDER GENERALIZED E-CONVEXITY

LE THANH TUNG

Department of Mathematics, College of Natural Sciences, Can Tho University, Can Tho 900000, Vietnam

Abstract. In this paper, optimality conditions and duality for E-differentiable semi-infinite programming with multiple interval-valued objective functions are investigated. We first establish Karush-Kuhn-Tucker necessary and sufficient optimality conditions for some types of efficient solutions of E-differentiable semi-infinite programming with multiple interval-valued objective functions. Then, we propose E-Wolfe and E-Mond-Weir type duality for the E-differentiable semi-infinite programming with multiple interval-valued objective functions, and explore weak and strong duality relations under generalized E-convexity.

Keywords. E-differentiable semi-infinite programming; Multiple interval-valued objective functions; Karush-Kuhn-Tucker optimality conditions; E-Mond-Weir duality; E-Wolfe duality.

1. Introduction

Some multiobjective optimization problems in practice deal with the infinite number of constraints. These problems are called the multiobjective semi-infinite programming problems. Because of their important meanings not only for theoretical aspect, but also for practical application, multiobjective semi-infinite programming problems have been studied by many researchers. For some recent results in this direction, we refer to, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein. As the coefficients of objective functions in some optimization problems in socio-economic are used to be uncertain or imprecise data, the interval-valued, or the range of coefficients, of the objective functions are employed in these models. To find the solution of these problems, optimality conditions and duality for optimization problems with one or multiple interval-valued objective functions have been considered recently in many papers; see, e.g., [11, 12, 13, 14, 15, 16]. The generalized convexity is often used in establishing optimality conditions and investigating duality problems; see, e.g., [17, 18] and references therein. In [19], the classes of nonconvex sets and nonconvex functions, called E-convex sets and E-convex

E-mail address: lttung@ctu.edu.vn.

Received April 12, 2020; Accepted May 19, 2020.

©2020 Journal of Nonlinear Functional Analysis

functions, were proposed. Some important properties and applications of these classes were developed in [18, 20, 21, 22, 23]. Recently, Megahed, Gomma and Youness [24] introduced the concept of E-differentiable convex functions and established the optimality conditions for mathematical programming. By utilizing the E-differentiable convexity, the optimality conditions and duality problems were considered for vector optimization problems with inequality and equality constraints in [25] and for multiobjective fractional programming problems in [26]. In [27], the optimality conditions for the E-differentiable vector optimization problem with the multiple interval-valued objective functions were given. However, to the best of our knowledge, there is no paper dealing with E-differentiable semi-infinite programming with the multiple interval-valued objective functions.

Motivated by the above observations, in this paper, we establish Karush-Kuhn-Tucker optimality conditions and investigate duality problems for E-differentiable semi-infinite programming with multiple interval-valued objective functions. The organization of the paper is as follows. Section 2 recalls basic notions and presents the differentiable E-convexity notion. Karush-Kuhn-Tucker necessary and sufficient optimality conditions for the E-differentiable semi-infinite programming with multiple interval-valued objective functions are discussed in Section 3. Section 4 is devoted to exploring E-Mond-Weir and E-Wolfe dual problems of semi-infinite programming with multiple interval-valued objective functions. Some examples are provided to illuminate the outcomes of the paper.

2. PRELIMINARIES

The following notations and definitions will be used throughout the paper. Let $\mathbb{R}^n$ be an Euclidean space. The notation $\langle\cdot,\cdot\rangle$ is utilized to denote the inner product. The notion $B(x,\delta)$ stands for the open ball with center $x\in\mathbb{R}^n$ and radius δ. For a subset $X\subseteq\mathbb{R}^n$, int$X$, cl$X$, aff$X$, and co$X$ stand for its interior, closure, affine hull, convex hull of X, respectively (shortly resp). The cone and the convex cone (containing the origin) generated by X are demonstrated resp by cone$X$, pos$X$. The negative polar cone and strictly negative polar cone of X are defined resp by

$$X^- := \{x^*\in\mathbb{R}^n | \langle x^*,x\rangle \le 0\ \forall x\in X\}, X^s := \{x^*\in\mathbb{R}^n | \langle x^*,x\rangle < 0\ \forall x\in X\setminus\{0\}\}.$$

It is easy to check that $X^s\subset X^-$ and if $X^s\neq\emptyset$, then cl$X^s = X^-$. Moreover, the bipolar theorem (see, e.g., [28]) states that $X^{--} = \text{cl cone}X$. The contingent cone [28] of X at $\bar{x}\in\text{cl}X$ is

$$T(X,\bar{x}) := \{x\in\mathbb{R}^n \mid \exists\tau_k\downarrow 0, \exists x_k\to x,\ \forall k\in\mathbb{N}, \bar{x}+\tau_k x_k\in X\}.$$

Definition 2.1. [19] A set $X\subseteq\mathbb{R}^n$ is said to be E-convex with respect to (shortly wrt) an operator $E:\mathbb{R}^n\to\mathbb{R}^n$ if $\lambda E(x)+(1-\lambda)E(x')\in X$ for all $x,x'\in X$ and $\lambda\in[0,1]$.

Note that every convex set is E-convex in the case that E is the identity map. If $E(X)$ is a convex set and $E(X)\subseteq X$, then X is E-convex. Moreover, if X is E-convex, then $E(X)\subseteq X$.

Definition 2.2. Let $E:\mathbb{R}^n\to\mathbb{R}^n$. Let X be an E-convex set in $\mathbb{R}^n$ and $\varphi:\mathbb{R}^n\to\mathbb{R}$.

(i) [19] φ is said to be E-convex wrt E on X if and only if (shortly iff)

$$\varphi(\lambda E(x)+(1-\lambda)E(x'))\le\lambda\varphi(E(x))+(1-\lambda)\varphi(E(x')),$$

for all $x,x'\in X$ and $\lambda\in[0,1]$.

(ii) [19] φ is said to be strictly E-convex wrt E on A iff

$$\varphi(\lambda E(x)+(1-\lambda)E(x'))<\lambda\varphi(E(x))+(1-\lambda)\varphi(E(x')),$$

for all $x,x'\in X, E(x)\neq E(x')$ and $\lambda\in(0,1)$.

For an E-convex function φ and an E-convex set X, the function $(\varphi\circ E):X\to\mathbb{R}$ is defined by $(\varphi\circ E)(x)=\varphi(E(x))$ for all $x\in X$.

Definition 2.3. [24] Let $E:\mathbb{R}^n\to\mathbb{R}^n$. Let X be an open E-convex set in $\mathbb{R}^n$, $\varphi:\mathbb{R}^n\to\mathbb{R}$ and $\bar{x}\in\mathbb{R}^n$. φ is said to be E-differentiable at $\bar{x}$ iff $\varphi\circ E$ is differentiable at $\bar{x}$ and

$$(\varphi\circ E)(x)=(\varphi\circ E)(\bar{x})+\langle\nabla(\varphi\circ E)(\bar{x}),x-\bar{x}\rangle+\theta(\bar{x},x-\bar{x})\|x-\bar{x}\|,$$

where $\theta(\bar{x},x-\bar{x})\to 0$ as $x\to\bar{x}$.

Definition 2.4. Let $E:\mathbb{R}^n\to\mathbb{R}^n$. Let X be an open E-convex set in $\mathbb{R}^n$, $\bar{x}\in\mathbb{R}^n$, $\varphi:\mathbb{R}^n\to\mathbb{R}$ and φ is E-differentiable at $\bar{x}$.

(i) φ is said to be differentiable E-convex at $\bar{x}$ on X iff

$$\varphi(E(x))-\varphi(E(\bar{x}))\geq\langle\nabla(\varphi\circ E)(\bar{x}),x-\bar{x}\rangle,\forall x\in X, \tag{2.1}$$

and, φ is said to be differentiable E-convex on X if φ is differentiable E-convex at each $x\in X$.

(ii) φ is said to be strictly differentiable E-convex at $\bar{x}$ on X iff

$$\varphi(E(x))-\varphi(E(\bar{x}))>\langle\nabla(\varphi\circ E)(\bar{x}),x-\bar{x}\rangle,\forall x\in X\setminus\{\bar{x}\}.$$

and, φ is said to be strictly differentiable E-convex on X if φ is strictly differentiable E-convex at each $x\in X$.

Proposition 2.5. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear operator. Let X be an open E-convex set in $\mathbb{R}^n$ and $\varphi:\mathbb{R}^n\to\mathbb{R}$.*

(i) If X is a convex set, φ is E-differentiable at each point $x\in X$ and φ is differentiable E-convex on X, then φ is E-convex on X.

(ii) If φ is E-convex at $\bar{x}\in X$ and φ is differentiable at $E(\bar{x})$, then φ is differentiable E-convex at $\bar{x}$ on X.

(iii) If X is a convex set, φ is strictly E-convex at $\bar{x}\in X$ and φ is differentiable at $E(\bar{x})$, then φ is strictly differentiable E-convex at $\bar{x}$ on X.

Proof. (i) Let $x',x''\in X$ and $\lambda\in[0,1]$. Setting $x=x',\bar{x}=(1-\lambda)x'+\lambda x''\in X$, one derives from (2.1) that

$$\varphi(E(x'))-\varphi(E((1-\lambda)x'+\lambda x''))\geq\langle\nabla(\varphi\circ E)((1-\lambda)x'+\lambda x''),x'-((1-\lambda)x'+\lambda x'')\rangle.$$

This together with the linearity of E implies that

$$\varphi(E(x'))-\varphi((1-\lambda)E(x')+\lambda E(x''))\geq\lambda\langle\nabla(\varphi\circ E)((1-\lambda)x'+\lambda x''),x'-x''\rangle. \tag{2.2}$$

Setting $x=x'',\bar{x}=(1-\lambda)x'+\lambda x''$, one implies from (2.1) that

$$\varphi(E(x''))-\varphi(E((1-\lambda)x'+\lambda x''))\geq\langle\nabla(\varphi\circ E)((1-\lambda)x'+\lambda x''),x''-((1-\lambda)x'+\lambda x'')\rangle,$$

which leads to

$$\varphi(E(x'))-\varphi((1-\lambda)E(x')+\lambda E(x''))\geq(1-\lambda)\langle\nabla(\varphi\circ E)((1-\lambda)x'+\lambda x'')),x''-x'\rangle. \tag{2.3}$$

Multiplying (2.2) and (2.3) with λ and $1-\lambda$, resp, and adding them up, one has

$$\varphi(\lambda E(x') + (1-\lambda)E(x'')) \leq \lambda\varphi(E(x')) + (1-\lambda)\varphi(E(x'')),$$

for all $x', x'' \in X$ and $\lambda \in [0,1]$.
(ii) Since X be an open E-convex, there exists $\delta > 0$ such that

$$\lambda E(x) + (1-\lambda)E(\bar{x}) \in B(E(\bar{x}), \delta) \subseteq X,$$

for all $x \in X$ and $\lambda \in (0,1)$. By using the E-convexity of φ at $\bar{x}$, one has

$$\varphi(\lambda E(x) + (1-\lambda)E(\bar{x})) \leq \lambda\varphi(E(x)) + (1-\lambda)\varphi(E(\bar{x})),$$

or equivalently,

$$\varphi(E(x)) - \varphi(E(\bar{x})) \geq \frac{\varphi(E(\bar{x}) + \lambda(E(x) - E(\bar{x}))) - \varphi(E(\bar{x}))}{\lambda}, \forall \lambda \in (0,1).$$

Letting $\lambda \downarrow 0$, we deduce from the differentiability of φ at $E(\bar{x})$, the linearity of E and the chain rule that

$$\begin{aligned}
\varphi(E(x)) - \varphi(E(\bar{x})) &\geq \langle \nabla\varphi(E(\bar{x})), E(x) - E(\bar{x})\rangle \\
&= \langle \nabla\varphi(E(\bar{x})), \nabla E(\bar{x})(x-\bar{x})\rangle \\
&= \nabla\varphi(E(\bar{x}))(\nabla E(\bar{x})(x-\bar{x})) \\
&= (\nabla\varphi(E(\bar{x})).\nabla E(\bar{x}))(x-\bar{x}) \\
&= \langle \nabla(\varphi\circ E)(\bar{x}), x-\bar{x}\rangle.
\end{aligned}$$

(iii) It follows from (ii) that

$$\varphi(E(x)) - \varphi(E(\bar{x})) \geq \langle \nabla(\varphi\circ E)(\bar{x}), x-\bar{x}\rangle, \forall x \in X. \tag{2.4}$$

Next, we prove that the inequality in (2.4) is strict if $x \neq \bar{x}$. Suppose to the contrary that there exists $\widetilde{x} \in X$ with $\widetilde{x} \neq \bar{x}$ and

$$\varphi(E(\widetilde{x})) - \varphi(E(\bar{x})) = \langle \nabla(\varphi\circ E)(\bar{x}), \widetilde{x} - \bar{x}\rangle. \tag{2.5}$$

From (2.5), the linearity of E, the strictly E-convexity of φ at $\bar{x}$ and $\widetilde{x} \neq \bar{x}$, one has, for $\lambda \in (0,1)$,

$$\begin{aligned}
\varphi(E(\lambda\widetilde{x} + (1-\lambda)\bar{x})) &= \varphi(\lambda E(\widetilde{x}) + (1-\lambda)E(\bar{x})) \\
&< \lambda\varphi(E(\widetilde{x})) + (1-\lambda)\varphi(E(\bar{x})) \\
&= \varphi(E(\bar{x})) + \lambda(\varphi(E(\widetilde{x})) - \varphi(E(\bar{x}))) \\
&= \varphi(E(\bar{x})) + \lambda\langle \nabla(\varphi\circ E)(\bar{x}), \widetilde{x} - \bar{x}\rangle \\
&= \varphi(E(\bar{x})) + \langle \nabla(\varphi\circ E)(\bar{x}), (\lambda\widetilde{x} + (1-\lambda)\bar{x}) - \bar{x}\rangle.
\end{aligned}$$

This, taking into account the fact on $\lambda\widetilde{x} + (1-\lambda)\bar{x} \in X$ and $\lambda\widetilde{x} + (1-\lambda)\bar{x} \neq \bar{x}$, gives us that

$$\varphi(E(\lambda\widetilde{x} + (1-\lambda)\bar{x})) - \varphi(E(\bar{x})) < \langle \nabla(\varphi\circ E)(\bar{x}), (\lambda\widetilde{x} + (1-\lambda)\bar{x}) - \bar{x}\rangle,$$

which contradicts (2.4). $\square$

Example 2.6. Let $X = (-1,1)$, $\varphi : X \to \mathbb{R}$ and $E : X \to X$ be defined by

$$\varphi(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}, \; E(x) = \frac{x}{2}.$$

Then, $E(X) = (-\frac{1}{2}, \frac{1}{2})$ is a convex set and $E(X) \subset X$. This shows that X is a E-convex set. Since

$$(\varphi \circ E)(x) = \begin{cases} \frac{x^2}{4}, & \text{if } x \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

φ is E-differentiable at any $\bar{x} \in X$. Taking $\bar{x} = 0$, one has $\varphi(E(\bar{x})) = E(\bar{x}) = 0$ and $\nabla(\varphi \circ E)(\bar{x}) = 0$. Hence, for $x = -\frac{1}{4} \in X$ and $x \neq \bar{x}$,

$$\varphi(E(x)) - \varphi(E(\bar{x})) = 0 \ngtr \langle \nabla(\varphi \circ E)(\bar{x}), x - \bar{x} \rangle.$$

Thus, φ is not strictly differentiable E-convex at $\bar{x}$. However, for all $x \in X$,

$$\varphi(E(x)) - \varphi(E(\bar{x})) \geq \langle \nabla(\varphi \circ E)(\bar{x}), x - \bar{x} \rangle,$$

which implies that φ is differentiable E-convex at $\bar{x}$.

Let $\mathscr{K}_C$ denote the class of all closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathscr{K}_C = \{[\underline{c}, \overline{c}] \mid \underline{c}, \overline{c} \in \mathbb{R} \text{ and } \underline{c} \leq \overline{c}\},$$

where $\underline{c}$ and $\overline{c}$ means the lower and upper bound of $[\underline{c}, \overline{c}]$, resp. Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be in $\mathscr{K}_C$ and $\lambda \in \mathbb{R}$. Then, by definition, one has

$$A + B := \{a + b \mid a \in A, b \in B\} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \lambda A := \lambda[\underline{a}, \overline{a}] = \begin{cases} [\lambda \underline{a}, \lambda \overline{a}], & \text{if } \lambda \geq 0, \\ [\lambda \overline{a}, \lambda \underline{a}], & \text{if } \lambda < 0. \end{cases}$$

Hence, $-A = [-\overline{a}, -\underline{a}]$ and $A - B = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$.

Definition 2.7. Let $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$ be two sets in $\mathscr{K}_C$.

(i) $A \leq_{LU} B$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$.

(ii) $A <_{LU} B$ if $A \leq_{LU} B$ and $A \neq B$, or equivalently, $A <_{LU} B$ if

$$\begin{cases} \underline{a} < \underline{b} \\ \overline{a} \leq \overline{b}, \end{cases} \text{or} \begin{cases} \underline{a} \leq \underline{b} \\ \overline{a} < \overline{b}, \end{cases} \text{or} \begin{cases} \underline{a} < \underline{b} \\ \overline{a} < \overline{b}. \end{cases}$$

Let X be a nonempty open subset of $\mathbb{R}^n$. A function $H : X \to \mathscr{K}_C$ is called an interval-valued function if $H(x) = [\underline{H}(x), \overline{H}(x)]$ with $\underline{H}, \overline{H} : X \to \mathbb{R}$ such that $\underline{H}(x) \leq \overline{H}(x)$ for each $x \in X$.

Definition 2.8. Let $E : \mathbb{R}^n \to \mathbb{R}^n$, X be an E-convex set in $\mathbb{R}^n$ and $H : X \to \mathscr{K}_C$ be an interval-valued function.

(i) H is said to be LU-E-convex [27] at $\bar{x} \in X$ if, for all $x \in X$ and $\lambda \in [0,1]$, $H(\lambda E(x) + (1-\lambda)E(\bar{x})) \leq_{LU} \lambda H(E(x)) + (1-\lambda)H(E(\bar{x}))$. H is said to be LU-E-convex on X if H is LU-E-convex at each $x \in X$.

(ii) H is said to be strictly LU-E-convex [27] at $\bar{x}$ if, for all $x \in X$ with $E(x) \neq E(\bar{x})$ and $\lambda \in (0,1)$, $H(\lambda E(x) + (1-\lambda)E(\bar{x})) <_{LU} \lambda H(E(x)) + (1-\lambda)H(E(\bar{x}))$. H is called strictly LU-E-convex on X if H is strictly LU-E-convex at each $x \in X$.

Definition 2.9. [27] Let $E : \mathbb{R}^n \to \mathbb{R}^n$. Let X be an open E-convex set in $\mathbb{R}^n$, $H : \mathbb{R}^n \to \mathscr{K}_C$ and $\bar{x} \in \mathbb{R}^n$. H is said to be E-differentiable at $\bar{x}$ iff $\underline{H} \circ E$ and $\overline{H} \circ E$ is differentiable at $\bar{x}$ and

$$(\underline{H} \circ E)(x) = (\underline{H} \circ E)(\bar{x}) + \langle \nabla(\underline{H} \circ E)(\bar{x}), x - \bar{x} \rangle + \underline{\theta}(\bar{x}, x - \bar{x})\|x - \bar{x}\|,$$

$$(\overline{H} \circ E)(x) = (\overline{H} \circ E)(\bar{x}) + \langle \nabla(\overline{H} \circ E)(\bar{x}), x - \bar{x} \rangle + \overline{\theta}(\bar{x}, x - \bar{x})\|x - \bar{x}\|,$$

where $\underline{\theta}(\bar{x}, x - \bar{x}), \overline{\theta}(\bar{x}, x - \bar{x}) \to 0$ as $x \to \bar{x}$.

Definition 2.10. Let $E:\mathbb{R}^n\to\mathbb{R}^n$. Let X be an open E-convex set in $\mathbb{R}^n$, $\bar{x}\in X$ and the interval-valued function $H:X\to\mathscr{K}_C$ be E-differentiable at $\bar{x}$.
(i) H is said to be differentiable LU-E-convex at $\bar{x}$ on X iff

$$\underline{H}(E(x))-\underline{H}(E(\bar{x}))\geq\langle\nabla(\underline{H}\circ E)(\bar{x}),x-\bar{x}\rangle,\overline{H}(E(x))-\overline{H}(E(\bar{x}))\geq\langle\nabla(\overline{H}\circ E)(\bar{x}),x-\bar{x}\rangle,$$

for all $x\in X$. If $X=\mathbb{R}^n$, we say that H is differentiable LU-E-convex at $\bar{x}$.
(ii) H is said to be strongly differentiable LU-E-convex at $\bar{x}$ on X iff

$$\underline{H}(E(x))-\underline{H}(E(\bar{x}))>\langle\nabla(\underline{H}\circ E)(\bar{x}),x-\bar{x}\rangle,\overline{H}(E(x))-\overline{H}(E(\bar{x}))>\langle\nabla(\overline{H}\circ E)(\bar{x}),x-\bar{x}\rangle,$$

$\forall x\in X$ with $x\neq\bar{x}$.

Lemma 2.11. [29] *Let $\{C_t|t\in\Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K:=\operatorname{pos}\left(\bigcup\limits_{t\in\Gamma}C_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

Lemma 2.12. [30] *Suppose that S,P are arbitrary (possibly infinite) index sets, $a_s=a(s)=(a_1(s),...,a_n(s))$ maps S onto $\mathbb{R}^n$, and so does a_p. Suppose that the set $\operatorname{co}\{a_s,s\in S\}+\operatorname{pos}\{a_p,p\in P\}$ is closed. Then, the following statements are equivalent:*

$$I:\quad\begin{cases}\langle a_s,x\rangle<0,s\in S,S\neq\emptyset\\ \langle a_p,x\rangle\leq 0,p\in P\end{cases}\quad\textit{has no solution } x\in\mathbb{R}^n;$$
$$II:\qquad 0\in\operatorname{co}\{a_s,s\in S\}+\operatorname{pos}\{a_p,p\in P\}.$$

Lemma 2.13. [31] *If X is a nonempty compact subset of $\mathbb{R}^n$, then*

(i) co$X$ *is a compact set;*
(ii) *If* $0\notin\operatorname{co}X$, *then* pos$X$ *is a closed cone.*

3. KKT OPTIMALITY CONDITIONS

In this section, we consider the following semi-infinite programming with multiple interval-valued objective functions:
(P): $\text{LU}-\min\mathscr{F}(x):=(F_1(x),...,F_m(x))=([\underline{F}_1(x),\overline{F}_1(x)],...,[\underline{F}_m(x),\overline{F}_m(x)])$
s.t. $g_t(x)\leq 0,\ t\in T,$
where $F_i:\mathbb{R}^n\to\mathscr{K}_C$ are interval-valued functions for $i\in I:=\{1,...,m\}$ and $g_t (t\in T)$ are functions from $\mathbb{R}^n$ to $\mathbb{R}$. The index set T is an arbitrary nonempty set, not necessary finite. Denote by $\Omega:=\{x\in\mathbb{R}^n\mid g_t(x)\leq 0,\ t\in T\}$ the feasible solution set of (P). Let $\mathbb{R}^{|T|}_+$ denote the collection of all the functions $\lambda:T\to\mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x}\in\Omega$, denote $T(\bar{x}):=\{t\in T|g_t(\bar{x})=0\}$ the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x}\in\Omega$ is

$$\Lambda(\bar{x}):=\{\lambda\in\mathbb{R}^{|T|}_+|\lambda_t g_t(\bar{x})=0,\forall t\in T\}.$$

Notice that $\lambda\in\Lambda(\bar{x})$ if there exists a finite index set $J\subset T(\bar{x})$ such that $\lambda_t>0$ for all $t\in J$ and $\lambda_t=0$ for all $t\in T\setminus J$.

Definition 3.1. Let $\bar{x}\in\Omega$.

(i) $\bar{x}$ is a type-1 (Pareto) optimal solution [15] of (P), denoted by $\bar{x} \in E(P,1)$, if there is no $x \in \Omega$ satisfying

$$\begin{cases} F_i(x) \leq_{LU} F_i(\bar{x}), & \forall i \in I, \\ F_{i_0}(x) <_{LU} F_{i_0}(\bar{x}), & \text{for at least one } i_0 \in I. \end{cases}$$

(ii) $\bar{x}$ is a weakly type-1 optimal solution [15] of (P), denoted by $\bar{x} \in WE(P,1)$, if there is no $x \in \Omega$ satisfying $F_i(x) <_{LU} F_i(\bar{x}), \forall i \in I$.

We can check that $E(P,1) \subseteq WE(P,1)$. In the sequel, we always assume that $E : \mathbb{R}^n \to \mathbb{R}^n$ is an one-to-one and onto operator, and $F_i (i \in I), g_t (t \in T)$ are E-differentiable on the $\mathbb{R}^n$. Let us consider the associated E-multiobjective semi-infinite programming (P_E) of (P) as follows:

$$\begin{aligned} (P_E) : \mathrm{LU-}\min (\mathscr{F} \circ E)(z) &:= ((F_1 \circ E)(z), \ldots, (F_m \circ E)(z)) \\ \text{s.t.} \quad & (g_t \circ E)(z) \leq 0, \quad t \in T. \end{aligned}$$

Denote by $\Omega_E := \{z \in \mathbb{R}^n \mid (g_t \circ E)(z) \leq 0,\ t \in T\}$ the feasible solution set of (P_E) and the index set of all active constraints at $\bar{z} \in \Omega_E$ is $T_E(\bar{z}) := \{t \in T | (g_t \circ E)(\bar{z}) = 0\}$. Proving similar to the proof of Lemma 3.5 in [27], we obtain that $E(\Omega_E) = \Omega$. The set of active constraint multipliers at $\bar{z} \in \Omega_E$ is

$$\Lambda_E(\bar{z}) := \{\lambda \in \mathbb{R}^{|T|}_+ | \lambda_t (g_t \circ E)(\bar{z}) = 0, \forall t \in T\}.$$

Definition 3.2. Let $\bar{z} \in \Omega_E$.

(i) $\bar{z}$ is a type-1 (Pareto) E-optimal solution [27] of (P_E), denoted by $\bar{z} \in E(P_E,1)$, if there is no $z \in \Omega_E$ satisfying

$$\begin{cases} (F_i \circ E)(z) \leq_{LU} (F_i \circ E)(\bar{z}), & \forall i \in I, \\ (F_{i_0} \circ E)(z) <_{LU} (F_{i_0} \circ E)(\bar{z}), & \text{for at least one } i_0 \in I. \end{cases}$$

(ii) $\bar{z}$ is a weakly type-1 E-optimal solution [27] of (P_E), denoted by $\bar{z} \in WE(P_E,1)$, if there is no $z \in \Omega_E$ satisfying $(F_i \circ E)(z) <_{LU} (F_i \circ E)(\bar{z}), \forall i \in I$.

Lemma 3.3. [27] *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a given one-to-one and onto operator, $\bar{x} \in \Omega$ and $\bar{z} \in \Omega_E$.*

(i) If $\bar{z} \in WE(P_E,1)$ ($\bar{z} \in E(P_E,1)$, resp), then $E(\bar{z}) \in WE(P,1)$ ($E(\bar{z}) \in E(P,1)$, resp).

(ii) If $\bar{x} \in WE(P,1)$ ($\bar{x} \in E(P,1)$, resp), then there exists $\bar{z} \in \Omega_E$ such that $\bar{x} = E(\bar{z})$ and $\bar{z} \in WE(P_E,1)$ ($\bar{z} \in E(P_E,1)$, resp).

Proposition 3.4. *The (ACQ_E) holds at $\bar{z} \in \Omega_E$ if $\left(\bigcup_{t \in T_E(\bar{z})} \nabla (g_t \circ E)(\bar{z}) \right)^{-} \subseteq T(\Omega_E, \bar{z})$ and the set $\Delta_E := \mathrm{pos} \bigcup_{t \in T_E(\bar{z})} \nabla (g_t \circ E)(\bar{z})$ is closed.*

Proposition 3.5. *Suppose that $\bar{z} \in WE(P_E,1)$ and (ACQ_E) holds at $\bar{z}$. Then, there exist $(\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ with $\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1$ and $\lambda \in \Lambda_E(\bar{z})$ such that*

$$\sum_{i=1}^m \alpha_i^L \nabla (\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^m \alpha_i^U \nabla (\overline{F}_i \circ E)(\bar{z}) + \sum_{t \in T} \lambda_t \nabla (g_t \circ E)(\bar{z}) = 0.$$

Proof. Since $\bar{z} \in WE(P_E,1)$, there is no $z \in \Omega_E$ satisfying

$$(F_i \circ E)(z) <_{LU} (F_i \circ E)(\bar{z}), \forall i \in I,$$

or equivalently, $\begin{cases} (\underline{F}_i \circ E)(z) \leq (\underline{F}_i \circ E)(\bar{z}) \\ (\overline{F}_i \circ E)(z) < (\overline{F}_i \circ E)(\bar{z}), \end{cases}$ or $\begin{cases} (\underline{F}_i \circ E)(z) < (\underline{F}_i \circ E)(\bar{z}) \\ (\overline{F}_i \circ E)(z) \leq (\overline{F}_i \circ E)(\bar{z}), \end{cases}$

$$\text{or} \left\{ \begin{array}{l} (\underline{F}_i \circ E)(z) < (\underline{F}_i \circ E)(\bar{z}) \\ (\overline{F}_i \circ E)(z) < (\overline{F}_i \circ E)(\bar{z}). \end{array} \right. \tag{3.1}$$

We first justify that

$$\left(\bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z})) \right)^s \cap T(\Omega_E, \bar{z}) = \emptyset. \tag{3.2}$$

If otherwise, there exists $d \in \left(\bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z})) \right)^s \cap T(\Omega_E, \bar{z})$. Hence, one has $\langle \nabla(\underline{F}_i \circ E)(\bar{z}), d \rangle < 0, \forall i \in I$ and $\langle \nabla(\overline{F}_i \circ E)(\bar{z}), d \rangle < 0, \forall i \in I$. Since $d \in T(\Omega_E, \bar{z})$, there exist $\tau_k \downarrow 0$ and $d_k \to d$ such that $\bar{z} + \tau_k d_k \in \Omega_E$ for all k. We derive from $\underline{F}_i (i \in I)$ are E-differentiable at $\bar{x}$ that

$$(\underline{F}_i \circ E)(\bar{z} + \tau_k d_k) = (\underline{F}_i \circ E)(\bar{z}) + \tau_k \langle \nabla(\underline{F}_i \circ E)(\bar{z}), d_k \rangle + \underline{\theta}_i(\bar{z}, \tau_k d_k) \|\tau_k d_k\|, \forall i \in I.$$

Consequently, for all $i \in I$,

$$\begin{aligned} \frac{(\underline{F}_i \circ E)(\bar{z} + \tau_k d_k) - (\underline{F}_i \circ E)(\bar{z})}{\tau_k} &= \langle \nabla(\underline{F}_i \circ E)(\bar{z}), d_k \rangle + \underline{\theta}_i(\bar{z}, \tau_k d_k) \|d_k\| \\ &\xrightarrow[k \to \infty]{} \langle \nabla(\underline{F}_i \circ E)(\bar{z}), d \rangle < 0. \end{aligned}$$

Thus, for each $i \in I$, there is $k_i^L > 0$ such that $\frac{1}{\tau_k}((\underline{F}_i \circ E)(\bar{z} + \tau_k d_k) - (\underline{F}_i \circ E)(\bar{z})) < 0$, for all $k > k_i^L$. Setting $\overline{k}^L = \max_{i \in I} k_i^L$, we have $(\underline{F}_i \circ E)(\bar{z} + \tau_k d_k) < (\underline{F}_i \circ E)(\bar{z}), \forall k > \overline{k}^L, \forall i \in I$. Similarly, there exists $\overline{k}^U > 0$ such that $(\overline{F}_i \circ E)(\bar{z} + \tau_k d_k) < (\overline{F}_i \circ E)(\bar{z}), \forall k > \overline{k}^U, \forall i \in I$. Setting $\overline{k} := \max\{\overline{k}^L, \overline{k}^U\}$, we assure the existence of $k > \overline{k}$ large enough such that $\bar{z} + \tau_k d_k \in \Omega_E$ and for all $i \in I$,

$$(\underline{F}_i \circ E)(\bar{z} + \tau_k d_k) < (\underline{F}_i \circ E)(\bar{z}) \text{ and } (\overline{F}_i \circ E)(\bar{z} + \tau_k d_k) < (\overline{F}_i \circ E)(\bar{z}), \forall i \in I,$$

which contradicts (3.1). Hence, (3.2) holds. We derive from (3.2) and (ACQ_E) that

$$\begin{aligned} &\left(\bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z})) \right)^s \cap \left(\bigcup_{t \in T_E(\bar{z})} \nabla(g_t \circ E)(\bar{z}) \right)^- \\ &\subset \left(\bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z})) \right)^s \cap T(\Omega_E, \bar{z}) = \emptyset. \end{aligned}$$

This implies that there is no $d \in \mathbb{R}^n$ such that

$$\left\{ \begin{array}{ll} \langle \nabla(\underline{F}_i \circ E)(\bar{z}), d \rangle < 0, & \forall i \in I, \\ \langle \nabla(\overline{F}_i \circ E)(\bar{z}), d \rangle < 0, & \forall i \in I, \\ \langle \nabla(g_t \circ E)(\bar{z}), d \rangle \leq 0, & \forall t \in T_E(\bar{z}). \end{array} \right.$$

Moreover, we deduce from Lemma 2.13 that $\operatorname{co} \bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z}))$ is a compact set. Hence, $\operatorname{co} \bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z})) + \Delta_E$ is closed. According to Lemma 2.12, one gets

$$0 \in \operatorname{co} \bigcup_{i \in I} (\nabla(\underline{F}_i \circ E)(\bar{z}) \cup \nabla(\overline{F}_i \circ E)(\bar{z})) + \operatorname{pos} \bigcup_{t \in T_E(\bar{z})} \nabla(g_t \circ E)(\bar{z}).$$

In view of Lemma 2.11, there exist $(\alpha^L, \alpha^U) := ((\alpha_1^L, ..., \alpha_m^L), (\alpha_1^U, ..., \alpha_m^U)) \in \mathbb{R}_+^m \times \mathbb{R}_+^m$ with $\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1$ and $\lambda \in \Lambda_E(\bar{z})$ such that

$$\sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(\bar{z}) + \sum_{t \in T} \lambda_t \nabla(g_t \circ E)(\bar{z}) = 0.$$

The conclusion is obtained. □

Proposition 3.6. *Let $\bar{z} \in \Omega_E$. Assume that there exist $(\alpha^L, \alpha^U) \in \mathbb{R}_+^m \times \mathbb{R}_+^m$ with $\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) =$ 1 and $\lambda \in \Lambda_E(\bar{z})$ such that*

$$\sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(\bar{z}) + \sum_{t \in T} \lambda_t \nabla(g_t \circ E)(\bar{z}) = 0. \tag{3.3}$$

If $F_i (i \in I)$ are strongly differentiable LU-E-convex at $\bar{z}$ and $g_t (t \in T)$ are differentiable E-convex at $\bar{z}$, then $\bar{z} \in E(P_E, 1)$ and $E(\bar{z}) \in E(P, 1)$.

Proof. Since $\bar{z} \in \Omega_E$ satisfying (3.3), we have that there exists a finite subset J_E of $T_E(\bar{z})$ such that

$$\sum_{t \in J_E} \lambda_t \nabla(g_t \circ E)(\bar{z}) = -\left(\sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(\bar{z}) \right). \tag{3.4}$$

Arguing by contradiction, suppose that $\bar{z} \notin E(P_E, 1)$. Then, there exists a $\widehat{z} \in \Omega_E$ satisfying

$$(F_i \circ E)(\widehat{z}) \leq_{LU} (F_i \circ E)(\bar{z}), \forall i \in I \text{ and } (F_{i_0} \circ E)(\widehat{z}) <_{LU} (F_{i_0} \circ E)(\bar{z}) \text{ for at least one } i_0 \in I,$$

or equivalently, $\begin{cases} (\underline{F}_i \circ E)(\widehat{z}) \leq (\underline{F}_i \circ E)(\bar{z}) \\ (\overline{F}_i \circ E)(\widehat{z}) \leq (\overline{F}_i \circ E)(\bar{z}) \end{cases}$ for all $i \in I$, and, for at least one $i_0 \in I$,

$$\begin{cases} (\underline{F}_{i_0} \circ E)(\widehat{z}) \leq (\underline{F}_{i_0} \circ E)(\bar{z}) \\ (\overline{F}_{i_0} \circ E)(\widehat{z}) < (\overline{F}_{i_0} \circ E)(\bar{z}), \end{cases} \text{ or } \begin{cases} (\underline{F}_{i_0} \circ E)(\widehat{z}) < (\underline{F}_{i_0} \circ E)(\bar{z}) \\ (\overline{F}_{i_0} \circ E)(\widehat{z}) \leq (\overline{F}_{i_0} \circ E)(\bar{z}), \end{cases}$$

$$\text{or } \begin{cases} (\underline{F}_{i_0} \circ E)(\widehat{z}) < (\underline{F}_{i_0} \circ E)(\bar{z}) \\ (\overline{F}_{i_0} \circ E)(\widehat{z}) < (\overline{F}_{i_0} \circ E)(\bar{z}). \end{cases}$$

Hence, $\widehat{z} \neq \bar{z}$. The above inequalities together with $\alpha^L, \alpha^U \in \mathbb{R}_+^m$ satisfying $\sum_{i=1}^m (\alpha_i^L + \alpha_i^U) = 1$ imply that

$$\sum_{i=1}^m \alpha_i^L ((\underline{F}_i \circ E)(\widehat{z}) - (\underline{F}_i \circ E)(\bar{z})) + \sum_{i=1}^m \alpha_i^U ((\overline{F}_i \circ E)(\widehat{z}) - (\overline{F}_i \circ E)(\bar{z})) \leq 0. \tag{3.5}$$

We deduce from the strongly LU-E-convexity of $F_i (i \in I)$ at $\bar{z}$ that

$$(\underline{F}_i \circ E)(\widehat{z}) - (\underline{F}_i \circ E)(\bar{z}) > \langle \nabla(\underline{F}_i \circ E)(\bar{z}), \widehat{z} - \bar{z} \rangle, i \in I,$$

$$(\overline{F}_i \circ E)(\widehat{z}) - (\overline{F}_i \circ E)(\bar{z}) > \langle \nabla(\overline{F}_i \circ E)(\bar{z}), \widehat{z} - \bar{z} \rangle, i \in I.$$

Hence, we derive from the above inequalities, (3.5) and (3.4) that

$$\sum_{t \in J_E} \lambda_t \langle \nabla(g_t \circ E)(\bar{z}), \widehat{z} - \bar{z} \rangle = -\left\langle \sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(\bar{z}), \widehat{z} - \bar{z} \right\rangle > 0. \tag{3.6}$$

As $\widehat{z} \in \Omega_E$ and $(g_t \circ E)(\bar{z}) = 0$ for all $t \in J_E$, we get $(g_t \circ E)(\widehat{z}) \le (g_t \circ E)(\bar{z}), \forall t \in J_E$. Thus, by the differentiable E-convexity of $g_t (t \in T)$ at $\bar{z}$, we have

$$\sum_{t \in J} \lambda_t \langle \nabla (g_t \circ E)(\bar{z}), \widehat{z} - \bar{z} \rangle \le \sum_{t \in J} \lambda_t ((g_t \circ E)(\widehat{z}) - (g_t \circ E)(\bar{z})) \le 0,$$

contradicting with (3.6). □

Example 3.7. Consider the following nonconvex and nonsmooth multiobjective semi-infinite programming

(P): $LU - \min \mathscr{F}(x) = (F_1(x), F_2(x)) = ([\sqrt[3]{x_1} + x_2^2, \sqrt[3]{x_1} + 2x_2^2], [x_2^2 + x_2, 2x_2^2 + x_2])$
s.t. $g_t(x) = -t\sqrt[3]{x_1} + (t-1)x_2 \le 0, \quad t \in T = [0,1]$.

We can check that $\Omega = \mathbb{R}^2_+$. Let $E : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $E(x_1, x_2) = (x_1^3, x_2)$. Hence, E is an one-to-one and onto map. Then,

$$\begin{aligned}(P_E)\text{: } LU - \min (\mathscr{F} \circ E)(z) &= ((F_1 \circ E)(z), (F_2 \circ E)(z)) \\ &= ([z_1 + z_2^2, z_1 + 2z_2^2], [z_2^2 + z_2, 2z_2^2 + z_2]) \\ \text{s.t.} \quad (g_t \circ E)(z) &= -tz_1 + (t-1)z_2 \le 0, \quad t \in T = [0,1].\end{aligned}$$

Therefore, $\Omega_E = \mathbb{R}^2_+$ and $E(\Omega_E) = \Omega$. Let us take $\bar{z} = (0,0) \in \Omega_E$. We can check that $\bar{z} \in WE(P_E, 1)$. Moreover, by some calculations, one has

$$\nabla(\underline{F}_1 \circ E)(\bar{z}) = \nabla(\overline{F}_1 \circ E)(\bar{z}) = (1,0), \nabla(\underline{F}_2 \circ E) \circ E)(\bar{z}) = \nabla(\overline{F}_2 \circ E)(\bar{z}) = (0,1),$$

$$T_E(\bar{z}) = T(\bar{z}) = T, \nabla(g_t \circ E)(z) = (-t, t-1), \forall t \in T, \bigcup_{t \in T_E(\bar{z})} \nabla(g_t \circ E)(\bar{z}) = [-1,0] \times [-1,0]$$

$$\text{pos} \bigcup_{t \in T_E(\bar{z})} \nabla(g_t \circ E)(\bar{z}) = -\mathbb{R}^2_+, \left(\bigcup_{t \in T(\bar{z})} \nabla(g_t \circ E)(\bar{z}) \right)^- = \mathbb{R}^2_+, T(\Omega, \bar{z}) = \mathbb{R}^2_+.$$

Thus, (ACQ_E) holds at $\bar{z}$ and all assumptions in Proposition 3.5 are satisfied. Now, let $\alpha^L = \alpha^U = (\frac{1}{4}, \frac{1}{4})$ and $\lambda : T \to \mathbb{R}$ be defined by $\lambda(t)$ equal to 1 if $t = 1/2$; and equal to zero otherwise. Then, $\alpha^L, \alpha^U \in \mathbb{R}^2_+$ with $\sum_{i=1}^{2} (\alpha_i^L + \alpha_i^U) = 1$, $\lambda \in \Lambda_E(\bar{z})$ and

$$\begin{aligned}&\sum_{i=1}^{2} \alpha_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^{2} \alpha_i^U \nabla(\overline{F}_i \circ E)(\bar{z}) + \sum_{t \in T} \lambda_t \nabla g_t(\bar{x}) \\ &= \frac{1}{4}.((1,0) + (0,1)) + \frac{1}{4}.((1,0) + (0,1)) + 1.(-\frac{1}{2}, -\frac{1}{2}) = (0,0),\end{aligned}$$

i.e., the conclusion of Proposition 3.5 is satisfied.

Moreover, we can check that $F_i (i = 1,2)$ are strongly differentiable LU-E-convex at $\bar{z}$ and $g_t (t \in T)$ are differentiable E-convex at $\bar{z}$. Hence, all assumptions in Proposition 3.6 hold. Then, it follows that $\bar{z} \in E(P_E, 1)$. By virtue of Lemma 3.3, one concludes that $\bar{x} = E(\bar{z}) = (0,0) \in E(P, 1)$.

4. DUALITY

Let A_i, B_i be in $\mathscr{K}_C$ for all $i = 1, \dots, m$ and $\mathscr{A} := (A_1, A_2, \dots, A_m)$, $\mathscr{B} := (B_1, B_2, \dots, B_m)$. In what follows, we use the following notations for the sake of convenience:

$$\mathscr{A} \preceq_{LU} \mathscr{B} \Leftrightarrow \begin{cases} A_i \le_{LU} B_i, & \forall i \in I, \\ A_k <_{LU} B_k, & \text{for at least one } k \in I, \end{cases}$$

$\mathscr{A} \not\preceq_{LU} \mathscr{B}$ is the negation of $\mathscr{A} \preceq_{LU} \mathscr{B}$.

Note that $\bar{x} \in E(P,1)$ if there is no $x \in \Omega$ satisfying $\mathscr{F}(x) \preceq_{LU} \mathscr{F}(\bar{x})$.

4.1. *E*-**Mond-Weir duality.** For $u \in \mathbb{R}^n, (\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ and $\lambda \in \mathbb{R}^{|T|}_+$, denote

$$\mathscr{L}_E(u, \alpha^L, \alpha^U, \lambda) := (\mathscr{F} \circ E)(u) = ((F_1 \circ E)(u), ..., (F_m \circ E)(u)).$$

We consider the E-Mond-Weir [32] dual problem (D_{MW_E}) of (P) as follows

$$\text{LU}-\max \mathscr{L}_E(u, \alpha^L, \alpha^U, \lambda) = ([(\underline{F}_1 \circ E)(u), (\overline{F}_1 \circ E)(u)], ..., [(\underline{F}_m \circ E)(u), \overline{F}_m \circ E)(u)])$$

$$\text{s.t.} \quad \sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(u) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(u) + \sum_{t\in T} \lambda_t \nabla(g_t \circ E)(u) = 0,$$

$$\sum_{t\in T} \lambda_t (g_t \circ E)(u) \geq 0, u \in \mathbb{R}^n, (\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}, \lambda \in \mathbb{R}^{|T|}_+.$$

The feasible set of (D_{MW_E}) is

$$\Omega_{MW_E} := \Big\{(u, \alpha^L, \alpha^U, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \mid (\alpha^L, \alpha^U) \neq (0,0),$$
$$\sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(u) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(u) + \sum_{t\in T} \lambda_t \nabla(g_t \circ E)(u) = 0, \sum_{t\in T} \lambda_t (g_t \circ E)(u) \geq 0\Big\}.$$

and its projection on $\mathbb{R}^n$ is $Y_{MW_E} := \{u \in \mathbb{R}^n \mid (u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}\}$.

Definition 4.1. The point $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{MW_E}$ is said to be a type-1 E-optimal solution of (D_{MW_E}), denoted by $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{MW_E}, 1)$, if there is no $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}$ such that $\mathscr{L}_E(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq_{LU} \mathscr{L}_E(u, \alpha^L, \alpha^U, \lambda)$.

Proposition 4.2. *(Weak duality between (P_E) and (D_{MW_E})) Let $z \in \Omega_E$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}$. If $F_i (i \in I)$ are strongly differentiable LU-E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$ and $g_t (t \in T)$ are differentiable E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$, then $(\mathscr{F} \circ E)(z) \not\preceq \mathscr{L}_E(u, \alpha^L, \alpha^U, \lambda)$.*

Proof. Since $z \in \Omega_E$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}$, we have

$$(g_t \circ E)(z) \leq 0, \forall t \in T, \tag{4.1}$$

$$\sum_{t\in T} \lambda_t (g_t \circ E)(u) \geq 0, \tag{4.2}$$

$$\sum_{i=1}^m \alpha_i^L \nabla(\underline{F}_i \circ E)(u) + \sum_{i=1}^m \alpha_i^U \nabla(\overline{F}_i \circ E)(u) + \sum_{t\in T} \lambda_t \nabla(g_t \circ E)(u) = 0. \tag{4.3}$$

Suppose to the contrary that $(\mathscr{F} \circ E)(z) \preceq_{LU} \mathscr{L}(u, \alpha^L, \alpha^U, \lambda) = (\mathscr{F} \circ E)(u)$, i.e.,

$$\begin{cases} (F_i \circ E)(z) \leq_{LU} (F_i \circ E)(u), & \forall i \in I, \\ (F_{i_0} \circ E)(z) <_{LU} (F_{i_0} \circ E)(u), & \text{for at least one } i_0 \in I. \end{cases}$$

This implies that $(\underline{F}_i \circ E)(z) \leq (\underline{F}_i \circ E)(u)$, $(\overline{F}_i \circ E)(x) \leq (\overline{F}_i \circ E)(u)$ for all $i \in I$, and for at least one $i_0 \in I$, we have

$$\begin{cases} (\underline{F}_{i_0} \circ E)(z) < \underline{F}_{i_0} \circ E)(u) \\ (\overline{F}_{i_0} \circ E)(z) \leq \overline{F}_{i_0} \circ E)(u), \end{cases} \text{or} \begin{cases} (\underline{F}_{i_0} \circ E)(z) \leq \underline{F}_{i_0} \circ E)(u) \\ (\overline{F}_{i_0} \circ E)(z) < \overline{F}_{i_0} \circ E)(u), \end{cases}$$

$$\text{or} \begin{cases} (\underline{F}_{i_0} \circ E)(z) < \underline{F}_{i_0} \circ E)(u) \\ (\overline{F}_{i_0} \circ E)(z) < \overline{F}_{i_0} \circ E)(u). \end{cases}$$

Hence, $z \neq u$. For $(\alpha^L, \alpha^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$, we obtain

$$\sum_{i=1}^m \alpha_i^L (\underline{F}_i \circ E)(z) + \sum_{i=1}^m \alpha_i^U (\overline{F}_i \circ E)(z) \leq \sum_{i=1}^m \alpha_i^L (\underline{F}_i \circ E)(u) + \sum_{i=1}^m \alpha_i^U (\overline{F}_i \circ E)(u). \tag{4.4}$$

Since $\underline{F}_i, \overline{F}_i (i \in I)$ are strictly differentiable E-convex and $z \neq u$, we have
$(\underline{F}_i \circ E)(z) - (\underline{F}_i \circ E)(u) > \langle \nabla(\underline{F}_i \circ E)(u), z-u \rangle, i \in I,$
$(\overline{F}_i \circ E)(z) - (\overline{F}_i \circ E)(u) > \langle \nabla(\overline{F}_i \circ E)(u), z-u \rangle, i \in I.$
In view of the above inequalities and (4.4), we arrive at

$$\left\langle \sum_{i=1}^{m} \alpha_i^L \nabla(\underline{F}_i \circ E)(u) + \sum_{i=1}^{m} \alpha_i^U \nabla(\overline{F}_i \circ E)(u), z-u \right\rangle < 0. \tag{4.5}$$

In addition, from (4.1), (4.2), one has $\sum_{t\in T} \lambda_t (g_t \circ E)(z) \leq 0 \leq \sum_{t\in T} \lambda_t (g_t \circ E)(u)$. This along with the differentiable E-convexity of $g_t (t \in T)$ leads that

$$\sum_{t\in T} \lambda_t \langle \nabla(g_t \circ E)(u), z-u \rangle \leq 0. \tag{4.6}$$

By adding (4.5) and (4.6), we have

$$\left\langle \sum_{i=1}^{m} \alpha_i^L \nabla(\underline{F}_i \circ E)(u) + \sum_{i=1}^{m} \alpha_i^U \nabla(\overline{F}_i \circ E)(u) + \sum_{t\in T} \lambda_t \nabla(g_t \circ E)(u), z-u \right\rangle < 0,$$

which contradicts (4.3). □

Corollary 4.3. *(Weak duality between (P) and (D_{MW_E})) Let $x \in \Omega$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}$. If $F_i (i \in I)$ are strongly differentiable LU-E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$ and $g_t (t \in T)$ are differentiable E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$, then $\mathscr{F}(x) \not\preceq \mathscr{L}_E(u, \alpha^L, \alpha^U, \lambda)$.*

Proof. Let $x \in \Omega$ and $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}$. Since $E(\Omega_E) = \Omega$, there exists $z \in \Omega_E$ such that $x = E(z)$ and $\mathscr{F}(x) = (\mathscr{F} \circ E)(z)$. The conclusion follows from Proposition 4.2. □

Proposition 4.4. *(Strong duality between (P_E) and (D_{MW_E})) Let $\bar{z} \in WE(P_E, 1)$ and (ACQ_E) hold at $\bar{z}$. Then, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in (\mathbb{R}^m_+ \times \mathbb{R}^m_+) \setminus \{(0,0)\}$ and $\bar{\lambda} \in \Lambda_E(\bar{z})$ such that $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{MW_E}$ and $(\mathscr{F} \circ E)(\bar{z}) = \mathscr{L}_E(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$. Furthermore, if $F_i (i \in I)$ are strongly differentiable LU-E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$ and $g_t (t \in T)$ are differentiable E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$, then $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{MW_E}, 1)$.*

Proof. By Proposition 3.5, we have that there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ with $\sum_{i=1}^{m} (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$ and $\bar{\lambda} \in \Lambda_E(\bar{z})$ such that

$$\sum_{i=1}^{m} \bar{\alpha}_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i=1}^{m} \bar{\alpha}_i^U \nabla(\overline{F}_i \circ E)(\bar{z}) + \sum_{t\in T} \bar{\lambda}_t \nabla(g_t \circ E)(\bar{z}) = 0.$$

Since $\bar{\lambda} \in \Lambda_E(\bar{z}), \bar{\lambda}_t (g_t \circ E)(\bar{z}) = 0$ for all $t \in T$, we have $\sum_{t\in T} \bar{\lambda}_t (g_t \circ E)(\bar{z}) = 0$. Thus, $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{MW_E}$ and $(\mathscr{F} \circ E)(\bar{z}) = \mathscr{L}_E(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$. If $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \notin E(D_{MW_E}, 1)$, then there exists $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{MW_E}$ such that

$$(\mathscr{F} \circ E)(\bar{z}) = \mathscr{L}_E(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq \mathscr{L}_E(u, \alpha^L, \alpha^U, \lambda),$$

which contradicts Proposition 4.2. Therefore, $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{MW_E}, 1)$. □

Corollary 4.5. *(Strong E-duality between (P) and (D_{MW_E})) Let $\bar{x} \in WE(P, 1)$ and $E(\bar{z}) = \bar{x}$ with $\bar{z} \in WE(P_E, 1)$ as in Lemma 3.3. Suppose that (ACQ_E) holds at $\bar{z}$. Then, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \setminus \{(0,0)\}$ and $\bar{\lambda} \in \Lambda_E(\bar{z})$ such that $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{MW_E}$. Furthermore, if $F_i (i \in I)$*

are strongly differentiable LU-E-convex at u on $\Omega_{MW_E} \cup Y_{MW_E}$ *and* $g_t (t \in T)$ *are differentiable E-convex at u on* $\Omega_{MW_E} \cup Y_{MW_E}$*, then* $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{MW_E}, 1)$.

4.2. ***E*-Wolfe duality.** For $u \in \mathbb{R}^n, \alpha^L, \alpha^U \in \mathbb{R}^m_+$ with $\sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1$, $\lambda \in \mathbb{R}^{|T|}_+$ and $e = (1,..,1) \in \mathbb{R}^m$, we denote

$$\widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda) := (\mathscr{F} \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u) e$$
$$= \left((F_1 \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u), \ldots, (F_m \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u) \right).$$

Note that, for $b \in \mathbb{R}$, $[(\underline{F}_i \circ E)(u), (\overline{F}_i \circ E)(u)] + b = [(\underline{F}_i \circ E)(u) + b, (\overline{F}_i \circ E)(u) + b], \forall i \in I$. We define the E-Wolfe [33] type dual problem as follows:

$$(D_{W_E}) \text{: } LU - \max \widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda) := (\mathscr{F} \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u) e$$
$$\text{s.t.} \quad \sum_{i=1}^{m} \alpha_i^L \nabla (\underline{F}_i \circ E)(u) + \sum_{i=1}^{m} \alpha_i^U \nabla (\overline{F}_i \circ E)(u) + \sum_{t \in T} \lambda_t \nabla (g_t \circ E)(u) = 0$$
$$u \in \mathbb{R}^n, \alpha^L, \alpha^U \in \mathbb{R}^m_+, \sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1, \lambda \in \mathbb{R}^{|T|}_+.$$

The feasible set of (D_{W_E}) is

$$\Omega_{W_E} := \Big\{ (u, \alpha^L, \alpha^U, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+ \mid \sum_{i=1}^{m} (\alpha_i^L + \alpha_i^U) = 1,$$
$$\sum_{i=1}^{m} \alpha_i^L \nabla (\underline{F}_i \circ E)(u) + \sum_{i=1}^{m} \alpha_i^U \nabla (\overline{F}_i \circ E)(u) + \sum_{t \in T} \lambda_t \nabla (g_t \circ E)(u) = 0 \Big\}$$

and its projection on $\mathbb{R}^n$ is $Y_{W_E} := \{ u \in \mathbb{R}^n \mid (u, \alpha^L, \alpha^U, \lambda) \in \Omega_{W_E} \}$.

Definition 4.6. The point $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{W_E}$ is said to be a type-1 E-optimal solution of (D_{W_E}), denoted by $(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{W_E}, 1)$, if there is no $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{W_E}$ such that $\widetilde{\mathscr{L}_E}(\bar{u}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq \widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda)$.

Proposition 4.7. *(Weak duality between* (P_E) *and* (D_{W_E})*) Let* $z \in \Omega_E$ *and* $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{W_E}$*. If* $F_i (i \in I)$ *are strongly differentiable LU-E-convex at u on* $\Omega_{W_E} \cup Y_{W_E}$ *and* $g_t (t \in T)$ *are differentiable E-convex at u on* $\Omega_{W_E} \cup Y_{W_E}$*, then* $(\mathscr{F} \circ E)(z) \npreceq \widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda)$.

Proof. Since $z \in \Omega_E$ and $(u, \alpha, \lambda) \in \Omega_{W_E}$, one has

$$(g_t \circ E)(z) \leq 0, \forall t \in T, \tag{4.7}$$

and

$$\sum_{i=1}^{m} \alpha_i^L \nabla (\underline{F}_i \circ E)(u) + \sum_{i=1}^{m} \alpha_i^U \nabla (\overline{F}_i \circ E)(u) + \sum_{t \in T} \lambda_t \nabla (g_t \circ E)(u) = 0. \tag{4.8}$$

Suppose to the contrary that $(\mathscr{F} \circ E)(z) \preceq_{LU} \widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda)$. This implies that $(\underline{F}_i \circ E)(z) \leq (\underline{F}_i \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u)$, and $(\overline{F}_i \circ E)(x) \leq (\overline{F}_i \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u)$ for all $i \in I$, and for at least one $i_0 \in I$,

$$\begin{cases} (\underline{F}_{i_0} \circ E)(z) < \underline{F}_{i_0} \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u) \\ (\overline{F}_{i_0} \circ E)(z) \leq (\overline{F}_{i_0} \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u), \end{cases}$$
$$\text{or} \begin{cases} (\underline{F}_{i_0} \circ E)(z) \leq \underline{F}_{i_0} \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u) \\ (\overline{F}_{i_0} \circ E)(z) < (\overline{F}_{i_0} \circ E)(u) + \sum_{t \in T} \lambda_t (g_t \circ E)(u), \end{cases}$$

$$\text{or}\left\{\begin{array}{l}(\underline{F}_{i_0}\circ E)(z)<\underline{F}_{i_0}\circ E)(u)+\sum\limits_{t\in T}\lambda_t(g_t\circ E)(u)\\(\overline{F}_{i_0}\circ E)(z)<(\overline{F}_{i_0}\circ E)(u)+\sum\limits_{t\in T}\lambda_t(g_t\circ E)(u).\end{array}\right.$$

Then, $z\neq u$. If not,

$$-\left(\sum_{t\in T}\lambda_t(g_t\circ E)(u)\right)e\preceq 0,$$

which is impossible since $\lambda\in\mathbb{R}_+^{|T|}$ and $(g_t\circ E)(u)=(g_t\circ E)(z)\leq 0$ for all $t\in T$. Moreover, for $(\alpha^L,\alpha^U)\in\mathbb{R}_+^m\times\mathbb{R}_+^m\setminus\{(0,0)\}$ with $\sum\limits_{i=1}^m(\alpha_i^L+\alpha_i^U)=1$, we obtain

$$\begin{aligned}&\sum_{i=1}^m\alpha_i^L(\underline{F}_i\circ E)(z)+\sum_{i=1}^m\alpha_i^U(\overline{F}_i\circ E)(z)+\sum_{t\in T}\lambda_t(g_t\circ E)(z)\\&\leq\sum_{i=1}^m\alpha_i^L(\underline{F}_i\circ E)(z)+\sum_{i=1}^m\alpha_i^U(\overline{F}_i\circ E)(z)\\&<\sum_{i=1}^m\alpha_i^L(\underline{F}_i\circ E)(u)+\sum_{i=1}^m\alpha_i^U(\overline{F}_i\circ E)(u)+\sum_{i=1}^m(\alpha_i^L+\alpha_i^U).\sum_{t\in T}\lambda_t(g_t\circ E)(u)\\&=\sum_{i=1}^m\alpha_i^L(\underline{F}_i\circ E)(u)+\sum_{i=1}^m\alpha_i^U(\overline{F}_i\circ E)(u)+\sum_{t\in T}\lambda_t(g_t\circ E)(u).\end{aligned}\tag{4.9}$$

Since $\underline{F}_i,\overline{F}(i\in I)$ are strictly differentiable E-convex and $z\neq u$, we have

$$(\underline{F}_i\circ E)(z)-(\underline{F}_i\circ E)(u)>\langle\nabla(\underline{F}_i\circ E)(u),z-u\rangle,i\in I,$$
$$(\overline{F}_i\circ E)(z)-(\overline{F}_i\circ E)(u)>\langle\nabla(\overline{F}_i\circ E)(u),z-u\rangle,i\in I,$$

and

$$(g_t\circ E)(z)-(g_t\circ E)(u)\geq\langle\nabla(g_t\circ E)(u),z-u\rangle,t\in T.$$

Taking into account the above inequalities and (4.9), we arrive at

$$\begin{aligned}&\left(\sum_{i=1}^m\alpha_i^L(\underline{F}_i\circ E)(z)+\sum_{i=1}^m\alpha_i^U(\overline{F}_i\circ E)(z)+\sum_{t\in T}\lambda_t(g_t\circ E)(z)\right)\\&\quad-\left(\sum_{i=1}^m\alpha_i^L(\underline{F}_i\circ E)(u)+\sum_{i=1}^m\alpha_i^U(\overline{F}_i\circ E)(u)+\sum_{t\in T}\lambda_t(g_t\circ E)(u)\right)\\&\geq\left\langle\sum_{i=1}^m\alpha_i^L\nabla(\underline{F}_i\circ E)(u)+\sum_{i=1}^m\alpha_i^U\nabla(\overline{F}_i\circ E)(u)+\sum_{t\in T}\lambda_t\nabla(g_t\circ E)(u),z-u\right\rangle.\end{aligned}\tag{4.10}$$

Combining (4.9) and (4.10), one obtains

$$\left\langle\sum_{i=1}^m\alpha_i^L\nabla(\underline{F}_i\circ E)(u)+\sum_{i=1}^m\alpha_i^U\nabla(\overline{F}_i\circ E)(u)+\sum_{t\in T}\lambda_t\nabla(g_t\circ E)(u),z-u\right\rangle<0,$$

contradicting 4.8). □

Corollary 4.8. *(Weak E-duality between (P) and (D_{W_E})) Let $x\in\Omega$ and $(u,\alpha^L,\alpha^U,\lambda)\in\Omega_{W_E}$. If $F_i(i\in I)$ are strongly differentiable LU-E-convex at u on $\Omega_{W_E}\cup Y_{W_E}$ and $g_t(t\in T)$ are differentiable E-convex at u on $\Omega_{W_E}\cup Y_{W_E}$, then $\mathscr{F}(x)\npreceq\widetilde{\mathscr{L}_E}(u,\alpha^L,\alpha^U,\lambda)$.*

Proposition 4.9. *(Strong duality between (P_E) and (D_{W_E})) Let $\bar{z} \in WE(P_E, 1)$ and (ACQ_E) hold at $\bar{z}$. Then, there exist $\bar{\alpha}^L, \bar{\alpha}^U \in \mathbb{R}^m_+$ with $\sum_{i=1}^{m} (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$ and $\bar{\lambda} \in \Lambda_E(\bar{z})$ such that $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{D_{W_E}}$ and $(\mathscr{F} \circ E)(\bar{z}) = \widetilde{\mathscr{L}_E}(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$. Furthermore, if $F_i (i \in I)$ are strongly differentiable LU-E-convex at u on $\Omega_{W_E} \cup Y_{W_E}$ and $g_t (t \in T)$ are differentiable E-convex at u on $\Omega_{W_E} \cup Y_{W_E}$, then $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{W_E}, 1)$.*

Proof. By Proposition 3.5, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ with $\sum_{i=1}^{m} (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$ and $\bar{\lambda} \in \Lambda_E(\bar{z})$ such that

$$\sum_{i \in I} \bar{\alpha}_i^L \nabla(\underline{F}_i \circ E)(\bar{z}) + \sum_{i \in I} \bar{\alpha}_i^U \nabla(\overline{F}_i \circ E)(\bar{z}) + \sum_{t \in T} \bar{\lambda}_t \nabla(g_t \circ E)(\bar{z}) = 0.$$

Since $\bar{\lambda} \in \Lambda_E(\bar{z})$, we have $\bar{\lambda}_t (g_t \circ E)(\bar{z}) = 0$ for all $t \in T$. Thus,

$$(\mathscr{F} \circ E)(\bar{z}) = (\mathscr{F} \circ E)(\bar{z}) + \left(\sum_{t \in T} \bar{\lambda}_t (g_t \circ E)(\bar{z}) \right) e = \widetilde{\mathscr{L}_E}(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}),$$

i.e.,

$$(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{W_E}$$

and

$$(\mathscr{F} \circ E)(\bar{z}) = \widetilde{\mathscr{L}_E}(\bar{x}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}).$$

Suppose to the contrary that $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \notin E(D_{W_E}, 1)$. By definition, there exists $(u, \alpha^L, \alpha^U, \lambda) \in \Omega_{W_E}$ such that

$$(\mathscr{F} \circ E)(\bar{z}) = \widetilde{\mathscr{L}_E}(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \preceq \widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda),$$

which contradicts Proposition 4.7. So, $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{W_E}, 1)$. □

Corollary 4.10. *(Strong E-duality between (P) and (D_{W_E})) Let $\bar{x} \in WE(P, 1)$ and $E(\bar{z}) = \bar{x}$ with $\bar{z} \in WE(P_E, 1)$ as in Lemma 3.3. Assume that (ACQ_E) holds at $\bar{z}$. Then, there exist $(\bar{\alpha}^L, \bar{\alpha}^U) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ with $\sum_{i=1}^{m} (\bar{\alpha}_i^L + \bar{\alpha}_i^U) = 1$ and $\bar{\lambda} \in \Lambda_E(\bar{z})$ such that $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in \Omega_{W_E}$ and $\mathscr{F}(\bar{x}) = \widetilde{\mathscr{L}_E}(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda})$. Furthermore, if $F_i (i \in I)$ are strongly differentiable LU-E-convex at u on $\Omega_{W_E} \cup Y_{W_E}$ and $g_t (t \in T)$ are differentiable E-convex at u on $\Omega_{W_E} \cup Y_{W_E}$, then $(\bar{z}, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}) \in E(D_{W_E}, 1)$.*

Example 4.11. Consider the following nonconvex and nonsmooth multiobjective semi-infinite programming

$$(P):\ LU - \min \mathscr{F}(x) = (F_1(x), F_2(x)) = ([\sqrt[3]{x_1} + x_2^2, \sqrt[3]{x_1} + 2x_2^2], [x_2^2 + x_2, 2x_2^2 + x_2])$$
$$\text{s.t.} \quad g_t(x) = -t\sqrt[3]{x_1} + (t-1)x_2 \leq 0, \quad t \in T = [0,1].$$

We can check that $\Omega = \mathbb{R}^2_+$. Let $E : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $E(x_1, x_2) = (x_1^3, x_2)$. Hence, E is an one-to-one and onto map. Then,

$$(P_E):\ LU - \min(\mathscr{F} \circ E)(z) = ([z_1 + z_2^2, z_1 + 2z_2^2], [z_2^2 + z_2, 2z_2^2 + z_2])$$
$$\text{s.t.} \quad (g_t \circ E)(z) = -tz_1 + (t-1)z_2 \leq 0, \quad t \in T = [0,1].$$

Hence, $\Omega_E = \mathbb{R}^2_+$ and $E(\Omega_E) = \Omega$.

$$(D_{W_E}):\ LU - \max \widetilde{\mathscr{L}_E}(u, \alpha^L, \alpha^U, \lambda) = ([u_1 + u_2^2, u_1 + 2u_2^2], [u_2^2 + u_2, 2u_2^2 + u_2]) + \sum_{t \in T} \lambda_t (-tu_1 + (1-t)u_2)(1,1)$$

$$\text{s.t.} \quad \alpha_1^L(u,0)+\alpha_2^L(0,u)+\alpha_1^U(u,0)+\alpha_2^U(0,u)+\sum_{t\in T}\lambda_t(-t,t-1)=0$$
$$u\in\mathbb{R}^2, \alpha^L,\alpha^U\in\mathbb{R}^2_+, \sum_{i=1}^{2}(\alpha_i^L+\alpha_i^U)=1, \lambda\in\mathbb{R}_+^{|T|}.$$

Let us take $\bar{z}=(0,0)\in\Omega_E$. We can check that $\bar{z}\in WE(P_E,1)$. In view of Example 3.7, all the assumptions in Proposition 3.5 are satisfied. Now, let $\bar{\alpha}^L=\bar{\alpha}^U=(\frac{1}{4},\frac{1}{4})$ and $\bar{\lambda}:T\to\mathbb{R}$ be defined by $\bar{\lambda}(t)$ equal to 1 if $t=1/2$; and equal to zero otherwise. Then, $\alpha^L,\alpha^U\in\mathbb{R}^2_+$ with $\sum_{i=1}^{2}(\alpha_i^L+\alpha_i^U)=1$, $\bar{\lambda}\in\Lambda_E(\bar{z})$ and

$$\sum_{i=1}^{2}\bar{\alpha}_i^L\nabla(\underline{F}_i\circ E)(\bar{z})+\sum_{i=1}^{2}\bar{\alpha}_i^U\nabla(\overline{F}_i\circ E)(\bar{z})+\sum_{t\in T}\bar{\lambda}_t\nabla g_t(\bar{x})=(0,0),$$

i.e., $(\bar{z},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\in\Omega_{W_E}$. Moreover, we can check that $F_i(i=1,2)$ are strongly differentiable LU-E-convex at $\bar{z}$ and $g_t(t\in T)$ are differentiable E-convex at $\bar{z}$. Hence, all the assumptions in Proposition 4.4 hold. Then, it follows that $(\bar{z},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda})\in E(D_{W_E},1)$.

REFERENCES

[1] G. Caristi, M. Ferrara, Necessary conditions for nonsmooth multiobjective semi-infinite problems using Michel-Penot subdifferential, Decisions Econ. Finan. 40 (2017), 103-113.

[2] T.D. Chuong, D.S. Kim, Nonsmooth semi-infinite multiobjective optimization problems, J. Optim. Theory Appl. 160 (2014), 748-762.

[3] T.D. Chuong, J.C. Yao, Isolated and proper efficiencies in semi-infinite vector optimization problems, J. Optim. Theory Appl. 162 (2014), 447-462.

[4] A. Kabgani, M. Soleimani-damaneh, Characterization of (weakly/properly/robust) efficient solutions in nonsmooth semi-infinite multiobjective optimization using convexificators, Optimization 67 (2017), 217-235.

[5] N. Kanzi, S. Nobakhtian, Optimality conditions for nonsmooth semi-infinite multiobjective programming, Optim. Lett. 8 (2014), 1517-1528.

[6] N. Kanzi, On strong KKT optimality conditions for multiobjective semi-infinite programming problems with Lipschitzian data, Optim. Lett. 8 (2015), 1121-1129.

[7] L.T. Tung, Strong Karush-Kuhn-Tucker optimality conditions for multiobjective semi-infinite programming via tangential subdifferential, RAIRO Oper. Res. 52 (2018), 1019-1041.

[8] L.T. Tung, Strong Karush-Kuhn-Tucker optimality conditions for Borwein properly efficient solutions of multiobjective semi-infinite programming, Bull. Braz. Math. Soc. (N.S.) (2019), doi: 10.1007/s00574-019-00190-9.

[9] L.T. Tung, Karush-Kuhn-Tucker optimality conditions and duality for multiobjective semi-infinite programming via tangential subdifferentials, Numer. Funct. Anal. Optim. 41 (2020), 659-684.

[10] L.T. Tung, Karush-Kuhn-Tucker optimality conditions for nonsmooth multiobjective semidefinite and semi-infinite programming, J. Appl. Numer. Optim. 1 (2019), 63-75.

[11] Y. Chalco-Cano, W.A. Lodwick, R. Osuna-Gómez, A. Rufián-Lizana, The Karush-Kuhn-Tucker optimality conditions for fuzzy optimization problems, Fuzzy Optim. Decis. Mak. 15 (2016), 57-73.

[12] D.V. Luu, T.T. Mai, Optimality and duality in constrained interval-valued optimization, 4OR 16 (2018), 311-337.

[13] R. Osuna-Gómez, B. Hernádez-Jiménez, Y. Chalco-Cano, G. Ruiz-Gazón, New efficiency conditions for multiobjective interval-valued programming problems, Inform. Sci. 420 (2017), 235-248.

[14] L.T. Tung, Karush-Kuhn-Tucker optimality conditions and duality for convex semi-infinite programming with multiple interval-valued objective functions, J. Appl. Math. Comput. 62 (2020), 67-91.

[15] H.C. Wu, The Karush-Kuhn-Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions, European J. Oper. Res. 196 (2009), 49-60.

[16] H.C. Wu, The optimality conditions for optimization problems with convex constraints and multiple fuzzy-valued objective functions, Fuzzy Optim. Decision Making 8 (2009), 295-321.

[17] M.A. Goberna, N. Kanzi, Optimality conditions in convex multiobjective SIP, Math. Program. 164 (2017), 67-191.
[18] M. Soleimani-damaneh, E-convexity and its generalizations, Int. J. Comput. Math. 88 (2011), 3335-3349.
[19] E. A. Youness, E-convex sets, E-convex functions, and E-convex programming, J. Optim. Theory Appl. 102 (1999), 439-450.
[20] G.R. Piao, L. G. Jiao, D. S. Kim, Optimality and mixed duality in multiobjective E-convex programming, J. Inequal. Appl. 2015 (2015), 1-13.
[21] Y.R. Syau, E.S. Lee, Some properties of E-convex functions, Appl. Math. Lett. 18 (2015), 1074-1080.
[22] X. M. Yang, On E-convex sets, E-convex functions and E-convex programming, J. Optim. Theory Appl. 109 (2001), 699-704.
[23] E. A. Youness, Quasi and strictly quasi E-convex functions, Stat. Manag. Syst. 4 (2001), 201-210.
[24] A.A. Megahed, H.G. Gomma, E. A. Youness, A. H. El-Banna, Optimality conditions of E-convex programming for an E-differentiable function, J. Inequal. Appl. 2013 (2013), 1-11.
[25] T. Antczak, N. Abdulaleem, E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with inequality and equality constraints, J. Nonlinear Sci. Appl. 12 (2018), 745-764.
[26] T. Antczak, N. Abdulaleem, Optimality and duality results for E-differentiable multiobjective fractional programming problems under E-convexity, J. Inequal. Appl. 2019 (2019), 1-24.
[27] T. Antczak, N. Abdulaleem, Optimality conditions for E-differentiable vector optimization problems with the multiple interval-valued objective function, J. Ind. Manag. Optim. 13 (2019), 1-19.
[28] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, 1990.
[29] R.T. Rockafellar, Convex Analysis, Princeton Math. Ser., Vol. 28, Princeton University Press, Princeton, New Jersey, 1970.
[30] M.A. Goberna,M.A. Lopéz, Linear Semi-Infinite Optimization, Wiley, Chichester, 1998.
[31] J.B. Hiriart-Urruty, C. Lemaréchal, Convex Analysis and Minimization Algorithms I, Springer, Berlin, 1993.
[32] B. Mond, T.Weir, Generalized concavity and duality. In: S. Schaible,W.T. Ziemba, (eds.) Generalized Concavity in Optimization and Economics, pp. 263-279, Academic Press, New York, 1981.
[33] P. Wolfe, A duality theorem for nonlinear programming, Quart. Appl. Math. 19 (1961), 239-244.