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THREE SOLUTIONS FOR FRACTIONAL $p(x,.)$-LAPLACIAN DIRICHLET PROBLEMS WITH WEIGHT

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Abstract. The aim of this paper is to investigate weak solutions for a problem involving the fractional $p(x,.)$-Laplacian operator with weight. The main tool used for obtaining the existence result is a recent three critical-points theorem established by Ricceri. Moreover, we establish some new continuous and compact embedding theorems of the fractional Sobolev spaces with variable exponent into the Hölder spaces.

Keywords. Fractional $p(.,.)$-Laplacian operator; Weighted variable exponent spaces; Continuous and compact embeddings; Three critical-points theorem.

1. Introduction

Recently, the study of partial differential equations and variational problems with nonstandard $p(x)$-growth conditions has undergone a fast development. The study of such problems has a strong motivation due to the fact that they can model various phenomena, which arise in the problem of elastic mechanics [1], electrorheological fluids [2] or image restoration [3]. We know that the $p(x)$-Laplacian operator, where $p(.)$ is a continuous function, possesses more complicated properties than the p-Laplacian operator mainly due to the fact that it is not homogeneous. There has been many results devoted to the existence of solutions for variable exponent problems; see, e.g., [4, 5, 6] and the references therein. More recently, some researchers extended the integer case to the fractional one. In particular, many authors generalized the last operator to the fractional case (fractional $p(x,.)$-Laplacian operator). Then, they introduced a suitable functional space to study the problems in which a fractional variable exponent operator is involved; see, e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the reference therein.

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In this paper, we are concerned with the following Dirichlet problem

$$\begin{cases} \big(-\Delta_{p(x,.)}\big)^s u(x) + w(x)|u|^{\bar{p}(x)-2}u = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathcal{P}^s_w$}$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain, $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1,+\infty)$ is a bounded continuous functions and $\bar{p}(x) = p(x,x)$ for all $x \in \overline{\Omega}$, $s \in (0,1)$, λ, μ are two positive real numbers. $w : \Omega \longrightarrow (0,+\infty)$ such that $w \in L^1(\Omega)$ with $\operatorname*{essinf}_{x\in\Omega} w(x) = w_0 > 0$, and $f, g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are two Carathéodory functions with subcritical growth conditions. Here, the operator $(-\Delta_{p(x,.)})^s$ is the fractional $p(x,.)$-Laplacian operator defined as follows

$$(-\Delta_{p(x,.)})^s u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy \text{ for all } x \in \mathbb{R}^N,$$

where $p.v.$ is a commonly used abbreviation in the principal value sense. Note that $(-\Delta_{p(x,.)})^s$ is a nonlocal pseudo-differential operator of elliptic type which can be seen as a generalization of the fractional p-Laplacian operator $(-\Delta_p)^s$ in the constant exponent case (i.e., when $p(x,.) = p = constant$) and is the fractional version of the well-known $p(x)$-Laplacian operator $\Delta_{p(x)}u(x) = div\big(|\nabla u(x)|^{p(x)-2}u(x)\big)$ (where $p(x) = p(x,x)$) which is associated with the variable exponent Sobolev space.

One typical feature of the problem ($\mathcal{P}^s_w$) is the nonlocality in the sense that the value of $(-\Delta_{p(x,.)})^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on Ω, but actually on the entire space $\mathbb{R}^N$. Therefore, the Dirichlet datum given in $\mathbb{R}^N \setminus \Omega$ (which is different from the classical case of the $p(x)$-Laplacian) and is not simply on $\partial\Omega$, which implies that the first equation in ($\mathcal{P}^s_w$) is no longer a pointwise equation. It is no longer a pointwise identity. Therefore it is often called nonlocal problem. This causes some mathematical difficulties, which make the study of such a problem particularly interesting and challenging.

Recently, there are many associated results on the existence of solutions for the problem ($\mathcal{P}^s_w$) with $\mu = 0$. For instance, in the constant case when $p(x,y) = p$, Afrouzi and Heidarkhani in [18] obtained the existence of three solutions for the problem

$$\begin{cases} -\Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > N$, $\lambda > 0$ and the function f satisfies suitable assumptions. While in [19], Bonanno and Candito studied the same problem with Neumann boundary conditions, and got the same results under different assumptions. Mihăilescu in [20] established the existence of three solutions for the problem

$$\begin{cases} -\Delta_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\inf_{x\in\overline{\Omega}} p(x) > N$, $\lambda > 0$ and $f(x,u) = |u|^{q(x)2}u - u$ with $q(x) < \inf_{x\in\overline{\Omega}} p(x)$, see also [21, 22].

In that context, for the fractional case, Azroul, Benkirane and Srati in [23] used the three critical points theorem to obtain the existence of three weak solutions for a Kirchhff-type problem with homogeneous Dirichlet boundary conditions involving the fractional p-Laplacian operator

$$\begin{cases} M\left([u]_{s,p}\right)(-\Delta)_p^s u(x) = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \qquad (\mathcal{P}_M^{s,p})$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded open domain, $0 < s < 1 < p < \infty$, λ and μ are two real parameters. $M : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous function, which satisfies the (polynomial growth) condition
(M_1) : There exist two constants $m_0 > 0$ and $1 < \alpha < \frac{p_s^*}{p}$ such that

$$M(t) \geqslant m_0 t^{\alpha-1} \quad \text{for all } t \in \mathbb{R}^+,$$

with $p_s^* = \frac{Np}{N-sp}$, and f satisfies some suitable growth conditions. For more related problems, we refer the reader to [24, 25].

This paper is organized as follows. In Section 2, we give some definitions and fundamental properties of generalized Lebesgue spaces $L^{q(x)}$ and fractional Sobolev spaces with variable exponent $W^{s,p(x,y)}$. In addition, we prove a continuous and compact embedding of these spaces into $C^0(\overline{\Omega})$ in the case $sp^- > N$. Moreover, we introduce the weighted fractional Sobolev spaces with variable exponent $W_w^{s,p(x,y)}$ and establish a continuous and compact embedding theorem of these spaces into $C^0(\overline{\Omega})$. In Section 3, we study the existence of at least three solutions of problem $(\mathcal{P}_w^s)$ by using as the main tool a variational principle due to Ricceri.

Throughout this paper, for simplicity, we use c_i to denote the general nonnegative or positive constant (the exact value may change from line to line).

2. Variatoinal setting and preliminaries results

To investigate problem $(\mathcal{P}_w^s)$, we need some basic properties of the Lebesgue spaces. Furthermore, we recall and establish some qualitative properties of the new fractional weighted Sobolev spaces with variable exponent.

2.1. **Variable exponent Lebesgue spaces.** In this subsection, we recall some useful properties of variable exponent spaces. For more details, we refer the reader to [26, 27], and the references therein. Denote by $\mathcal{M}(\Omega)$ the set of all measurable real functions on Ω and consider the set

$$C_+(\overline{\Omega}) = \left\{ q \in C(\overline{\Omega}) : q(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}.$$

For all $q \in C_+(\overline{\Omega})$, we define

$$q^+ = \sup_{x \in \overline{\Omega}} q(x) \quad \text{and} \quad q^- = \inf_{x \in \overline{\Omega}} q(x),$$

such that

$$1 < q^- \leqslant q(x) \leqslant q^+ < +\infty. \tag{2.1}$$

For any $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega |u(x)|^{q(x)} dx < +\infty \right\}.$$

This vector space endowed with the *Luxemburg norm*, which is defined by

$$\|u\|_{L^{q(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leqslant 1 \right\}$$

is a separable reflexive Banach space.

Let $\hat{q} \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\frac{1}{q(x)} + \frac{1}{\hat{q}(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.1. (Hölder's inequality). *If $u \in L^{q(x)}(\Omega)$ and $v \in L^{\hat{q}(x)}(\Omega)$, then*

$$\left| \int_\Omega uv dx \right| \leqslant \left(\frac{1}{q^-} + \frac{1}{\hat{q}^-} \right) \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{\hat{q}(x)}(\Omega)} \leqslant 2\|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{\hat{q}(x)}(\Omega)}.$$

An immediate consequence of the Hölder's inequality is the following.

Corollary 2.2. *If $r(.), q(.) \in C_+(\overline{\Omega})$, define $p(.) \in C_+(\overline{\Omega})$ by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a positive constant C such that, for all $u \in L^{q(x)}(\Omega)$ and $v \in L^{r(x)}(\Omega)$, $uv \in L^{p(x)}(\Omega)$ and

$$\|uv\|_{L^{p(x)}(\Omega)} \leqslant C\|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{r(x)}(\Omega)}.$$

We define the modular of the space $L^{q(x)}(\Omega)$ by

$$\begin{aligned} \rho_{q(.)} : L^{q(x)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \rho_{q(.)}(u) = \textstyle\int_\Omega |u(x)|^{q(x)} dx. \end{aligned}$$

Proposition 2.3. *Let $u \in L^{q(x)}(\Omega)$ and $\{u_k\} \subset L^{q(x)}(\Omega)$. Then*

(i) $\|u\|_{L^{q(x)}(\Omega)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_{q(.)}(u) < 1$ *(resp.* $= 1, > 1$*)*,
(ii) $\|u\|_{L^{q(x)}(\Omega)} < 1 \Rightarrow \|u\|^{q+}_{L^{q(x)}(\Omega)} \leqslant \rho_{q(.)}(u) \leqslant \|u\|^{q-}_{L^{q(x)}(\Omega)}$,
(iii) $\|u\|_{L^{q(x)}(\Omega)} > 1 \Rightarrow \|u\|^{q-}_{L^{q(x)}(\Omega)} \leqslant \rho_{q(.)}(u) \leqslant \|u\|^{q+}_{L^{q(x)}(\Omega)}$,
(iv) $\lim\limits_{k\to+\infty} \|u_k - u\|_{L^{q(x)}(\Omega)} = 0 \quad \Leftrightarrow \quad \lim\limits_{k\to+\infty} \rho_{q(.)}(u_k - u) = 0.$

Now, let $w \in \mathcal{M}(\Omega)$ with $w(x) > 0$ for a.e. $x \in \Omega$. We define the weighted variable exponent Lebesgue space $L^{q(x)}_w(\Omega)$ by

$$L^{q(x)}_w(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega w(x)|u(x)|^{q(x)} dx < +\infty \right\},$$

with the norm

$$\|u\|_{q,w} = \inf\left\{ \gamma > 0 : \int_\Omega w(x) \left| \frac{u(x)}{\gamma} \right|^{q(x)} dx \leqslant 1 \right\}.$$

From now on, we suppose that

$$(w_1): \quad w \in L^1(\Omega) \quad \text{and} \quad \operatorname*{essinf}_{x\in\Omega} w(x) = w_0 > 0.$$

Then $L^{q(x)}_w(\Omega)$ is a Banach space obviously (see [28, 29] for more details). Moreover, the weighted modular on $L^{q(x)}_w(\Omega)$ is defined as follows

$$\begin{aligned} \rho_{q,w} : L^{q(x)}_w(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \rho_{q,w}(u) = \textstyle\int_\Omega w(x)|u(x)|^{q(x)} dx. \end{aligned}$$

As an example of w, we can take $w(x)=(1+|x|)^{\alpha}$, for all $x\in\Omega$, where $\alpha\in\mathbb{R}^{+}$, or $w(x)=(1+|x|)^{\alpha(x)}$ with $\alpha\in C_{+}(\overline{\Omega})$.

The following proposition is similar to Proposition 2.3, and it follows easily from the definition of $\|u\|_{q,w}$ and $\rho_{q,w}$.

Proposition 2.4. *Let $u\in L_{w}^{q(x)}(\Omega)$ and $\{u_n\}\subset L_{w}^{q(x)}(\Omega)$. Then*

(i) $\|u\|_{q,w}<1$ $(resp.=1,>1)\Leftrightarrow\rho_{q,w}(u)<1$ $(resp.=1,>1)$,
(ii) $\|u\|_{q,w}<1\Rightarrow\|u\|_{q,w}^{q+}\leqslant\rho_{q,w}(u)\leqslant\|u\|_{q,w}^{q-}$,
(iii) $\|u\|_{q,w}>1\Rightarrow\|u\|_{q,w}^{q-}\leqslant\rho_{q,w}(u)\leqslant\|u\|_{q,w}^{q+}$,
(iv) $\lim\limits_{n\to+\infty}\|u_n\|_{q,w}=0\quad\Leftrightarrow\quad\lim\limits_{n\to+\infty}\rho_{q,w}(u_n)=0$,
(v) $\lim\limits_{n\to+\infty}\|u_n\|_{q,w}=\infty\quad\Leftrightarrow\quad\lim\limits_{n\to+\infty}\rho_{q,w}(u_n)=\infty$.

2.2. **Fractional Sobolev spaces with variable exponent.** In this subsection, we present the definition and some results on fractional Sobolev spaces with variable exponent that was introduced in [8, 10, 14, 17], and prove a continuous and compact embedding of these spaces into $C^{0}(\overline{\Omega})$ in the case $sp^{-}>N$. Moreover, we introduce the new fractional weighted Sobolev space $W_{w}^{s,p(x,y)}$ and establish a continuous and compact embedding theorem of these spaces into $C^{0}(\overline{\Omega})$.

We first recall some useful results in constant case. Let Ω be an open set in $\mathbb{R}^{N}$. Let $s\in(0,1)$ and $p\in(1,+\infty)$. Then the fractional Sobolev space is defined as follows

$$W^{s,p}(\Omega)=\left\{u\in L^{p}(\Omega):\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p}}{|x-y|^{sp+N}}\,dxdy<+\infty\right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)}=\|u\|_{L^{p}(\Omega)}+[u]_{s,p},$$

where the term defined by

$$[u]_{s,p}=[u]_{s,p}(\Omega)=\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p}}{|x-y|^{sp+N}}\,dxdy\right)^{\frac{1}{p}}.$$

is the so-called *Gagliardo seminorm* of u.

For any nonnegative integer m, let $C^{m}(\Omega)$ be the space of continuous functions whose partial derivatives up to order m are continuous on Ω. If Ω is open, then the continuous functions on Ω are not necessarily bounded. The following defines a useful and important subspace of $C^{m}(\Omega)$.

Definition 2.5. For an open subset Ω of $\mathbb{R}^{N}$, let $C_{b}^{m}(\Omega)$ be the subset of $C^{m}(\Omega)$ consisting of the functions whose partial derivatives of order m are bounded and uniformly continuous on Ω. By endowing this subspace with the norm

$$\|\varphi\|_{C_{b}^{m}(\Omega)}=\sup_{|\alpha|\leqslant m}\sup_{x\in\Omega}|D^{\alpha}\varphi(x)|,$$

we obtain a Banach space.

Remark 2.6. Note that when Ω is a bounded open subset, any function on this space $(C_{b}^{m}(\Omega))$, as well as all its partial derivatives, admits a continuous extension to $\overline{\Omega}$. The space $C_{b}^{m}(\Omega)$ is therefore identical to $C^{m}(\overline{\Omega})$.

Consider now the following important subspace of $C_{b}^{m}(\Omega)$.

Definition 2.7. For $\theta \in (0,1]$, $C_b^{0,\theta}(\Omega)$ denotes the space of Hölder continuous functions of order θ on Ω, defined as follows:

$$C_b^{0,\theta}(\Omega) = \left\{ \varphi \in C_b(\Omega) : \exists c > 0 \text{ for all } (x,y) \in \Omega^2,\ |\varphi(x) - \varphi(y)| \leqslant c|x-y|^\theta \right\},$$

where $C_b(\Omega) = C_b^0(\Omega)$.

Proposition 2.8. ([30, Corollary 4.53 and Theorem 4.54]). *Let $s \in (0,1)$ and let $p \in (1,+\infty)$ such that $sp > N$. Then*

(i) if Ω be a Lipschitz open set in $\mathbb{R}^N$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and, more precisely, $W^{s,p}(\Omega) \hookrightarrow C_b^{0,s-N/p}(\Omega)$,

(ii) if Ω be a Lipschitz bounded open set in $\mathbb{R}^N$, then the embedding of $W^{s,p}(\Omega)$ into $C_b^{0,\theta}(\Omega)$ is compact for all $\theta < s - N/p$.

For more details on fractional Sobolev space, we refer the reader to [30, 31].

Now, we assume that Ω is a Lipschitz bounded open set in $\mathbb{R}^N$ and let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous bounded function such that

$$1 < p^- = \min_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) < +\infty \tag{2.2}$$

and

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \text{ for all } (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{2.3}$$

We set

$$\bar{p}(x) = p(x,x) \text{ for any } x \in \overline{\Omega}.$$

Throughout this paper, s is a fixed real number such that $0 < s < 1$.

We define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$W = W^{s,p(x,y)}(\Omega) = \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy < +\infty, \text{ for some } \gamma > 0 \right\}.$$

The space $W^{s,p(x,y)}(\Omega)$ is a Banach space (see [17]) if it is equipped with the norm

$$\|u\|_W = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $[.]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} := [u]_{s,p(x,y)}(\Omega) = \inf \left\{ \gamma > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy \leqslant 1 \right\}.$$

The space $(W, \|.\|_W)$ is separable reflexive (see, [14, Lemma 3.1]).

In [17], Kaufmann, Rossi and Vidal introduced the variable exponent Sobolev fractional space as follows

$$E = W^{s,q(x),p(x,y)}(\Omega) = \left\{ u \in L^{q(x)}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy < +\infty, \text{ for some } \gamma > 0 \right\},$$

where $q : \overline{\Omega} \longrightarrow (1,+\infty)$ is a continuous function satisfying (2.1).

We would like to mention that a continuous and compact embedding theorem was proved in [17] under the assumption $q(x) > \bar{p}(x) = p(x,x)$ for all $x \in \overline{\Omega}$ and $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. The authors in [8] give a slightly different version of continuous and compact embedding theorem assuming that $q(x) = \bar{p}(x) = p(x,x)$ for all $x \in \overline{\Omega}$ and $sp^+ < N$. In this paper, we establish a new continuous and compact embedding theorem of the space W into $C^0(\overline{\Omega})$, under the assumption $sp^- > N$.

Theorem 2.9. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous function satisfying (2.2) and (2.3) with $sp^- > N$. Then W is continuously embedded in $L^\infty(\Omega)$ and, more precisely,*

$$W \hookrightarrow C_b^{0,s-N/p^-}(\Omega).$$

Proof. We first need to prove that $W \hookrightarrow W^{s,p^-}$, that is, there exists $C > 0$ such that

$$\|u\|_{W^{s,p^-}(\Omega)} = \|u\|_{L^{p^-}(\Omega)} + [u]_{s,p^-} \leqslant C\|u\|_W = C\left[\|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p(x,y)}\right].$$

Since $p^- \leqslant \bar{p}(x)$ for all $x \in \overline{\Omega}$, we have $L^{\bar{p}(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$. That is, there exists $c_1 > 0$ such that

$$\|u\|_{L^{p^-}(\Omega)} \leqslant c_1\|u\|_{L^{\bar{p}(x)}(\Omega)}.$$

Hence, we just need to show that there exists $c_2 > 0$ such that

$$[u]_{s,p^-} \leqslant c_2[u]_{s,p(x,y)}. \tag{2.4}$$

Indeed, let us set

$$U(x,y) = \frac{|u(x)-u(y)|}{|x-y|^{s+1}}.$$

Let $\varepsilon \in (0,1)$. Then, by the Hölder inequality in Corollary 2.2, we have

$$\begin{aligned}
[u]_{s,p^- - \varepsilon} &= \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p^- -\varepsilon}}{|x-y|^{N+s(p^- -\varepsilon)}}\, dxdy\right)^{\frac{1}{p^- -\varepsilon}} \\
&= \left(\int_{\Omega\times\Omega} \left(\frac{|u(x)-u(y)|}{|x-y|^{s+1}}\right)^{p^- -\varepsilon} \frac{dxdy}{|x-y|^{N-p^- +\varepsilon}}\right)^{\frac{1}{p^- -\varepsilon}} \\
&= \left(\int_{\Omega\times\Omega} |U(x,y)|^{p^- -\varepsilon} \frac{dxdy}{|x-y|^{N-p^- +\varepsilon}}\right)^{\frac{1}{p^- -\varepsilon}} \\
&= \|U\|_{L^{p^- -\varepsilon}(\Omega\times\Omega,\nu_\varepsilon)} \\
&\leqslant c_3\|U\|_{L^{p(x,y)}(\Omega\times\Omega,\nu_\varepsilon)}\|1\|_{L^{r(x,y)}(\Omega\times\Omega,\nu_\varepsilon)} \\
&\leqslant c_4\|U\|_{L^{p(x,y)}(\Omega\times\Omega,\nu_\varepsilon)},
\end{aligned}$$

where

$$\frac{1}{r(x,y)} = \frac{1}{p^- -\varepsilon} - \frac{1}{p(x,y)} \quad \text{and} \quad d\nu_\varepsilon(x,y) = \frac{dxdy}{|x-y|^{N-p^- +\varepsilon}}.$$

Thus, as $\varepsilon \to 0^+$, we obtain

$$[u]_{s,p^-} \leqslant c_4\|U\|_{L^{p(x,y)}(\Omega\times\Omega,\nu)}.$$

Next, our aim is to show that

$$\|U\|_{L^{p(x,y)}(\Omega\times\Omega,\nu)} \leqslant c_5[u]_{s,p(x,y)}.$$

Let $\eta > 0$ be such that

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy \leqslant 1.$$

Choose

$$h = \sup\left\{1, \sup_{(x,y)\in\Omega\times\Omega} |x-y|^{\frac{p^- - p^+}{p^+}}\right\} \quad \text{and} \quad \bar{\eta} = \eta h.$$

Then

$$\begin{aligned}\int_{\Omega\times\Omega}\left(\frac{|u(x)-u(y)|}{\bar{\eta}|x-y|^{s+1}}\right)^{p(x,y)} \frac{dxdy}{|x-y|^{N-p^-}} &= \int_{\Omega\times\Omega}\frac{|x-y|^{p^-}}{h^{p(x,y)}|x-y|^{p(x,y)}}\frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy \\ &\leqslant \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy \leqslant 1.\end{aligned}$$

It follows that

$$\|U\|_{L^{p(x,y)}(\Omega\times\Omega,\nu)} \leqslant \bar{\eta} = \eta h,$$

which implies the inequality (2.4). Now, since $sp^- > N$, we conclude from Proposition 2.8-(i) that $W^{s,p^-}(\Omega) \hookrightarrow L^\infty(\Omega)$. Hence $W \hookrightarrow L^\infty(\Omega)$. Moreover, $W^{s,p^-}(\Omega) \hookrightarrow C_b^{0,s-N/p^-}(\Omega)$, which implies that W is continuously embedded in $C_b^{0,s-N/p^-}(\Omega)$. □

Remark 2.10. For all θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < 1$, we have

$$C_b^{0,\theta_2}(\Omega) \hookrightarrow C_b^{0,\theta_1}(\Omega) \hookrightarrow C_b^0(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

Combining Remark 2.10 and Theorem 2.9, we conclude the following embedding result.

Corollary 2.11. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous function satisfies (2.2) and (2.3) with $sp^- > N$. Then the space W is continuously embedded in $C^0(\overline{\Omega})$.*

Corollary 2.12. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous function satisfies (2.2) and (2.3) with $sp^- > N$. Then embedding of W into $C_b^{0,\theta}(\Omega)$ is compact for any $\theta < s - N/p^-$.*

Proof. From Theorem 2.9, one has $W \hookrightarrow W^{s,p^-}$. Now, since $sp^- > N$, one obtains from Proposition 2.8-(ii) that the embedding $W^{s,p^-}(\Omega) \hookrightarrow C_b^{0,\theta}(\Omega)$ is compact for any $\theta < s - N/p^-$. Hence, the embedding $W \hookrightarrow C_b^{0,\theta}(\Omega)$ is compact for any $\theta < s - N/p^-$. □

Remark 2.13. By use of Remark 2.10 and Corollary 2.12, if $sp^- > N$, we have that the embedding $W \hookrightarrow C^0(\overline{\Omega})$ is compact .

Next, we introduce the fractional weighted variable exponent Sobalev space as follows:

$$W_w = W_w^{s,p(x,y)}(\Omega) = \left\{u \in L_w^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy < +\infty, \text{ for some } \gamma > 0\right\},$$

which endowed with the norm

$$\|u\|_w = \|u\|_{W_w} = \inf\left\{\gamma > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}}\,dxdy + \int_\Omega w(x)\left|\frac{u(x)}{\gamma}\right|^{\bar{p}(x)} dx \leqslant 1\right\}.$$

The norms $\|.\|_w$ and $\|.\|_W$ are equivalent in W_w. Moreover, the space $(W_w, \|.\|_w)$ is a separable reflexive Banach space.
We set

$$\rho^w_{p(.,.)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{sp(x,y)+N}}\,dxdy + \int_\Omega w(x)\,|u(x)|^{\bar{p}(x)}\,dx,$$

which is a modular on W_w, and it satisfies the following inequalities.

Proposition 2.14. *For all $u \in W_w$, we have*

(i) $\|u\|_w < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho^w_{p(.,.)}(u) < 1$ $(resp. = 1, > 1)$,
(ii) $\|u\|_w < 1 \Rightarrow \|u\|_w^{p+} \leqslant \rho^w_{p(.,.)}(u) \leqslant \|u\|_w^{p-}$,
(iii) $\|u\|_w > 1 \Rightarrow \|u\|_w^{p-} \leqslant \rho^w_{p(.,.)}(u) \leqslant \|u\|_w^{p+}$.

Lemma 2.15. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous function satisfies (2.2) and (2.3) with $sp^- > N$. Assume that w satisfies (w_1). Then the embedding of W_w into $C^0(\overline{\Omega})$ is continuous and compact. Hence, there exists a positive constant $c_0 > 0$ such that*

$$\|u\|_\infty \leqslant c_0\|u\|_w.$$

Proof. By (w_1), we have that $L^{\bar{p}(x)}_w(\Omega) \hookrightarrow L^{\bar{p}(x)}(\Omega)$. Thus, $W_w \hookrightarrow W$, as $sp^- > N$. By use of Corollary 2.11 we have $W_w \hookrightarrow C^0(\overline{\Omega})$. Furthermore, this embedding is compact. □

To prove the existence of at least three weak solutions for problem $(\mathcal{P}^s_w)$, we will use the Recceri's theorem of three critical points proved in [32].

Theorem 2.16. *Let X be a separable and reflexive real Banach space. Let $\Phi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|u\|_X\to+\infty} \Phi(u) + \lambda\Psi(u) = +\infty$ *for all* $\lambda > 0$,
(ii) *there exist* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$,
(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number ξ such that, for each $\lambda \in \Lambda$ and every continuously Gâteaux differentiable functional $\mathcal{J} : X \longrightarrow \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that for each $\mu \in [0,\mu^*]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu\mathcal{J}'(u) = 0$$

has at least three solutions in X whose norms are less than ξ.

3. MAIN RESULTS

In this section, we will prove that problem $(\mathcal{P}_w^s)$ has at least three weak solutions by means of the three critical-points theorem. We first denote by $\mathcal{A}$ the class of all Carathéodory functions $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfying the following condition:

$$|f(x,t)|\leqslant c_5+c_6|t|^{\alpha(x)-1} \quad \text{for all } (x,t)\in\Omega\times\mathbb{R},$$

where $\alpha\in C_+(\overline{\Omega})$ such that $1<\alpha^-\leqslant\alpha^+<p^-$ and c_5,c_6 are two positive constants.

We assume that there exists a constant $t_0\in\mathbb{R}$ such that

$$(f_1):\begin{cases} f(x,t)\leqslant 0, & \text{when } \quad |t|\in(0,t_0),\\[2mm] f(x,t)\geqslant 0, & \text{when } \quad |t|\in(t_0,+\infty).\end{cases}$$

Definition 3.1. We say that $u\in W_w$ is a weak solution of problem $(\mathcal{P}_w^s)$ if

$$\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp(x,y)}}\,dxdy+\int_\Omega w(x)|u|^{\bar{p}(x)-2}u\varphi dx$$
$$-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega g(x,u)\varphi dx=0, \tag{3.1}$$

for all $\varphi\in W_w$.

The existence result of this paper is given by the following theorem.

Theorem 3.2. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and let $s\in(0,1)$. Let $p:\overline{\Omega}\times\overline{\Omega}\longrightarrow(1,+\infty)$ be a continuous function satisfying (2.2) and (2.3) with $sp^->N$. Assume that w satisfies (w_1) and $f\in\mathcal{A}$ satisfies (f_1). Then there exist an open $\Lambda\subset(0,+\infty)$ and a positive real number $\xi>0$ such that, for each $\lambda\in\Lambda$ and every function $g\in\mathcal{A}$, there exists $\mu^*>0$ such that for any $\mu\in[0,\mu^*]$, problem $(\mathcal{P}_w^s)$ has at least three weak solutions whose norms are less than ξ.*

In order to prove Theorem 3.2, we need to show some auxiliary lemmas. But, we first define the functionals Φ, Ψ, $\mathcal{J}$: $W_w\longrightarrow\mathbb{R}$ as follows:

$$\Phi(u)=\int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy+\int_\Omega w(x)\frac{1}{\bar{p}(x)}|u(x)|^{\bar{p}(x)}dx,$$
$$\Psi(u)=-\int_\Omega F(x,u)dx \quad\text{and}\quad \mathcal{J}(u)=-\int_\Omega G(x,u)dx,$$

where $F(x,t)=\int_0^t f(x,\tau)d\tau$ and $G(x,t)=\int_0^t g(x,\tau)d\tau$.

Lemma 3.3. *Assume that the assumptions of Theorem 3.2 are satisfied. Then*

(i) the functional Φ is sequentially weakly lower semi-continuous, and its Gâteaux derivative $\Phi':W_w\longrightarrow W_w^*$ is given by

$$\langle\Phi'(u),\varphi\rangle=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp(x,y)}}\,dxdy+\int_\Omega w(x)|u|^{\bar{p}(x)-2}u\varphi dx,$$

(ii) the functional $\Phi':W_w\longrightarrow W_w^*$ is a homeomorphism,

where $\langle .,.\rangle$ denotes the usual duality between W_w and its dual space W_w^*

Proof. (i)- By a standard argument (see, for instance, [8, Lemma 3.1]), we have that $\Phi \in C^1(W_w,\mathbb{R})$ with the Gâteaux derivative is given by

$$\begin{aligned}\langle \Phi'(u),\varphi\rangle &= \langle \mathcal{L}(u),\varphi\rangle + \int_\Omega w(x)|u|^{\bar{p}(x)-2}u\varphi dx\\ &= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp(x,y)}}\,dxdy\\ &\quad+\int_\Omega w(x)|u|^{\bar{p}(x)-2}u\varphi dx,\end{aligned}$$

for all $u,\varphi\in W_w$.

On the other hand, since $t\longrightarrow |t|^{p(x,y)}$ is convex, we have that Φ is also convex on W_w. Now, let $\{u_n\}\subset W_w$ such that $u_n \rightharpoonup u$ in W_w. By use of the convexity of Φ, we have

$$\Phi(u_n)-\Phi(u)\geqslant \langle \Phi'(u),u_n-u\rangle.$$

Hence, by passing to limit, we get

$$\Phi(u)\leqslant \liminf_{n\to+\infty}\Phi(u_n),$$

that is, Φ is sequentially weakly lower semi-continuous.

(ii)- Firstly, we prove that $\Phi' : W_w \longrightarrow W_w^*$ has an inverse mapping $(\Phi')^{-1} : W_w^* \longrightarrow W_w$, by means of Minty-Browder Theorem (see [33, Theorem 26.A-(c), pp 557]). Indeed, similar to [14, Lemma 4.2], we obtain that Φ is strictly monotone. Since $\Phi\in C^1(W_w,\mathbb{R})$, the Φ is hemicontinuous. By Proposition 2.14, for any $u\in W_w$ with $\|u\|_w>1$, we have

$$\frac{\langle \Phi'(u),u\rangle}{\|u\|_w} = \frac{\rho^w_{p(x,y)}(u)}{\|u\|_w}\geqslant \|u\|_w^{p^--1}.$$

Thus, $\lim_{\|u\|_w\to+\infty}\frac{\langle \Phi'(u),u\rangle}{\|u\|_w}=+\infty$, that is, Φ' is coercive. Hence, in the light of Minty-Browder Theorem, Φ' has an inverse mapping $(\Phi')^{-1} : W_w^*\longrightarrow W_w$. Therefore, the continuity of $(\Phi')^{-1}$ is sufficient to ensure that Φ' is a homeomorphism. By [33, Theorem 26.A-(d), pp 557], we just need to show that Φ' is uniformly monotone. Indeed, using the following elementary inequality

$$\left(|\xi|^{q-2}|\xi-|\zeta|^{q-2}\zeta\right).(\xi-\zeta)\geqslant \frac{1}{2^q}|\xi-\zeta|^q \quad \text{for all } \xi,\zeta\in\mathbb{R} \text{ and } q\geqslant 2,$$

for all $u,v\in W_w$, we deduce that

$$\begin{aligned}<\Phi'(u)-\Phi'(v),u-v> &\geqslant \frac{\rho^w_{p(.,.)}(u-v)}{2^{p^+}}\\ &\geqslant \frac{1}{2^{p^+}}\begin{cases}\|u-v\|_w^{p^+-1}\|u-v\|_w, & \text{if} \quad \|u-v\|_w<1,\\ \|u-v\|_w^{p^--1}\|u-v\|_w, & \text{if} \quad \|u-v\|_w>1.\end{cases}\end{aligned}$$

We define the function

$$\kappa(t)=\begin{cases}\frac{t^{p^+-1}}{2^{p^+}}, & \text{if} \quad t<1,\\ \frac{t^{p^--1}}{2^{p^+}}, & \text{if} \quad t>1.\end{cases}$$

It is easy to verify that κ is an increasing function. Thus, for all $u,v\in W_w$, we get

$$\langle \Phi'(u)-\Phi'(v),u-v\rangle \geqslant \kappa\left(\|u-v\|_w\right)\|u-v\|_w,$$

that is, Φ' is uniformly monotone. We conclude that $(\Phi')^{-1}$ exists and it is continuous. □

Lemma 3.4. *Let $f,g \in \mathcal{A}$ then $\Psi, \mathcal{J} \in C^1(W_w, \mathbb{R})$ with the derivatives given by*

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u)vdx \quad \textit{and} \quad \langle \mathcal{J}'(u), v\rangle = \int_\Omega g(x,u)vdx,$$

for all $u, v \in W_w$. Moreover, the mappings $\Psi', \mathcal{J}' : W_w \longrightarrow W_w^*$ are compacts.

Proof. We prove the result for the operator Ψ. For $\mathcal{J}$, the proof is similar. First, from $f \in \mathcal{A}$ and the embedding result, we have that Ψ is well defined on W_w. Using a standard argument as in [8, Lemma 3.1], we have that

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u(x))vdx.$$

Now, we claim that Ψ' is continuous. Indeed, let $\{u_n\} \subset W_w$ be a sequence converges strongly to $u \in W_w$. By Lemma 2.15, W_w is compactly embedded in $L^\infty(\Omega)$. It follows that W_w is compactly embedded in $L^{\alpha(x)}(\Omega)$, where $\alpha(.)$ as in the definition of the class $\mathcal{A}$. Hence, we can define the operator $u \longmapsto f(.,u(.))$ from $L^{\alpha(x)}(\Omega)$ into $L^{\hat{\alpha}(x)}(\Omega)$ with $\frac{1}{\alpha(x)} + \frac{1}{\hat{\alpha}(x)} = 1$. Then, fixing $v \in W_w$ with $\|v\|_w \leqslant 1$, by the Hölder inequality and Lemma 2.15, we get

$$\begin{aligned} \big|\langle \Psi'(u_n) - \Psi'(u)\rangle\big| &= \left| \int_\Omega (f(x,u_n(x)) - f(x,u(x)))v(x)dx \right| \\ &\leqslant \|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{\alpha}(x)}(\Omega)} \|v\|_{L^{\alpha(x)}(\Omega)} \\ &\leqslant c_0 \|v\|_w \|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{\alpha}(x)}(\Omega)}, \end{aligned}$$

where c_0 is given in Lemma 2.15. Thus, for $\|v\|_w < 1$, we obtain

$$\|\Psi'(u_n) - \Psi'(u)\|_{W_w^*} \leqslant c_0 \|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{\alpha}(x)}(\Omega)}. \tag{3.2}$$

On the other hand, since W_w is compactly embedded in $L^{\alpha(x)}(\Omega)$, we have that $\{u_n\}$ converges strongly in $L^{\alpha(x)}(\Omega)$. It follows that there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $h \in L^{\alpha(x)}(\Omega)$ such that

$$u_n(x) \underset{n\to+\infty}{\longrightarrow} u(x) \text{ a.e. } x \in \Omega \quad \text{and} \quad |u_n(x)| \leqslant h(x) \text{ a.e. } x \in \Omega.$$

This fact combining with $f \in \mathcal{A}$ implies that

$$f(x,u_n(x)) - f(x,u(x)) \underset{n\to+\infty}{\longrightarrow} 0$$

and

$$|f(x,u_n(x)) - f(x,u(x))| \leqslant 2c_5 + c_6|u(x)|^{\alpha(x)-1} + c_6|h(x)|^{\alpha(x)-1} \in L^{\hat{\alpha}(x)}(\Omega),$$

for almost everywhere $x \in \Omega$, where $\alpha(.)$, c_5 and c_6 as in the definition of the class $\mathcal{A}$. Hence, in the light of the dominated convergence theorem, we conclude that

$$\|f(.,u_n(.)) - f(.,u(.))\|_{L^{\hat{\alpha}(x)}(\Omega)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Consequently, from (3.2), it follows that

$$\Psi'(u_n) \underset{n\to+\infty}{\longrightarrow} \Psi'(u) \quad \text{in } W_w^*,$$

which implies that Ψ' is continuous. Now, in order to verify the compactness of Ψ', take $\{u_n\} \subset W_w$ bounded. Then, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, weakly converges in W_w. Lemma 2.15 implies that $\{u_n\}$ converge strongly in $L^\infty(\Omega)$, it follows that W_w is compactly embedded in $L^{\alpha(x)}(\Omega)$. Using the same argument as above, we deduce that $\Psi'(u_n)$ converge strongly in W_w^*. Consequently, Ψ' is compact. □

Now, we are ready to prove our existence result.

Proof of Theorem 3.2 For proving our result, it is enough to verify that Φ, Ψ and $\mathcal{J}$ satisfy the hypotheses of Theorem 2.16. Indeed, By use of Lemma 3.3, we have that $\Phi \in C^1(W_w,\mathbb{R})$ is sequentially weakly lower semi-continuous, and its Gâteaux derivative $\Phi' : W_w \longrightarrow W_w^*$ admits a continuous inverse $(\Phi')^{-1} : W_w^* \longrightarrow W_w$. Moreover, Lemma 3.4 implies that $\Psi \in C^1(W_w,\mathbb{R})$ and its derivative $\Psi' : W_w \longrightarrow W_w^*$ is compact.

Next, we will show that the condition (i) of Theorem 2.16 is fulfilled. Indeed, By Proposition 2.14-(ii), for all $\|u\|_w > 1$, we have

$$\begin{aligned} \Phi(u) &\geqslant \frac{1}{p^+}\left\{\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy+\int_\Omega w(x)|u(x)|^{\bar{p}(x)}dx\right\} \\ &= \frac{1}{p^+}\rho^w_{p(.,.)}(u) \geqslant \frac{1}{p^+}\|u\|_w^{p^-}. \end{aligned} \tag{3.3}$$

On the other hand, as $f \in \mathcal{A}$, it follows that

$$|F(x,t)| \leqslant c_5|t| + c_6\frac{1}{\alpha(x)}|t|^{\alpha(x)}.$$

Therefore, by Lemma 2.15, one has

$$\begin{aligned} \Psi(u) &\geqslant -c_5\int_\Omega |u|dx - c_6\int_\Omega \frac{1}{\alpha(x)}|t|^{\alpha(x)}dx \\ &\geqslant -c_7\|u\|_w - \frac{c_6}{\alpha^+}\int_\Omega\left(|u|^{\alpha^+}+|u|^{\alpha^-}\right)dx \\ &= -c_7\|u\|_w - \frac{c_6}{\alpha^+}\int_\Omega\left(\|u\|^{\alpha^+}_{L^{\alpha^+}(\Omega)}+\|u\|^{\alpha^-}_{L^{\alpha^-}(\Omega)}\right). \end{aligned}$$

Since $1 < \alpha^- \leqslant \alpha^+ < p^-$, we have that W_w is continuously embedded in $L^{\alpha^+}(\Omega)$ and $L^{\alpha^-}(\Omega)$. Furthermore, there exist two positive constants $c_8, c_9 > 0$ such that

$$\|u\|_{L^{\alpha^+}(\Omega)} \leqslant c_8\|u\|_w \quad \text{and} \quad \|u\|_{L^{\alpha^-}(\Omega)} \leqslant c_9\|u\|_w.$$

Hence,

$$\Psi(u) \geqslant -c_7\|u\|_w - \frac{c_6}{\alpha^+}c_8\|u\|_w^{\alpha^+} - \frac{c_6}{\alpha^+}c_9\|u\|_w^{\alpha^-}. \tag{3.4}$$

From (3.3) and (3.4), for all $\lambda > 0$, it follows that

$$\Phi(u)+\lambda\Psi(u) \geqslant \frac{1}{p^+}\|u\|_w^{p^-} - \lambda c_7\|u\|_w - \lambda\frac{c_6}{\alpha^+}c_8\|u\|_w^{\alpha^+} - \lambda\frac{c_6}{\alpha^+}c_9\|u\|_w^{\alpha^-}.$$

Since $1 < \alpha^- \leqslant \alpha^+ < p^-$, one has

$$\lim_{\|u\|_w\to\infty} \Phi(u)+\lambda\Psi(u) = \infty.$$

Consequently, the assertion (i) of Theorem 2.16 is verified.

Now, we verify the conditions (ii) and (iii) of Theorem 2.16. As $\frac{\partial F}{\partial t}(x,t) = f(x,t)$ and by assumption (f_1), it follows that $t \longmapsto F(x,t)$ is increasing for all $t \in (t_0,+\infty)$ and decreasing for all $t \in (0,t_0)$ uniformly with respect to x. Obviously, $F(x,0) = 0$ and $F(x,t) \underset{t\to\infty}{\longrightarrow} \infty$. Thus, by use of assumption (f_1), there exists a positive number $\eta > 0$ such that

$$F(x,t) > 0 = F(x,0) \geqslant F(x,\tau) \quad \text{for all } x \in \Omega, t > \eta \text{ and } \tau \in (0,t). \tag{3.5}$$

Let a and b be two real numbers such that $0 < a < \min\{t_0, c_0\}$, with c_0 being given in Lemma 2.15, and $b > \eta$ satisfying

$$b^{p^-}\|w\|_{L^1(\Omega)} > 1, \tag{3.6}$$

$$b^{p^+}\|w\|_{L^1(\Omega)} > 1. \tag{3.7}$$

Let $b > 1$. When $t \in (0,a]$, we have from (3.5) that $F(x,t) \leqslant F(x,0)$. It follows that

$$\int_\Omega \sup_{t\in[0,a]} F(x,t)dx \leqslant \int_\Omega F(x,0)dx = 0. \tag{3.8}$$

Since $b > \eta$, we obtain from (3.5) that

$$\int_\Omega F(x,b)dx > 0.$$

It follows that

$$\frac{1}{c_0^{p^+}}\frac{a^{p+}}{b^{p^-}}\int_\Omega F(x,b)dx > 0. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\int_\Omega \sup_{t\in[0,a]} F(x,t)dx \leqslant 0 < \frac{1}{c_0^{p^+}}\frac{a^{p+}}{b^{p^-}}\int_\Omega F(x,b)dx. \tag{3.10}$$

Consider $u_0, u_1 \in W_w$ with $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \Omega$. We set $r = \frac{1}{p^+}\left(\frac{a}{c_0}\right)^{p^+}$. Clearly, $r \in (0,1)$. It is easy to see that

$$\Phi(u_0) = \Psi(u_0) = 0.$$

Using (3.6), we get

$$\begin{cases} \Phi(u_1) &= \int_\Omega \frac{1}{\bar{p}(x)}w(x)b^{\bar{p}(x)}dx \geqslant \frac{1}{p^+}b^{p^-}\|w\|_{L^1(\Omega)} > \frac{1}{p^+} > \frac{1}{p^+}\left(\frac{a}{c_0}\right)^{p^+} = r, \\ \Psi(u_1) &= -\int_\Omega F(x,u_1(x))dx = -\int_\Omega F(x,b)dx. \end{cases}$$

Similar to the case that $b < 1$, using (3.7), we get

$$\Phi(u_1) = \int_\Omega \frac{1}{\bar{p}(x)}w(x)b^{\bar{p}(x)}dx \geqslant \frac{1}{p^+}b^{p^+}\|w\|_{L^1(\Omega)} > \frac{1}{p^+} > \frac{1}{p^+}\left(\frac{a}{c_0}\right)^{p^+} = r.$$

Consequently,

$$\Phi(u_0) < r < \Phi(u_1).$$

Hence, the condition (ii) in Theorem 2.16 is verified.

On the other hand, we have

$$\begin{aligned} -\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)} &= -r\frac{\Psi(u_1)}{\Phi(u_1)} \\ &= r\frac{\int_\Omega F(x,b)dx}{\int_\Omega \frac{1}{\bar{p}(x)}w(x)b^{\bar{p}(x)}dx} > 0. \end{aligned} \tag{3.11}$$

We consider the case when $u \in W_w$ with $\Phi(u) \leqslant r < 1$. Since

$$\frac{1}{p^+}\rho^w_{p(.,.)}(u) \leqslant \Phi(u) \leqslant r < 1,$$

it follows that $\rho^w_{p(.,.)}(u) \leqslant p^+ r = \left(\frac{a}{c_0}\right)^{p^+} < 1$. This fact and Proposition 2.14-(i) imply that $\|u\|_{W_w} < 1$. Furthermore, it is clear that

$$\frac{1}{p^+}\|u\|^{p^+}_{W_w} \leqslant \frac{1}{p^+}\rho^w_{p(.,.)}(u) \leqslant \Phi(u) \leqslant r.$$

Thus, using Lemma 2.15, we have

$$|u(x)| \leqslant c_0\|u\|_w \leqslant c_0(p^+ r)^{\frac{1}{p^+}} = a \quad \text{for all } x \in \Omega, u \in W_w \text{ and } \Phi(u) \leqslant r.$$

By (3.8), it follows that

$$-\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) = \sup_{u\in\Phi^{-1}((-\infty,r])} -\Psi(u) \leqslant \int_\Omega \sup_{0\leqslant u(x)\leqslant a} F(x,u(x))dx \leqslant 0.$$

Combining this inequality with (3.11), we obtain

$$-\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) < r\frac{\int_\Omega F(x,b)dx}{\int_\Omega \frac{1}{\bar{p}(x)}w(x)b^{\bar{p}(x)}dx},$$

that is,

$$\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}.$$

which means that condition (iii) in Theorem 2.16 is fulfilled. Consequently, there exist an open interval $\Lambda \subset (0,+\infty)$ and a positive real number $\xi > 0$ such that for each $\lambda \in \Lambda$ and every function $g \in \mathcal{A}$, there exists $\mu^* > 0$ such that, for any $\mu \in [0,\mu^*]$, the equation

$$\Phi'(u)+\lambda\Psi'(u)+\mu\mathcal{J}'(u) = 0$$

has at least three weak solutions whose norms are less than ξ.

Remark 3.5. Applying ([34, Theorem 2.1]) in the proof of Theorem 3.2, we have an upper bound of the interval of parameters λ for which $(\mathcal{P}^s_w)$ has at least three weak solutions is obtained when $\mu = 0$. To be precise, from the conclusion of Theorem 3.2, one has

$$\Lambda \subset \left(0, k\frac{\int_\Omega \frac{1}{\bar{p}(x)}w(x)b^{\bar{p}(x)}dx}{\int_\Omega F(x,b)dx}\right),$$

for any $k > 1$, where b is as in the proof of Theorem 3.2 (namely, b satisfies (3.6) and (3.7) such that $F(x,b) > 0$).

4. Examples

In this section, we give two particular cases of the previous results.

• First, we consider the problem $(\mathcal{P}_w^s)$ in the case

$$f(x,t)=f(t)=l|t|^{q-2}t-m|t|^{r-2}t,$$

where l and m are two positive constants, $2<r<q<p^-$ and $N<sp^-$. Then problem $(\mathcal{P}_w^s)$ becomes

$$\begin{cases}\left(-\Delta_{p(x,.)}\right)^s u(x)+w(x)|u|^{\bar{p}(x)-2}u=\lambda l|t|^{q-2}t-\lambda m|t|^{r-2}t+\mu g(x,t), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega.\end{cases} \tag{$\mathcal{P}_1$}$$

In this case, we obtain

$$F(x,t)=F(t)=\frac{l}{q}t^q-\frac{m}{r}t^r \quad \text{for all } t\in\mathbb{R}.$$

It is easy to see that $f\in\mathcal{A}$. On the other hand, in assumption (f_1) for $t_0=\left(\frac{m}{l}\right)^{\frac{1}{q-r}}$, we have that

$$\begin{cases} f(x,t)\leqslant 0, \quad \text{when} \quad |t|\in\left(0,\left(\frac{m}{l}\right)^{\frac{1}{q-r}}\right),\\[2ex] f(x,t)\geqslant 0, \quad \text{when} \quad |t|\in\left(\left(\frac{m}{l}\right)^{\frac{1}{q-r}},+\infty\right).\end{cases}$$

Hence, we have the following result.

Corollary 4.1. *Assume that $2<r<q<p^-$ and $N<sp^-$. Then there exist an open interval $\Lambda\subset(0,+\infty)$ and a positive real number $\xi>0$ such that, for each $\lambda\in\Lambda$ and every function $g\in\mathcal{A}$, there exists $\mu^*>0$ such that, for any $\mu\in[0,\mu^*]$, problem $(\mathcal{P}_1)$ has at least three weak solutions whose norms are less than ξ.*

• Second, we consider problem $(\mathcal{P}_w^s)$ in the case

$$f(x,t)=|t|^{q(x)-2}t-|t|^{r(x)-2}t,$$

where $q,r\in C_+(\overline{\Omega})$ such that $2<r^-\leqslant r^+<q^-\leqslant q^+<p^-$ and $sp^->N$. Hence, problem $(\mathcal{P}_w^s)$ becomes

$$\begin{cases}\left(-\Delta_{p(x,.)}\right)^s u(x)+w(x)|u|^{\bar{p}(x)-2}u=\lambda|t|^{q(x)-2}t-\lambda|t|^{r(x)-2}t+\mu g(x,t), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega.\end{cases} \tag{$\mathcal{P}_2$}$$

In this case, we obtain

$$F(x,t)=\frac{1}{q(x)}t^{q(x)}-\frac{1}{r(x)}t^{r(x)} \quad \text{for all } (x,t)\in\Omega\times[0,+\infty).$$

It is easy to see that $f\in\mathcal{A}$. Moreover, in assumption (f_1) for $t_0=1$, we have that

$$\begin{cases} f(x,t)\leqslant 0 \quad \text{when} \quad |t|\in(0,1),\\[2ex] f(x,t)\geqslant 0 \quad \text{when} \quad |t|\in(1,+\infty).\end{cases}$$

Then, we obtain the next result.

Corollary 4.2. *Assume that* $q, r \in C_+(\overline{\Omega})$ *such that* $2 < r^- \leqslant r^+ < q^- \leqslant q^+ < p^-$ *and* $sp^- > N$. *Then there exist an open interval* $\Lambda \subset (0, +\infty)$ *and a positive real number* $\xi > 0$ *such that, for each* $\lambda \in \Lambda$ *and every function* $g \in \mathcal{A}$, *there exists* $\mu^* > 0$ *such that for any* $\mu \in [0, \mu^*]$, *problem* $(\mathcal{P}_1)$ *has at least three weak solutions whose norms are less than* ξ.

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REFERENCES

[1] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR. Izv. 9 (1987), 33-66.

[2] M. Råužička, Electrorheological fluids: modeling and mathematical theory. In: Lecture Notes in Mathematics, vol. 1748. Springer, Berlin, 2000.

[3] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image processing, SIAM J. Appl. Math. 66 (2006), 1383-1406.

[4] E. Azroul, A. Boumazourh, M. Srati, On positive weak solutions for a class of weighted $(p(.), q(.))$-Laplacian systems, Moroccan J. Pure Appl. Anal. 5 (2019), 125-139.

[5] F. Cammaroto, L. Vilasi, Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator, Nonlinear Anal. 74 (2011), 1841-1852.

[6] X. L. Fan, On nonlocal $p(x)$-Laplacian Dirichlet problems, Nonlinear Anal. 72 (2010), 3314-3323.

[7] K. B. Ali, M. Hsini, K. Kefi, and N. T. Chung, On a nonlocal fractional $p(.,.)$-Laplacian problem with competing nonlinearities, Complex Anal. Operator Theory 13 (2019), 1377-1399.

[8] E. Azroul, A. Benkirane, M. Shimi, Eigenvalue problems involving the fractional $p(x)$-Laplacian operator, Adv. Oper. Theory 4 (2019), 539-555.

[9] E. Azroul, M. Shimi, Nonlocal eigenvalue problems with variable exponent, Moroccan J. Pure Appl. Anal. 4 (2018), 46-61.

[10] E. Azroul, A. Benkirane, M. Shimi, M. Srati, On a class of fractional $p(x)$-Kirchhoff type problems, Appl. Anal. doi:10.1080/00036811.2019.1603372.

[11] E. Azroul, A. Benkirane and M. Shimi, Existence and Multiplicity of solutions for fractional $p(x,.)$-Kirchhoff type problems in $\mathbb{R}^N$, Appl. Anal. (2019), DOI: 10.1080/00036811.2019.1673373.

[12] E. Azroul , A. Benkirane, A. Boumazourh, M. Shimi, Existence results for fractional $p(x,.)$-Laplacian problem via the Nehari manifold approach, Appl. Math. Optim. (2020) DOI: 10.1007/s00245-020-09686-z.

[13] E. Azroul , A. Benkirane, M. Shimi, General fractional Sobolev space with variable exponent and applications to nonlocal problems. Adv. Oper. Theory (2020), DOI: 10.1007/s43036-020-00062-w.

[14] A. Bahrouni, V. Rădulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, Discrete Contin. Dyn. Syst. 11 (2018), 379-389.

[15] N.T Chung, Eigenvalue Problems for Fractional $p(x,y)$-Laplacian Equations with Indefinite Weight, Taiwanese J. Math. 23 (2019), 1153-1173.

[16] N.T Chung and H.Q. Toan, On a class of fractional Laplacian problems with variable exponents and indefinite weights, Collectanea Math. 71 2019, 223-237.

[17] U. Kaufmann, J. D. Rossi, R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, Electron. J. Qual. Theory Diff. Equ. 76 (2017), 1-10.

[18] G.A. Afrouzi, S. Heidarkhani, Three solutions for a Dirichlet boundary value problem involving the p-Laplacian, Nonlinear Anal. 66 (2007), 2281-2288.

[19] G. Bonanno, P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian, Arch. Math. (Basel) 80 (2003), 424-429.

[20] M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, Nonlinear Anal. 67 (2007), 2007, 1419-1425.

[21] R. A. Mashiyev, Three Solutions to a Neumann Problem for Elliptic Equations with Variable Exponent, Arab. J. Sci. Eng. 36 (2011), 1559-1567.

[22] W. W. Pan, G. A. Afrouzi, L. Li, Three solutions to a $p(x)$-Laplacian problem in weighted-variable-exponent Sobolev space, An. St. Univ. Ovidius Constanta, Vol. 21 (2013), 195-205.

[23] E. Azroul, A. Benkirane, M. Srati, Three solutions for a Kirchhoff type problem involving nonlocal fractional p-Laplacian Adv. Operator Theory 4 (2019), 821-835.

[24] E. Azroul, A. Benkirane, A. Boumazourh, M. Srati, Three solutions for a nonlocal fractional p-Kirchhoff Type elliptic system, Appl. Anal. 2019 Doi: 10.1080/00036811.2019.1670347.

[25] E. Azroul, A. Benkirane, M. Srati, Three solutions for a Schrodinger-Kirchhoff type equation involving nonlocal fractional integro-defferential operators, J. Pseudo-Differ. Oper. Appl. (2020). https://doi.org/10.1007/s11868-020-00331-5.

[26] X.L. Fan, D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424-446.

[27] O. Kováčik, J. Rákosník, On Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, Czechoslovak Math. J. 41 (1991), 592-618.

[28] D. Cruz-Uribe, L. Diening, P. Hästö, The maximal operator on weighted variable lebesgue spaces, Frac. Calc. Appl. Anal. 14 (2011), 361-374.

[29] I. Aydin, Weighted Variable Sobolev Spaces and Capacity, J. Funct. Spaces Appl. 2012 (2012), Article ID 132690.

[30] F. Demengel, G. Demengel, Functional Spaces for the Theory of Elliptic Partial Differential Equations, Springer (2007), DOI 10.1007/978-1-4471-2807-6

[31] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521-573.

[32] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal., 70 (2009), 3084-3089.

[33] M. Struwe, Variational methods: Applications to Nonlinear Partial Differential Equatios and Hamiltonian systems, Springer-Verlag, Berlin, 1996.

[34] G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Anal. 54 (2003), 651-665.