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GENERAL SECOND ORDER FUNCTIONAL DIFFERENTIAL INCLUSION DRIVEN BY THE SWEEPING PROCESS WITH SUBSMOOTH SETS

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Abstract. In this paper, we consider a perturbed second order nonconvex sweeping process for a class of subsmooth moving sets depending jointly on time, state and velocity. The perturbation, known as the external forces applied on the system, is general and takes the form of a sum of a single-valued continuous mapping and a set-valued unbounded mapping. We generalize earlier results obtained for this kind of problems. We deal also with a delayed perturbation, we extend a discretization approach known for the time-dependent sweeping process to the case when the moving set depends also on state and velocity.

Keywords. Differential inclusion; Sweeping process; Subsmooth sets; Delayed perturbation.

1. Introduction

In this paper, we consider a perturbed second order nonconvex sweeping process for a class of subsmooth moving sets depending jointly on time, state and velocity. The perturbations, which are known as the external forces applied on the system, are general and take the form of a sum of a single-valued continuous mapping and a set-valued unbounded mapping. The problem is the following:

$$(\mathscr{P})\begin{cases} -\ddot{y}(t) \in N_{D\big(t,y(t),\dot{y}(t)\big)}(\dot{y}(t)) + G\big(t,y(t),\dot{y}(t)\big) + h\big(t,y(t),\dot{y}(t)\big), \text{ a.e. } t \in [0,T]; \\ y(0) = v_0;\ \dot{y}(0) = u_0 \in D(0,v_0,u_0), \end{cases}$$

where D and G are set-valued mappings with nonempty closed values, which map $[0,T]\times H\times H$ to H, $N_{D(t,y(t),\dot{y}(t))}(\dot{y}(t))$ denotes the normal cone to the moving set $D(t,y(t),\dot{y}(t))$ at $\dot{y}(t)$ and $h:[0,T]\times H\times H\to H$ is a continuous mapping. G and h represent set-valued and single-valued perturbation forces, respectively. The study of this kind of evolution problems was initiated by Moreau [1] for the Lagrangian system to frictionless unilateral constraints and Castaing [2]

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when the moving set depends on the state and takes convex compact values. Since then, various generalizations have been obtained, see e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein. The nonconvex case was considered by [12], the authors proved the existence of solutions to $(\mathscr{P})$ for uniformly prox-regular sets $D(t,v(t)))$ with absolutely continuous variation in space and Lipschitz variation in time and with a single-valued perturbation. By means of a generalized version of Schauder's fixed point theorem, Castaing, Ibrahim and Yarou [11] provided another approach to prove the existence for uniformly prox regular and ball-compact sets $D(t,v(t))$ with absolutely continuous variation in time, without perturbation and for the perturbed problem (even in presence of a delay). The existence of solutions for such a problem was established by proving the convergence of the Moreau's catching-up algorithm. For other approaches, we refer to [13, 14, 15, 16, 17].

Recently, the case when the moving set depends jointly on time, state and velocity, was addressed in [15] by using a new approach, the Galerkin-Like method. Roughly speaking, one approachs the original problem by projecting the state into a finite dimensional space but not the velocity. Then, one proves the existence of solutions to the approached problems and their strong convergence to a solution of the original problem under compactness assumptions.

In this paper, we give, in the setting of an infinite dimensional Hilbert space, a new existence result for the perturbed second order sweeping process $(\mathscr{P})$. We generalize some known results, and also extend our results in [5] in many directions: we show that the approach in [5] can be adapted to yield the existence of solutions to $(\mathscr{P})$ for the general class of equi-uniformly subsmooth sets, which generalizes the convex case and the uniform prox-regularity case. Furthermore, these sets depend jointly on time, state and velocity. Using a different approach, we weaken the hypotheses on the perturbation by taking a continuous mapping satisfying a linear growth condition and an unbounded set-valued perturbation for which only the element of minimum norm satisfies a linear growth condition.

On the other hand, for dealing with functional differential inclusions, we extend another reduction approach, which is known for the time-dependent sweeping process in presence of delay. It reduces a second order sweeping process with delayed perturbation to a problem without delay and apply our results obtained in this case. We show that this approach is still valid in the case of time, state and velocity-dependent sweeping process. The paper is organized as follows. In Section 2, we recall some basic notations, definitions and useful results, which are used throughout the paper. In Section 3, we provide the existence result for problem $(\mathscr{P})$. The delayed problem is studied in the last section.

2. Notation and preliminaries

This section is devoted to notations, definitions and preliminary results, that will be needed in the sequel. Let H be a real separable Hilbert space with its inner product $\langle\cdot,\cdot\rangle$ and the associate norm $\|\cdot\|$. We denote by $\overline{\mathbf{B}}$ the unit closed ball of space H. $B(a,r)$ (respectively, $\overline{B}(a,r)$) is used to denote the open (respectively, closed) ball of center $a\in H$ and radius $r>0$, and $\mathscr{C}_H([0,T])$ is used to denote the Banach space of all continuous mappings $u:[0,T]\to H$ endowed with the norm of uniform convergence. By $W_H^{2,1}([0,T])$, we denote the space of all continuous mappings in $\mathscr{C}_H([0,T])$ such that their first derivatives are continuous and their second weak derivatives belong to $L_H^1([0,T])$. For a nonempty closed subset S of H, we denote by $d(\cdot,S)$ the usual distance function associated with S, i.e.,

$$d(u,S) = \inf_{y\in S}\|u-y\|.$$

$\mathrm{Proj}_S(u)$, the projection of u onto S, is defined by

$$\mathrm{Proj}_S(u) = \{y \in S: \quad d(u,S) = \|u-y\|\}.$$

We denote by $\overline{co}(S)$ the closed convex hull of S, characterized by

$$\overline{co}(S) = \{x \in H: \ \forall x' \in H, \ \langle x',x\rangle \leq \delta^*(x',S)\},$$

where $\delta^*(x',S) = \sup_{y\in S}\langle x',y\rangle$ stands for the support function of S at $x' \in H$. Recall that for a closed convex subset S,

$$d(x,S) = \sup_{x'\in\overline{\mathbf{B}}}[\langle x',x\rangle - \delta^*(x',S)].$$

A subset S is said to be relatively ball compact if, for any closed ball $\overline{B}(x,r)$ of H, the set $\overline{B}(x,r)\bigcap S$ is relatively compact.

Let $\varphi : H \to (-\infty,+\infty]$ be a lower semicontinuous function and $x \in \mathrm{dom}\,\varphi = \{x \in H : \varphi(x) < +\infty\}$, the proximal subdifferential $\partial^P\varphi(x)$, of φ at x (see [18]) is the set of all proximal subgradients of φ at x, any $\xi \in H$ is a proximal subgradient of φ at x if there exist positive numbers η and ς such that

$$\varphi(y) - \varphi(x) + \eta\,\|y-x\|^2 \geq \langle \xi, y-x\rangle, \ \forall y \in x+\varsigma\overline{\mathbf{B}}.$$

Let x be a point of $S \subset H$, we recall (see [18]) that the proximal normal cone to S at x is defined by $N^P(S,x) = \partial^P\delta_S(x)$, where δ_S denotes the indicator function of S, i.e. $\delta_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N_S^P(x) = \{\xi \in H: \ \exists\rho > 0 \ \text{ s.t. } \ x \in \mathrm{Proj}_S(x+\rho\xi)\}.$$

If φ is a real-valued locally-Lipschitz function defined on H, the Clarke subdifferential $\partial^C\varphi(x)$ of φ at x is the nonempty convex compact subset of H and given by

$$\partial^C\varphi(x) = \{\xi \in H: \ \varphi^\circ(x;v) \geq \langle \xi, v\rangle, \forall v \in H\},$$

where

$$\varphi^\circ(x;v) = \lim_{y\to x,}\sup_{t\downarrow 0}\frac{\varphi(y+tv)-\varphi(y)}{t}$$

is the generalized directional derivative of φ at x in the direction v (see [18]). The Clarke normal cone $N^C(S,x)$ to S at $x \in S$ is defined by polarity with T_S^C, that is,

$$N_S^C(x) = \{\xi \in H: \ \langle \xi, v\rangle \leq 0, \ \forall v \in \mathrm{T}_S^C\},$$

where T_S^C denotes the Clarke tangent cone and is given by

$$\mathrm{T}_S^C = \{v \in H: \ d^\circ(x,S;v) = 0\}.$$

The Clarke subdifferential $\partial\varphi(x)$ of φ at a point x (where φ is finite) can be defined in terms of Clarke normal cones to the epigraph of the function by

$$\partial\varphi(x) = \{v \in H : (v,-1) \in N_{\mathrm{epi}\,\varphi}((x,\varphi(x)))\},$$

where $\mathrm{epi}\,\varphi$ denotes the epigraph of φ, that is,

$$\mathrm{epi}\,\varphi = \{(x,r) \in H \times \mathbf{R} : \varphi(\bar{x}) \leq r\}.$$

Furthermore,

$$\partial d(x,S) \subset N_S(x) \bigcap \overline{\mathbf{B}}, \text{ for all } x \in S.$$

Another important concept of the Fréchet subdifferential will be also needed. A vector $v \in H$ is said to be in the Fréchet subdifferential $\partial^F \varphi(x)$ of φ at x (see [19]) provided that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $y \in B(x,\delta)$,

$$\langle v, y-x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon\|y-x\|.$$

It is well known that $\partial^F \varphi(x) \subset \partial\varphi(x)$, and, for all $x \in S$, $N_S^F(x) \subset N_S(x)$ and

$$\partial^F d(x,S) = N_S^F(x) \cap \overline{\mathbf{B}}. \tag{2.1}$$

Another important property is that whenever $y \in \mathrm{Proj}_S(x)$, one has

$$x - y \in N_S^F(y), \text{ so } x-y \in N_S(y), \tag{2.2}$$

Recall that for a given $a \in]0,+\infty]$ the subset S is uniformly a-prox-regular (see [20]) or equivalently a-proximally smooth if every nonzero proximal to S can be realized by an a-ball. This means that, for all $\bar{x} \in S$ and all $0 \neq \xi \in N_S^P(\bar{x})$,

$$\langle \frac{\xi}{\|\xi\|}, x - \bar{x}\rangle \leq \frac{1}{2a}\|x-\bar{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{a} = 0$ for $a = +\infty$. Recall that for $a = +\infty$ the uniform a-prox-regularity of S is equivalent to the convexity of S.

Now, we introduce the definition of the equi-uniform subsmoothness for a family of sets. It is important to emphasize that this class of sets, introduced in [21] (see also [22]), is an extension of the convexity, the prox-regularity of a set. In this way, the result concerning existence of solutions to the second-order differential inclusion is more general. We begin with some basic definitions from the subsmoothness while referring the reader to [21]. Let Ω be a closed subset of H. We say that Ω is subsmooth at $x \in \Omega$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \xi_1 - \xi_2, x_1 - x_2\rangle \geq -\varepsilon\|x_1 - x_2\|, \tag{2.3}$$

whenever $x_1, x_2 \in \overline{B}(x,\delta) \cap \Omega$ and $\xi_i \in N_\Omega(x_i) \cap \overline{\mathbf{B}}$, $i = 1,2$. The set Ω is subsmooth if it is subsmooth at each point of Ω. We further say that Ω is uniformly subsmooth if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that (2.3) holds for all $x_1, x_2 \in \Omega$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_\Omega(x_i) \cap \overline{\mathbf{B}}$.

The following subdifferential regularity of the distance function also holds true for subsmooth sets.

Proposition 2.1. *([22]) Let Ω be a closed set of a Hilbert space. If Ω is subsmooth at $x \in S$, then*

$$N_\Omega(x) = N_\Omega^F(x)$$

and

$$\partial d(x,\Omega) = \partial^F d(x,\Omega).$$

Next, we give the definition of the equi-uniform subsmoothness.

Definition 2.2. Let $(S(q))_{q\in Q}$ be a family of closed sets of H with parameter $q \in Q$. This family is called equi-uniformly subsmooth if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $q \in Q$, the inequality (2.3) holds, for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and for all $\xi_i \in N_{S(q)}(x_i) \bigcap \overline{\mathbf{B}}, i = 1,2$.

These sets enjoy the following closure property, see [22, 23].

Proposition 2.3. *Let $\{C(t,v,u) :\ (t,v,u) \in [0,T]\times H\times H\}$ be a family of nonempty closed sets of H, which is equi-uniformly subsmooth and let η be a positive real number. Assume that there exist positive real constants L, Λ_1 and Λ_2 such that, for any $x,y,u,v,z \in H$ and $s,t \in [0,T]$,*

$$|d\big(z,C(t,v,u)\big) - d\big(z,C(s,x,y)\big)| \le L|t-s| + \Lambda_1\|v-x\| + \Lambda_2\|u-y\|.$$

Then the following assertions hold:

- *(a) for all $(t,v,u,y) \in \mathrm{Gph}(C)$, we have $\eta\partial d(y,C(t,v,u)) \subset \eta\overline{\mathbf{B}}$;*
- *(b) the convex weakly compact valued mapping $(s,v,u) \to \partial^p d(y,C(s,v,u))$ satisfies the upper semicontinuity property: for any sequences $(s_n)_n$ in $[0,T]$ converging to s, $(v_n,u_n)_n$ converging to (v,u), $(y_n)_n$ converging to $y \in C(s,v,u)$ with $y_n \in C(s_n,v_n,u_n)$ and for any $\xi \in H$, we have*

$$\limsup_{n\to\infty} \delta^*\big(\xi, \eta\partial d(y_n,C(s_n,v_n,u_n))\big) \le \delta^*\big(\xi, \eta\partial d(y,C(s,v,u))\big).$$

We will also need the following result, which is a discrete version of Gronwall's Lemma.

Lemma 2.4. [24] *Let $\alpha > 0$, (a_i) and (b_i) non-negative sequences satisfying*

$$a_i \le \alpha + \sum_{k=0}^{i-1} \beta_k a_k,\ \forall i \in \mathbf{N}.$$

Then,

$$a_i \le \alpha \exp\left(\sum_{k=0}^{i-1} \beta_k\right),\ \forall j \in \mathbf{N}.$$

3. Main results

We begin by an existence result for the problem $(\mathscr{P})$

Theorem 3.1. *Let $D : [0,T]\times H\times H \rightrightarrows H$ be a set-valued mapping with nonempty closed values satisfying:*

- $(\mathscr{A}_{D_1})$ *the family $\big\{D(t,x,x');\ (t,x,x') \in [0,T]\times H\times H\big\}$ is equi-uniformly subsmooth;*
- $(\mathscr{A}_{D_2})$ *there exist positive real constants L, Λ_1 and Λ_2 with $0 < \Lambda_1 < 1$, such that for all $(t_i,x_i,x_i',z) \in [0,T]\times H\times H\times H$, $(i = 1,2)$*

$$|d(z,D(t_1,x_1,x_1')) - d(z,D(t_2,x_2,x_2'))| \le L|t_1-t_2| + \Lambda_1\|x_1-x_2\| + \Lambda_2\|x_1'-x_2'\|;$$

- $(\mathscr{A}_{D_3})$ *for any bounded subsets A,B of H, the set $D([0,T]\times A\times B)$ is relatively ball compact.*

Let $G : [0,T]\times H\times H \rightrightarrows H$ be a set-valued mapping with nonempty closed convex values such that:

- $(\mathscr{A}_{G_1})$ *G is $\mathscr{L}([0,T])\otimes\mathscr{B}(H)\otimes\mathscr{B}(H)$-measurable and for all $t \in [0,T]$, $G(t,\cdot,\cdot)$ is scalarly upper semicontinuous on $H\times H$;*

$(\mathscr{A}_{G_2})$ *there exists a real* $\beta > 0$, *such that, for all* $(t,u,v) \in [0,T] \times H \times H$,

$$d\big(0, G(t,v,u)\big) \leq \beta(1+\|u\|+\|v\|).$$

Let $h : [0,T] \times H \times H \to H$ *be a continuous mapping such that*

$(\mathscr{A}_h)$ *there exists a real* $\gamma > 0$ *such that, for all* $t \in [0,T]$ *and for all* $(u,v) \in H \times H$,

$$\|h(t,v,u)\| \leq \gamma(1+\|u\|+\|v\|).$$

Then, for every $(u_0,v_0) \in H \times H$ *with* $u_0 \in D(0,v_0,u_0)$, *there exists at least a* $W_H^{2,1}([0,T])-$*solution* $y(\cdot)$ *of problem* $(\mathscr{P})$. *Furthermore,*

$$\|\dot{y}(t)\| \leq \Delta \quad \textit{and} \quad \|\ddot{y}(t)\| \leq \Theta, \quad \textit{a.e. } t \in [0,T],$$

where

$$\Delta = \Big(\|u_0\| + T\big(\frac{2\kappa(2+2\|v_0\|+\|u_0\|)+2L}{1-\Lambda_1}\big)\Big) \exp\Big(T\big(\frac{\Lambda_2 + 2\kappa(1+T)}{1-\Lambda_1}\big)\Big),$$

$$\Theta = \frac{L + \Lambda_2 \Delta + 2\kappa(1+\Delta+\Upsilon)}{1-\Lambda_1} + L + 2\kappa(1+\|u_0\|+\|v_0\|),$$

$$\kappa = \gamma + \beta \quad \textit{and} \quad \Upsilon = \|v_0\| + T\Delta.$$

Proof. For each $(t,x,y) \in [0,T] \times H \times H$, we put $P(t,x,y) = \mathrm{Proj}_{G(t,x,y)}(0)$ the element of minimal norm of $G(t,x,y)$ and $m(t,x,y) = P(t,x,y) + h(t,x,y)$. It follows that

$$\|m(t,x,y)\| \leq \kappa(1+\|x\|+\|y\|) \tag{3.1}$$

with $\kappa = \gamma + \beta$. Consider, for every $n \geq 1$, a partition of $[0,T]$ by the points $t_k^n = ke_n$, $e_n = \frac{T}{n}, k = 0, 1, \cdots, n$.

Step 1. Define inductively the sequences $(u_k^n)_{0\leq k\leq n}$ and $(v_k^n)_{0\leq k\leq n}$. Putting $u_0^n = u_0$, $v_0^n = v_0$, we define

$$u_{k+1}^n \in \mathrm{Proj}_{D\left(t_{k+1}^n, v_k^n, u_k^n\right)} \Big(u_k^n - e_n m(t_k^n, v_k^n, u_k^n)\Big) \tag{3.2}$$

and

$$v_{k+1}^n = v_k^n + e_n u_{k+1}^n. \tag{3.3}$$

By (2.2) and (3.2), we obtain

$$u_{k+1}^n - u_k^n + e_n m(t_k^n, v_k^n, u_k^n) \in -N_{D\left(t_{k+1}^n, v_k^n, u_k^n\right)}(u_{k+1}^n). \tag{3.4}$$

This algorithm is well defined. Indeed, for $k = 0$, by the closedness of the set $D(t_1^n, v_0^n, u_0^n)$, we can take

$$u_1^n \in \mathrm{Proj}_{D\left(t_1^n, v_0^n, u_0^n\right)} \Big(u_0^n - e_n m(t_0^n, v_0^n, u_0^n)\Big),$$

and write $v_1^n = v_0^n + e_n u_1^n$. Using $(\mathscr{A}_{D_2})$ and (3.1), we get

$$\begin{aligned}
\|u_1^n - u_0^n\| &\leq \|u_1^n - u_0^n + e_n m(t_0^n, v_0^n, u_0^n)\| + \|e_n m(t_0^n, v_0^n, u_0^n)\| \\
&= d\big(u_0^n - e_n m(t_0^n, v_0^n, u_0^n), D(t_1^n, v_0^n, u_0^n)\big) + \|e_n m(t_0^n, v_0^n, u_0^n)\| \\
&\leq |d\big(u_0^n, D(t_0^n, v_0^n, u_0^n)\big) - d\big(u_0^n, D(t_1^n, v_0^n, u_0^n)\big)| + 2e_n \|m(t_0^n, v_0^n, u_0^n)\| \\
&\leq \big(L + 2\kappa(1+\|u_0\|+\|v_0\|)\big)e_n.
\end{aligned}$$

Assume that $u_0^n,\cdots,u_k^n$ as well as $v_0^n,\cdots,v_k^n$ have been constructed satisfying (3.2) and (3.3). Since the set $D(t_{k+1}^n,v_k^n,u_k^n)$ is closed, we can take

$$u_{k+1}^n \in \mathrm{Proj}_{D\left(t_{k+1}^n,v_k^n,u_k^n\right)}\left(u_k^n - e_n m(t_k^n,v_k^n,u_k^n)\right).$$

For all $0\leq i\leq k$, we have

$$\begin{aligned}\|u_{i+1}^n-u_i^n\| &\leq \|u_{i+1}^n-u_i^n+e_nm(t_i^n,v_i^n,u_i^n)\|+\|e_nm(t_i^n,v_i^n,u_i^n)\|\\ &= d\big(u_i^n-e_nm(t_i^n,v_i^n,u_i^n),D(t_{i+1}^n,v_i^n,u_i^n)\big)+\|e_nm(t_i^n,v_i^n,u_i^n)\|\\ &\leq |d\big(u_i^n,D(t_{i+1}^n,v_i^n,u_i^n)\big)-d\big(u_i^n,D(t_i^n,v_{i-1}^n,u_{i-1}^n)\big)|+2e_n\|m(t_i^n,v_i^n,u_i^n)\|\\ &\leq Le_n+\Lambda_1\|u_i^n-u_{i-1}^n\|+\Lambda_2\|v_i^n-v_{i-1}^n\|+2e_n\|m(t_i^n,v_i^n,u_i^n)\|.\end{aligned}$$

According to (3.3), we obtain, for all $0\leq i\leq k\leq n$,

$$\|v_i^n-v_{i-1}^n\|\leq e_n\|u_i^n\|$$

and

$$\begin{aligned}\|v_i^n\| = \|v_{i-1}^n+e_nu_i^n\| &= \|v_0^n+e_n(u_1^n+\cdots+u_i^n)\|\\ &\leq \|v_0^n\|+e_n\bigg(\|u_1^n\|+\cdots+\|u_i^n\|\bigg).\end{aligned} \tag{3.5}$$

Then, for all $0\leq i\leq k\leq n$, we have

$$\begin{aligned}\|u_{i+1}^n-u_i^n\| &\leq Le_n+\Lambda_1\|u_i^n-u_{i-1}^n\|+\Lambda_2\|v_i^n-v_{i-1}^n\|+2e_n\kappa\big(1+\|u_i^n\|+\|v_i^n\|\big)\\ &\leq Le_n\sum_{m=1}^{i}\Lambda_1^{i-m}+\Lambda_1^i\|u_1^n-u_0^n\|+\Lambda_2e_n\sum_{m=1}^{i}\Lambda_1^{i-m+1}\|u_m^n\|\\ &\quad +2e_n\kappa\sum_{m=1}^{i}\Lambda_1^{i-m}\big(1+\|u_m^n\|+\|v_m^n\|\big).\end{aligned}$$

By use of the last inequality, (3.5) and the fact that $\Lambda_1\in]0,1[$, we get

$$\begin{aligned}\|u_{i+1}^n-u_0^n\| &\leq T\left(\frac{2\kappa(2+2\|v_0\|+\|u_0\|)+2L}{1-\Lambda_1}\right)\\ &\quad +e_n\left(\frac{\Lambda_2+2\kappa(1+T)}{1-\Lambda_1}\right)\sum_{m=1}^{i}\|u_m^n\|.\end{aligned}$$

It follows that

$$\|u_{i+1}^n\|\leq \rho+\rho_n\sum_{m=1}^{i}\|u_m^n\|,$$

with

$$\rho=\|u_0\|+T\big(\frac{2\kappa(2+2\|v_0\|+\|u_0\|)+2L}{1-\Lambda_1}\big)$$

and

$$\rho_n=e_n\left(\frac{\Lambda_2+2\kappa(1+T)}{1-\Lambda_1}\right).$$

By use of Lemma 2.4, we can write, for $1\leq i\leq k\leq n$,

$$\|u_{i+1}^n\|\leq \rho\exp\left(T\big(\frac{\Lambda_2+2\kappa(1+T)}{1-\Lambda_1}\big)\right)=\Delta.$$

Then, for $1 \leq i \leq k \leq n$,

$$\|u_i^n\| \leq \Delta, \tag{3.6}$$

$$\|v_i^n\| \leq \|v_0^n\| + T\Delta = \Upsilon \tag{3.7}$$

and

$$\frac{\|u_{i+1}^n - u_i^n\|}{e_n} \leq \frac{L + \Lambda_2\Delta + 2\kappa(1+\Delta+\Upsilon)}{1-\Lambda_1} + L + 2\kappa(1+\|u_0\|+\|v_0\|) = \Theta. \tag{3.8}$$

Step 2. Construction of $u_n(\cdot)$ and $v_n(\cdot)$.

For any $t \in [t_k^n, t_{k+1}^n]$, $k = 0, 1, \cdot\cdot, n-1$, we define

$$v_n(t) = v_k^n + (t - t_k)u_{k+1}^n,$$

and

$$u_n(t) = \frac{t_{k+1}^n - t}{e_n} u_k^n + \frac{t - t_k^n}{e_n} u_{k+1}^n.$$

Thus, for almost all $t \in [t_k^n, t_{k+1}^n]$,

$$\dot{u}_n(t) = \frac{u_{k+1}^n - u_k^n}{e_n}, \tag{3.9}$$

by relations (3.4) and (3.9) we obtain

$$-\dot{u}_n(t) \in N_{D\left(t_{k+1}^n, v_k^n, u_k^n\right)}(u_{k+1}^n) + m\big(t_k^n, v_k^n, u_k^n\big). \tag{3.10}$$

From (3.8), we get

$$||\dot{u}_n(t)|| \leq \Theta. \tag{3.11}$$

For each $t \in [0,T]$ and each $n \geq 1$, we define the functions

$$p_n(t) = \begin{cases} t_k^n, & \text{if} \quad t \in [t_k^n, t_{k+1}^n[, \\ t_{n-1}^n, & \text{if} \quad t = T, \end{cases}$$

$$q_n(t) = \begin{cases} t_{k+1}^n, & \text{if} \quad t \in [t_k^n, t_{k+1}^n[, \\ t_n^n, & \text{if} \quad t = T. \end{cases}$$

Observe that, for all $t \in [0,T]$,

$$\lim_{n\to\infty} |p_n(t) - t| = \lim_{n\to\infty} |q_n(t) - t| = 0.$$

So, (3.10) gives

$$-\dot{u}_n(t) \in N_{D(q_n(t), v_n(p_n(t)), u_n(p_n(t)))}(u_n(q_n(t)) + m(p_n(t), v_n(p_n(t)), u_n(p_n(t)))$$

for a.e. $t \in [0,T]$. It is obvious that, for all $n \geq 1$ and for all $t \in [0,T]$, the following hold:

$$\|m(p_n(t), v_n(p_n(t)), u_n(p_n(t)))\| \leq \kappa\big(1 + \|u_n(p_n(t))\| + \|v_n(p_n(t))\|\big)$$

$$u_n(q_n(t)) \in D(q_n(t), v_n(p_n(t)), u_n(p_n(t))); \tag{3.12}$$

$$v_n(t) = v_0 + \int_0^t u_n(p_n(s))ds, \tag{3.13}$$

$$\lim_{n\to\infty} p_n(t) = \lim_{n\to\infty} q_n(t) = t. \tag{3.14}$$

For $k = 1, \cdots, n$, by use of the relations (3.6), (3.7) and (3.12), we have

$$(u_n(q_n(t)))_n \subset D([0,T], \Upsilon\mathbf{B}, \Delta\mathbf{B}) \bigcap \Delta\mathbf{B}.$$

By $(\mathscr{A}_{D_3})$, it results that $(u_n(q_n(t)))_n$ is relatively compact and

$$\|u_n(q_n(t))-u_n(t)\| \leq \int_t^{q_n(t)} \|\dot{u}(s)\|ds \leq \Theta(q_n(t)-t) \to 0, \text{ as } n\to\infty.$$

That is, for all $t\in[0,T], (u_n(t))_n$ is relatively compact and $(u_n(\cdot))_n$ is equicontinuous. From Ascoli-Arzelà theorem, we have that $(u_n(\cdot))_n$ is relatively compact in $\mathscr{C}_H([0,T])$, so that we can extract from it a subsequence, that we do not relabel, which converges uniformly to some mapping $u\in\mathscr{C}_H([0,T])$. It is obvious that $u(0)=u_0$ and that $u(\cdot)$ is absolutely continuous. From the inequality (3.11), we have that there exists a subsequence (again denote by $(\dot{u}_n(\cdot))_n$), which converges $\sigma\Big(L^\infty_H\big([0,T]\big),L^1_H\big([0,T]\big)\Big)$ in $L^\infty_H\big([0,T]\big)$ to w with $\|w\|\leq\Theta$ and $w=\dot{u}$. Indeed, fixing $t\in[0,T]$ and taking any $\xi(\cdot)\in L^\infty_H\big([0,T]\big)$, we have

$$\lim_{n\to\infty}\big\langle \dot{u}_n(\cdot),\xi(\cdot)\big\rangle=\big\langle w(\cdot),\xi(\cdot)\big\rangle,$$

and

$$\lim_{n\to\infty}\int_0^t\big\langle \dot{u}_n(s),\xi(s)\big\rangle ds=\int_0^t\big\langle w(s),\xi(s)\big\rangle ds.$$

In particular, for $\xi(\cdot)=\mathbf{1}_{[0,t]}(\cdot)ej$, with $\mathbf{1}_{[0,t]}$ the characteristic function of the interval $[0,t]$ and (e_j) a basis of the separable Hilbert space H, we get, for all j

$$\big\langle\lim_{n\to\infty}\int_0^t \dot{u}_n(s)ds,e_j\big\rangle=\big\langle\int_0^t w(s)ds,e_j\big\rangle,$$

which ensures

$$\lim_{n\to\infty}\int_0^t \dot{u}_n(s)ds=\int_0^t w(s)ds.$$

Since $(u(\cdot))$ is a sequence of absolutely continuous maps, we get

$$\lim_{n\to\infty}(u_n(t)-u_n(0))=\lim_{n\to\infty}\int_0^t \dot{u}_n(s)ds=\int_0^t w(s)ds.$$

Then $u(t)=u(0)+\int_0^t w(s)ds$ and $\dot{u}=w$. So $(\dot{u}_n(\cdot))$ converges $\sigma\Big(L^\infty_H\big([0,T]\big),L^1_H\big([0,T]\big)\Big)$ in $L^\infty_H\big([0,T]\big)$ to $\dot{u}(\cdot)\in L^\infty_H\big([0,T]\big)=\Big(L^1_H\big([0,T]\big)\Big)'$. We obtain, for all $\xi_1(\cdot)\in L^1_H\big([0,T]\big)$,

$$\lim_{n\to\infty}\big\langle \dot{u}_n(\cdot),\xi_1(\cdot)\big\rangle=\big\langle \dot{u}(\cdot),\xi_1(\cdot)\big\rangle.$$

Note that $L^\infty_H\big([0,T]\big)\subset L^1_H\big([0,T]\big)$. From the last inequality, we deduce that, for all $\xi_1(\cdot)\in L^\infty_H\big([0,T]\big)$,

$$\lim_{n\to\infty}\big\langle \dot{u}_n(\cdot),\xi_1(\cdot)\big\rangle=\big\langle \dot{u}(\cdot),\xi_1(\cdot)\big\rangle.$$

Then, $(\dot{u}_n(\cdot))$ converges $\sigma\Big(L^1_H\big([0,T]\big),L^\infty_H\big([0,T]\big)\Big)$ in $L^1_H\big([0,T]\big)$ to $\dot{u}(\cdot)$. From (3.13), (3.14) and the convergence of $(u_n)_n$, we deduce that $(v_n)_n$ converges uniformly to an absolutely continuous function v with $v(t)=v_0+\displaystyle\int_0^t u(s)\,ds$.

On the other hand, we have

$$||h(p_n(t),v_n(p_n(t)),u_n(p_n(t)))\|\leq\gamma\big(1+\Delta+\Upsilon\big)$$

and

$$||P(p_n(t),v_n(p_n(t)),u_n(p_n(t)))\|\leq\beta\big(1+\Delta+\Upsilon\big).$$

For all $n \geq 0$ and for all $t \in [0,T]$, we put

$$\Big(h\big(p_n(\cdot), v_n(p_n(\cdot)), u_n(p_n(\cdot))\big)\Big)_n = \big(\zeta_n(\cdot)\big)_n,$$

and

$$\Big(P\big(p_n(\cdot), v_n(p_n(\cdot)), u_n(p_n(\cdot))\big)\Big)_n = \big(\eta_n(\cdot)\big)_n.$$

By the continuity of the mapping $h(\cdot,\cdot,\cdot)$, we get that $\zeta_n(\cdot)$ converges to $\zeta(\cdot) = h(\cdot, v(\cdot), u(\cdot))$ and for all $t \in [0,T]$,

$$\|\zeta(t)\| \leq \gamma\big(1+\Delta+\Upsilon\big),$$

and $(\eta_n(\cdot))_n$ is bounded. Taking a subsequence if necessary, we conclude that $(\eta_n(\cdot))_n$ converges $\sigma\big(L^1_H([0,T]), L^\infty_H([0,T])\big)$ to some mapping $\eta \in L^1_H\big([0,T]\big)$ with $\|\eta(t)\| \leq \beta\big(1+\Delta+\Upsilon\big)$.

Step 3. We prove that the mapping (u,v) is a solution to the following problem

$$\begin{cases} -\dot{u}(t) \in N_{D\left(t,v(t),u(t)\right)}\big(u(t)\big) + G\big(t,v(t),u(t)\big) + h\big(t,v(t),u(t)\big) \text{ a.e. } t \in [0,T]; \\ v(t) = v_0 + \int_0^t u(s)ds, \ \text{ for all } t \in [0,T]; \\ u(t) = u_0 + \int_0^t \dot{u}(s)ds, \ \text{ for all } t \in [0,T]; \\ u(t) \in D\big(t,v(t),u(t)\big), \ \text{ for all } t \in [0,T]. \end{cases}$$

Observe that

$$\begin{aligned} & d\big(u_n(t), D(t,v(t),u(t))\big) \\ & \leq ||u_n(t) - u_n(q_n(t))|| + d\big(u_n(q_n(t)), D(t,v(t),u(t))\big) \\ & \leq ||u_n(t) - u_n(q_n(t))|| + |d\big(u_n(q_n(t)), D(t,v(t),u(t))\big) - d\big(u_n(q_n(t)), D(q_n(t), v(p_n(t)), u(p_n(t)))\big)| \\ & \leq ||u_n(t) - u_n(q_n(t))|| + L|q_n(t) - t| + \lambda_1 ||u(t) - u_n(p_n(t))|| + \lambda_2 ||v(t) - v_n(p_n(t))||. \end{aligned}$$

Since

$$\lim_{n\to\infty} ||u_n(q_n(t)) - u_n(t)|| = \lim_{n\to\infty} ||v_n(p_n(t)) - v(t)|| = \lim_{n\to\infty} |q_n(t) - t| = 0,$$

and $D\big(t,v(t),u(t)\big)$ is closed, we get by passing to the limit in the preceding inequality that $u(t) \in D\big(t,v(t),u(t)\big)$. It follows that

$$\|\dot{u}_n(t) + \zeta_n(t) + \eta_n(t)\| \leq \|\dot{u}_n(t)\| + \|\zeta_n(t)\| + \|\eta_n(t)\| \leq \Theta + \kappa\big(1+\Delta+\Upsilon\big) = l,$$

that is,

$$\dot{u}_n(t) + \zeta_n(t) + \eta_n(t) \in l\overline{\mathbf{B}}.$$

Since

$$\dot{u}_n(t) + \zeta_n(t) + \eta_n(t) \in -N_{D\left(q_n(t), v_n(p_n(t)), u_n(p_n(t))\right)}\big(u_n(p_n(t))\big)$$

we get from (2.1) that

$$\dot{u}_n(t) + \zeta_n(t) + \eta_n(t) \in -l\partial d\Big(u_n(p_n(t)), D\big(q_n(t), v_n(p_n(t)), u_n(p_n(t))\big)\Big).$$

Note that $(\dot{u}_n + \zeta_n + \eta_n, \eta_n)_n$ weakly converges in $L^1_{H\times H}\big([0,T]\big)$ to $(\dot{u} + \zeta + \eta, \eta)_n$. An application of the Mazur's Theorem to $(\dot{u}_n + \zeta_n + \eta, \eta_n)_n$ provides a sequence $(w_n, \varpi_n)_n$ with

$$w_n \in co\big\{\dot{u}_m + \zeta_m + \eta_m : \ m \geq n\big\} \ \text{ and } \ \varpi_n \in co\big\{\eta_m : \ m \geq n\big\}$$

such that $(w_n, \varpi_n)_n$ converges strongly in $L^1_{H\times H}([0,T])$ to $(\dot{u}+\zeta+\eta, \eta)$. We can extract from $(w_n, \varpi_n)_n$ a subsequence which converges a.e. to $(\dot{u}+\zeta+\eta, \eta)$. Then, there is a Lebesgue negligible set $\mathscr{N} \subset [0,T]$ such that, for every $t \in [0,T]\setminus\mathscr{N}$,

$$\dot{u}(t)+\zeta(t)+\eta(t) \in \bigcap_{n\geq 0}\overline{\{w_m(t):\, m\geq n\}} \subset \bigcap_{n\geq 0}\overline{co}\{\dot{u}_m(t)+\zeta_m(t)+\eta_m(t):\, m\geq n\} \tag{3.15}$$

and

$$\eta(t) \in \bigcap_{n\geq 0}\overline{\{\varpi_m(t):\, m\geq n\}} \subset \bigcap_{n\geq 0}\overline{co}\{\eta_m(t):\, m\geq n\}. \tag{3.16}$$

Fixing any $t \in [0,T]\setminus\mathscr{N}, n\geq n_0$ and $\mu \in H$, we have that relation (3.15) gives

$$\langle \mu, \dot{u}(t)+\zeta(t)+\eta(t)\rangle \leq \limsup_{n\to\infty} \delta^*\Big(\mu, -l\partial d\big(u_n(p_n(t)), D(q_n(t), v_n(p_n(t)), u_n(p_n(t)))\big)\Big).$$

By use of Proposition 2.3, we obtain

$$\langle \mu, \dot{u}(t)+\zeta(t)+\eta(t)\rangle \leq \delta^*\Big(\mu, -l\partial d\big(u(t), D(t, v(t), u(t))\big)\Big),$$

which entails

$$\dot{u}(t)+\zeta(t)+\eta(t) \in -l\partial d\Big(u(t), D\big(t, v(t), u(t)\big)\Big) \subset -N_{D\left(t,v(t),u(t)\right)}\big(u(t)\big).$$

Further, the relation (3.16) gives

$$\langle \mu, \eta(t)\rangle \leq \limsup_{n\to\infty} \delta^*\Big(\mu, G\big(p_n(t), v_n(p_n(t)), u_n(p_n(t))\big)\Big).$$

Since $\delta^*\big(\mu, G(\cdot,\cdot,\cdot)\big)$ is upper semicontinuous on $[0,T]\times H\times H$, we have

$$\langle \mu, \eta(t)\rangle \leq \delta^*\Big(\mu, G\big(t, v(t), u(t)\big)\Big).$$

So, we get $d\Big(\eta(t), G\big(t, v(t), u(t)\big)\Big) \leq 0$, which further implies

$$\eta(t) \in G\big(t, v(t), u(t)\big) \text{ a.e. } t \in [0,T].$$

This completes the proof of the Theorem. □

4. DELAYED SWEEPING PROCESS

Now, we proceed, in the infinite dimensional setting, to an existence result for the second order functional differential inclusion governed by the time and state-dependent nonconvex sweeping process, that is, the perturbation contains a finite delay. This problem was addressed in [25] by using the discretization approach based on the Moreau's catching-up algorithm. Here, we provide another technique initiated in [9] for the first order time-dependent case, which consists to subdivide the interval $[0,T]$ in a sequence of subintervals and to reformulate the problem with delay to a sequence of problems without delay. For the second order functional problems regarding the time-dependent sweeping process, we refer to [8, 26]. We will extend this approach for the case of time and state-dependent sweeping process with unbounded delayed perturbation. Let $\tau > 0$ be a positive number and $\mathscr{C}_0 = \mathscr{C}_H([-\tau,0])$ (resp. $\mathscr{C}_T = \mathscr{C}_H([-\tau,T])$ the Banach space of H-valued continuous functions defined on $[-\tau,0]$ (resp. $[-\tau,T]$) equipped with the norm of uniform convergence. Let $u:[-\tau,T]\to H$. Then, for every $t\in[0,T]$, we define the function $u_t = \mathscr{T}(t)u$ on $[-\tau,0]$ by $(\mathscr{T}(t)u)(s) = u(t+s)$, $\forall s \in [-\tau,0]$. Clearly, if $u \in \mathscr{C}_T$, then $u_t \in \mathscr{C}_0$ and the mapping $u \to u_t$ is continuous.

Consider the following problem

$$(\mathscr{P}_\tau)\quad \begin{cases} -\dot{u}(t) \in N_{D(t,v(t),u(t))}(u(t)) + G(t,\mathscr{T}(t)v,\mathscr{T}(t)u) + h(t,\mathscr{T}(t)v,\mathscr{T}(t)u) \text{ a.e. } t \in [0,T]; \\ u(t) = \psi(0) + \int_0^t \dot{u}(s)ds, \;\; v(t) = \varphi(0) + \int_0^t u(s)ds, \;\; \forall t \in [0,T]; \\ u(t) \in D\big(t,v(t),u(t)\big), \;\; \forall t \in [0,T]; \\ u \equiv \psi \;\text{ and }\; v \equiv \varphi \;\text{ on } [-\tau,0]. \end{cases}$$

Theorem 4.1. *Assume that $D : [0,T] \times H \rightrightarrows H$ satisfies hypothesis $(\mathscr{A}_{D_1}),(\mathscr{A}_{D_2})$ and $(\mathscr{A}_{D_3})$. Let $G : [0,T] \times \mathscr{C}_0 \times \mathscr{C}_0 \rightrightarrows H$ be a set-valued mapping with nonempty closed convex values such that:*

- $(\mathscr{A}_{G_1})$ *G is $\mathscr{L}([0,T]) \otimes \mathscr{B}(\mathscr{C}_0) \otimes \mathscr{B}(\mathscr{C}_0)$-measurable and for all $t \in \mathbf{R}_+$, $G(t,\cdot,\cdot)$ is scalarly upper semicontinuous on $\mathscr{C}_0 \times \mathscr{C}_0$;*
- $(\mathscr{A}_{G_2})$ *there exists a real $\beta > 0$, such that, for all $(t,\varphi,\psi) \in [0,T] \times \mathscr{C}_0 \times \mathscr{C}_0$,*

$$d(0,G(t,\varphi,\psi)) \le \beta(1 + \|\varphi(0)\| + \|\psi(0)\|).$$

Let $h : [0,T] \times \mathscr{C}_0 \times \mathscr{C}_0 \to H$ be a continuous mapping such that

- $(\mathscr{A}_h)$ *there exists a real $\gamma > 0$ such that, for all $t \in [0,T]$ and for all $(t,\varphi,\psi) \in [0,T] \times \mathscr{C}_0 \times \mathscr{C}_0$,*

$$\|h(t,\varphi,\psi)\| \le \gamma(1 + \|\varphi(0)\| + \|\psi(0)\|).$$

Then, for every $(\varphi,\psi) \in \mathscr{C}_0 \times \mathscr{C}_0$ verifying $\psi(0) \in D(0,\varphi(0),\psi(0))$, there exist two Lipschitz mappings $u : [0,T] \to H$ and $v : [0,T] \to H$ solution of $(\mathscr{P}_\tau)$.

Proof. Letting $u_0 = \psi(0)$ and $v_0 = \varphi(0)$, we have $u_0 \in D(0,v_0,u_0)$. We consider the same partition of $[0,T]$ by the points $t_k^n = ke_n$, $e_n = \frac{T}{n}, (k = 0,1,...,n)$. For each $(t,v,u) \in \left[-\tau,t_1^n\right] \times H \times H$, we define $f_0^n : \left[-\tau,t_1^n\right] \times H \to H$ and $g_0^n : \left[-\tau,t_1^n\right] \times H \to H$ by

$$f_0^n(t,v) = \begin{cases} \varphi(t), & \forall t \in [-\tau,0], \\ \varphi(0) + \frac{n}{T}t(v - \varphi(0)), & \forall t \in \left]0,t_1^n\right], \end{cases}$$

$$g_0^n(t,u) = \begin{cases} \psi(t), & \forall t \in [-\tau,0], \\ \psi(0) + \frac{n}{T}t(u - \psi(0)), & \forall t \in \left]0,t_1^n\right]. \end{cases}$$

We have $f_0^n\left(t_1^n,v\right) = v$ and $g_0^n\left(t_1^n,u\right) = u$ for all $(u,v) \in H \times H$. Observe that the mapping $(v,u) \to \left(\mathscr{T}(t_1^n)f_0^n(\cdot,v), \mathscr{T}(t_1^n)g_0^n(\cdot,u)\right)$ from $H \times H$ to $\mathscr{C}_0 \times \mathscr{C}_0$ is nonexpansive since, for all $(v_1,v_2) \in H \times H$,

$$\begin{aligned} \|\mathscr{T}(t_1^n)f_0^n(\cdot,v_1) - \mathscr{T}(t_1^n)f_0^n(\cdot,v_2)\|_{\mathscr{C}_0} &= \sup_{s\in[-\tau,0]} \|f_0^n(s+t_1^n,v_1) - f_0^n(s+t_1^n,v_2)\| \\ &= \sup_{s\in\left[-\tau+\frac{T}{n},\frac{T}{n}\right]} \|f_0^n(s,v_1) - f_0^n(s,v_2)\| \\ &= \sup_{0\le s\le\frac{T}{n}} \left\|\frac{n}{T}s(v_1 - \varphi(0)) - \frac{n}{T}s(v_2 - \varphi(0))\right\| \\ &= \sup_{0\le s\le\frac{T}{n}} \left\|\frac{n}{T}s(v_1 - v_2)\right\| = \|v_1 - v_2\|. \end{aligned}$$

Similarly, for all $(u_1,u_2)\in H\times H$, we get

$$\|\mathscr{T}(t_1^n)g_0^n(\cdot,u_1)-\mathscr{T}(t_1^n)g_0^n(\cdot,u_2)\|_{\mathscr{C}_0}=\|u_1-u_2\|.$$

Hence, the mapping $(v,u)\to\left(\mathscr{T}(t_1^n)f_0^n(\cdot,v),\mathscr{T}(t_1^n)g_0^n(\cdot,u)\right)$ from $H\times H$ to $\mathscr{C}_0\times\mathscr{C}_0$ is nonexpansive. So, the set-valued mapping with nonempty closed convex values $G_0^n:\left[0,t_1^n\right]\times H\times H\rightrightarrows H$ defined by

$$G_0^n(t,v,u)=G(t,\mathscr{T}(t_1^n)f_0^n(\cdot,v),\mathscr{T}(t_1^n)g_0^n(\cdot,u))$$

is globally measurable and scalarly upper semicontinuous on $H\times H$ thanks to $(\mathscr{A}_{G_1})$ and

$$d(0,G_0^n(t,v,u))=d(0,G(t,\mathscr{T}(t_1^n)f_0^n(\cdot,v),\mathscr{T}(t_1^n)g_0^n(\cdot,u))$$

$$\leq\beta(1+\|v\|+\|u\|),$$

for all $(t,v,u)\in\left[0,t_1^n\right]\times H\times H$. The mapping $h_0^n:\left[0,t_1^n\right]\times H\times H\rightrightarrows H$ defined by

$$h_0^n(t,v,u)=h(t,\mathscr{T}(t_1^n)f_0^n(\cdot,v),\mathscr{T}(t_1^n)g_0^n(\cdot,u))$$

is continuous on $[0,T]\times H\times H$, and

$$h_0^n(t,v,u)=h(t,\mathscr{T}(t_1^n)f_0^n(\cdot,v),\mathscr{T}(t_1^n)g_0^n(\cdot,u))\leq\gamma(1+\|v\|+\|u\|),$$

Hence G_0^n and h_0^n verify the conditions of Theorem 3.1. Then there exist two absolutely continuous mappings $u_0^n:\left[0,t_1^n\right]\to H$ and $v_0^n:\left[0,t_1^n\right]\to H$ such that

$$\begin{cases} -\dot{u}_0^n(t)\in N_{D(t,v_0^n(t),u_0^n(t))}+G_0^n(t,v_0^n(t),u_0^n(t))+h_0^n(t,v_0^n(t),u_0^n(t)) \quad \text{a.e on } \left[0,t_1^n\right]; \\ v_0^n(t)=v_0+\int_0^t u_0^n(s)\,ds,\ u_0^n(t)=u_0+\int_0^t \dot{u}_0^n(s)\,ds,\ \forall t\in[0,t_1^n]; \\ u_0^n(t)\in D(t,v_0^n(t),u_0^n(t)),\ \forall t\in\left[0,t_1^n\right]; \\ v_0^n(0)=\varphi(0),\ u_0^n(0)=\psi(0), \end{cases}$$

with

$$\|v_0^n(t)\|\leq\Upsilon,\ \|u_0^n(t)\|\leq\Delta,\ \|\dot{u}_0^n(t)\|\leq\Theta.$$

Set

$$v_n(t)=\begin{cases}\varphi(t), & \forall t\in[-\tau,0], \\ v_0^n(t), & \forall t\in\left]0,t_1^n\right]; \end{cases}$$

and

$$u_n(t)=\begin{cases}\psi(t), & \forall t\in[-\tau,0], \\ u_0^n(t), & \forall t\in\left]0,t_1^n\right]. \end{cases}$$

Then, u_n and v_n are well defined on $\left[-\tau,t_1^n\right]$, with $v_n=\varphi$, $u_n=\psi$ on $[-\tau,0]$, and

$$\begin{cases} -\dot{u}_n(t)\in N_{D(t,v_n(t),u_n(t))}(u_n(t))+G_0(t,v_n(t),u_n(t))+h_0(t,v_n(t),u_n(t)) \quad \text{a.e } \left[0,t_1^n\right]; \\ v_n(t)=v_0+\int_0^t u_n(s)\,ds,\ u_n(t)=u_0+\int_0^t \dot{u}_n(s)\,ds,\ \forall t\in[0,t_1^n], \\ v_n(0)=v_0=\varphi(0),\ u_n(0)=u_0=\psi(0). \end{cases}$$

By induction, suppose that u_n and v_n are defined on $\left[-\tau,t_k^n\right]$ $(k\geq 1)$ with $v_n=\varphi, u_n=\psi$ on $[-\tau,0]$ satisfying

$$v_n(t)=\begin{cases} v_0^n(t)=v_0+\int_0^t u_n(s)\,ds,\ \ \forall t\in[0,t_1^n], \\ v_1^n(t)=v_n\left(t_1^n\right)+\int_{t_1^n}^t u_n(s)\,ds,\ \ \forall t\in]t_1^n,t_2^n], \\ \cdots \\ v_{k-1}^n(t)=v_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t u_n(s)\,ds,\ \ \forall t\in\left]t_{k-1}^n,t_k^n\right], \end{cases}$$

$$u_n(t)=\begin{cases} u_0^n(t)=u_0+\int_0^t \dot u_n(s)\,ds,\ \ \forall t\in[0,t_1^n]; \\ u_1^n(t)=u_n\left(t_1^n\right)+\int_{t_1^n}^t \dot u_n(s)\,ds,\ \ \forall t\in]t_1^n,t_2^n]; \\ \cdots \\ u_{k-1}^n(t)=u_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t \dot u_n(s)\,ds,\ \ \forall t\in\left]t_{k-1}^n,t_k^n\right], \end{cases}$$

u_n and v_n are solution of

$$\begin{cases} -\dot u_n(t)\in N_{D(t,v_n(t),u_n(t))}(u_n(t))+G(t,\mathscr{T}(t_k^n)f_{k-1}^n(\cdot,v_n(t)),\mathscr{T}(t_k^n)g_{k-1}^n(\cdot,u_n(t))) \\ \qquad +h(t,\mathscr{T}(t_k^n)f_{k-1}^n(\cdot,v_n(t)),\mathscr{T}(t_k^n)g_{k-1}^n(\cdot,u_n(t))); \\ v_n(t)=v_{k-1}^n(t)=v_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t u_n(s)\,ds; \\ u_n(t)=u_{k-1}^n(t)=u_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t \dot u_n(s)\,ds; \\ u_n(t)\in D(t,v_n(t),u(t)) \end{cases}$$

on $\left]t_{k-1}^n,t_k^n\right]$, where f_{k-1}^n and g_{k-1}^n are defined for any $(v,u)\in H\times H$ as follows

$$f_{k-1}^n(t,v)=\begin{cases} v_n(t), & \forall t\in\left[-\tau,t_{k-1}^n\right], \\ v_n\left(t_{k-1}^n\right)+\dfrac{n}{T}\left(t-t_{k-1}^n\right)\left(v-v_n\left(t_{k-1}^n\right)\right), & \forall t\in\left]t_{k-1}^n,t_k^n\right]. \end{cases}$$

$$g_{k-1}^n(t,u)=\begin{cases} u_n(t), & \forall t\in\left[-\tau,t_{k-1}^n\right], \\ u_n\left(t_{k-1}^n\right)+\dfrac{n}{T}\left(t-t_{k-1}^n\right)\left(u-u_n\left(t_{k-1}^n\right)\right), & \forall t\in\left]t_{k-1}^n,t_k^n\right]. \end{cases}$$

Similarly, we can define $f_k^n,g_k^n:\left[-\tau,t_{k+1}^n\right]\times H\to H$ as

$$f_k^n(t,v)=\begin{cases} v_n(t), & \forall t\in\left[-\tau,t_k^n\right], \\ v_n\left(t_k^n\right)+\dfrac{n}{T}(t-t_k^n)(v-v_n(t_k^n)), & \forall t\in\left]t_k^n,t_{k+1}^n\right]. \end{cases}$$

$$g_k^n(t,u)=\begin{cases} u_n(t), & \forall t\in\left[-\tau,t_k^n\right], \\ u_n\left(t_k^n\right)+\dfrac{n}{T}(t-t_k^n)(u-u_n(t_k^n)), & \forall t\in\left]t_k^n,t_{k+1}^n\right]. \end{cases}$$

for any $(u,v)\in H\times H$. Note that, for all $(u,v)\in H\times H$,

$$\mathscr{T}(t_{k+1}^n)f_k^n(0,v)=f_k^n\left(t_{k+1}^n,v\right)=v,$$

$$\mathscr{T}(t_{k+1}^n)g_k^n(0,u)=g_k^n\left(t_{k+1}^n,u\right)=u.$$

Note also that, for all $(u_1, v_1), (u_2, v_2) \in H \times H$,

$$\begin{aligned}\left\|\mathscr{T}(t_{k+1}^n) f_k^n(\cdot, v_1) - \mathscr{T}(t_{k+1}^n) f_k^n(\cdot, v_2)\right\|_{\mathscr{C}_0} &= \sup_{s \in [-\tau, 0]} \left\| f_k^n\left(s + t_{k+1}^n, v_1\right) - f_k^n\left(s + t_{k+1}^n, v_2\right)\right\| \\ &= \sup_{s \in \left[-\tau + \frac{(k+1)T}{n}, \frac{(k+1)T}{n}\right]} \left\| f_k^n(s, v_1) - f_k^n(s, v_2)\right\|,\end{aligned}$$

and

$$\begin{aligned}\left\|\mathscr{T}(t_{k+1}^n) g_k^n(\cdot, u_1) - \mathscr{T}(t_{k+1}^n) g_k^n(\cdot, u_2)\right\|_{\mathscr{C}_0} &= \sup_{s \in [-\tau, 0]} \left\| g_k^n\left(s + t_{k+1}^n, u_1\right) - g_k^n\left(s + t_{k+1}^n, u_2\right)\right\| \\ &= \sup_{s \in \left[-\tau + \frac{(k+1)T}{n}, \frac{(k+1)T}{n}\right]} \left\| g_k^n(s, u_1) - g_k^n(s, u_2)\right\|.\end{aligned}$$

We have two cases. 1) If $-\tau + \dfrac{(k+1)T}{n} < \dfrac{kT}{n}$, then

$$\begin{aligned}\sup_{s \in \left[-\tau + \frac{(k+1)T}{n}, \frac{(k+1)T}{n}\right]} \left\| f_k^n(s, v_1) - f_k^n(s, v_2)\right\| &= \sup_{s \in \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right]} \left\| f_k^n(s, v_1) - f_k^n(s, v_2)\right\| \\ &= \sup_{\frac{kT}{n} \le s \le \frac{(k+1)T}{n}} \left\| \frac{n}{T}(s - t_k^n)(v_1 - v_2)\right\| = \|v_1 - v_2\|,\end{aligned}$$

and

$$\begin{aligned}\sup_{s \in \left[-\tau + \frac{(k+1)T}{n}, \frac{(k+1)T}{n}\right]} \left\| g_k^n(s, u_1) - g_k^n(s, u_2)\right\| &= \sup_{s \in \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right]} \left\| g_k^n(s, u_1) - g_k^n(s, u_2)\right\| \\ &= \sup_{\frac{kT}{n} \le s \le \frac{(k+1)T}{n}} \left\| \frac{n}{T}(s - t_k^n)(u_1 - u_2)\right\| = \|u_1 - u_2\|.\end{aligned}$$

2) If $\frac{kT}{n} \le -\tau + \frac{(k+1)T}{n} \le \frac{(k+1)T}{n}$, then

$$\begin{aligned}\sup_{s \in \left[-\tau + \frac{(k+1)T}{n}, \frac{(k+1)T}{n}\right]} \left\| f_k^n(s, v_1) - f_k^n(s, v_2)\right\| &= \sup_{s \in \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right]} \left\| f_k^n(s, v_1) - f_k^n(s, v_2)\right\| \\ &= \sup_{\frac{kT}{n} \le s \le \frac{(k+1)T}{n}} \left\| \frac{n}{T}(s - t_k^n)(v_1 - v_2)\right\| \\ &= \|v_1 - v_2\|,\end{aligned}$$

and

$$\begin{aligned}\sup_{s \in \left[-\tau + \frac{(k+1)T}{n}, \frac{(k+1)T}{n}\right]} \left\| g_k^n(s, u_1) - g_k^n(s, u_2)\right\| &= \sup_{s \in \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right]} \left\| g_k^n(s, u_1) - g_k^n(s, u_2)\right\| \\ &= \sup_{\frac{kT}{n} \le s \le \frac{(k+1)T}{n}} \left\| \frac{n}{T}(s - t_k^n)(u_1 - u_2)\right\| \\ &= \|u_1 - u_2\|.\end{aligned}$$

So the mapping $(v,u)\to\left(\mathscr{T}(t_{k+1}^n)f_k^n(\cdot,v),\mathscr{T}(t_{k+1})g_k^n(\cdot,u)\right)$ from $H\times H$ to $\mathscr{C}_0\times\mathscr{C}_0$ is nonexpansive. Hence the set-valued mapping $G_k^n:\left[t_k^n,t_{k+1}^n\right]\times H\times H\rightrightarrows H$ defined by

$$G_k^n(t,v,u)=G\left(t,\mathscr{T}(t_{k+1}^n)f_k^n(\cdot,v),\mathscr{T}(t_{k+1}^n)g_k^n(\cdot,u)\right)$$

is globally measurable and scalarly upper semicontinuous on $H\times H$, with nonempty closed convex values. As above, we can easily check that

$$d(0,G_k^n(t,v,u))\le\beta\left(1+\|u\|+\|v\|\right),\ \forall(t,u,v)\in\left[t_k^n,t_{k+1}^n\right]\times H\times H.$$

And the mapping $h_k^n:\left[t_k^n,t_{k+1}^n\right]\times H\times H\rightrightarrows H$ defined by

$$h_k^n(t,v,u)=h\left(t,\mathscr{T}(t_{k+1}^n)f_k^n(\cdot,v),\mathscr{T}(t_{k+1}^n)g_k^n(\cdot,u)\right)$$

is continuous on $H\times H$, and

$$h_k^n(t,v,u)\le\gamma(1+\|u\|+\|v\|),\ \forall(t,u,v)\in\left[t_k^n,t_{k+1}^n\right]\times H\times H.$$

Applying Theorem 3.1, there exist two absolutely continuous mappings $u_k^n:\left[t_k^n,t_{k+1}^n\right]\to H$ and $v_k^n:\left[t_k^n,t_{k+1}^n\right]\to H$ such that

$$\begin{cases}-\dot{u}_k^n(t)\in N_{D(t,v_k^n(t),u_k^n(t))}\left(u_k^n(t)\right)+G_k^n(t,v_k^n(t),u_k^n(t))+h_k^n(t,v_k^n(t),u_k^n(t))\text{ a.e. }\left[t_k^n,t_{k+1}^n\right];\\ v_k^n(t)=v_n\left(t_k^n\right)+\int_{t_k^n}^t u_k^n(s)\,ds,\ \ \forall t\in\left[t_k^n,t_{k+1}^n\right];\\ u_k^n(t)=u_n\left(t_k^n\right)+\int_{t_k^n}^t \dot{u}_k^n(s)\,ds,\ \ \forall t\in\left[t_k^n,t_{k+1}^n\right];\\ u_k^n(t)\in D(t,v_k^n(t),u_k^n(t))\ \ \forall t\in\left[t_k^n,t_{k+1}^n\right],\end{cases}$$

with

$$\|u_k^n(t)\|\le\Delta,\ \ \|v_k^n(t)\|\le\Upsilon,\ \ \|\dot{u}_k^n(t)\|\le\Theta.$$

Thus, by induction, we can construct two continuous mappings $u_n,v_n:[-\tau,T]\to H\times H$ with

$$v_n(t)=\begin{cases}\varphi(t), & \forall t\in[-\tau,0],\\ v_k^n(t), & \forall t\in\left]t_k^n,t_{k+1}^n\right],\ \forall k=0,\cdots,n-1;\end{cases}$$

$$u_n(t)=\begin{cases}\psi(t), & \forall t\in[-\tau,0],\\ u_k^n(t), & \forall t\in\left]t_k^n,t_{k+1}^n\right],\ \forall k=0,\cdots,n-1,\end{cases}$$

such that their restriction on each interval $\left[t_k^n,t_{k+1}^n\right]$ is a pair solution to

$$\begin{cases}-\dot{u}(t)\in N_{D(t,v(t),u(t))}(u(t))+G(t,\mathscr{T}(t_{k+1}^n)f_k^n(\cdot,v(t)),\mathscr{T}(t_{k+1}^n)g_k^n(\cdot,u(t)))\\ \qquad +h(t,\mathscr{T}(t_{k+1}^n)f_k^n(\cdot,v(t)),\mathscr{T}(t_{k+1}^n)g_k^n(\cdot,u(t)));\\ v(t)=v_n\left(t_k^n\right)+\int_{t_k^n}^t u(s)\,ds,\ u(t)=u_n\left(t_k^n\right)+\int_{t_k^n}^t \dot{u}(s)\,ds\\ u(t)\in D(t,v(t),u(t)).\end{cases}$$

Let $P_k^n : \left[t_k^n, t_{k+1}^n\right] \times \mathscr{C}_0 \times \mathscr{C}_0$ be the element of minimal norm of G_k^n. Then

$$\begin{cases} P_k^n\left(t, v_k^n(t), u_k^n(t)\right) \in G_k^n(t, v_k^n(t), u_k^n(t)) \text{ a.e. on } \left[t_k^n, t_{k+1}^n\right], \\ -\dot{u}_k^n(t) \in N_{D(t, v_k^n(t), u_k^n(t))}\left(u_k^n(t)\right) + P_k^n\left(t, v_k^n(t), u_k^n(t)\right) + h_k^n\left(t, v_k^n(t), u_k^n(t)\right), \\ v_k^n\left(t_k^n\right) = v_n\left(t_k^n\right), u_k^n\left(t_k^n\right) = u_n\left(t_k^n\right) \\ u_k^n(t) \in D(t, v_k^n(t), u_k^n(t)), \ \forall t \in \left[t_k^n, t_{k+1}^n\right]. \end{cases}$$

For notational convenience, set $P_n(t,v,u) = P_k^n(t,v,u)$, $h_n(t,v,u) = h_k^n(t,v,u)$, $\theta_n(t) = t_{k+1}^n$ and $\delta_n(t) = t_k^n$, for all $t \in \left]t_k^n, t_{k+1}^n\right]$. Then, we get for almost every $t \in [0,T]$

$$\begin{cases} P_n(t, v_n(t), u_n(t)) \in G(t, \mathscr{T}(\theta_n(t)) f^n_{\frac{n}{T}\delta_n(t)}(\cdot, v_n(t)), \mathscr{T}(\theta_n(t)) g^n_{\frac{n}{T}\delta_n(t)}(\cdot, u_n(t))); \\ -\dot{u}_n(t) \in N_{D(t, v_n(\theta_n(t)), u_n(\theta_n(t)))}(u_n(\theta_n(t))) + P_n(t, v_n(t), u_n(t)) + h_n(t, v_n(t), u_n(t)); \\ v_n(0) = \varphi(0), \ u_n(0) = \psi(0) \in D(0, \varphi(0), \psi(0)), \\ u_n(t) \in D(t, v_n(\theta_n(t)), u_n(\theta_n(t))), \forall t \in [0,T] \end{cases}$$

with for all $t \in [0,T]$

$$d\left(0, G(t, \mathscr{T}(\theta_n(t)) f^n_{\frac{n}{T}\delta_n(t)}(\cdot, v_n(t)), \mathscr{T}(\theta_n(t)) g^n_{\frac{n}{T}\delta_n(t)}(\cdot, u_n(t)))\right)$$
$$\leq \beta\left(1 + \|u_n(t)\| + \|v_n(t)\|\right)$$

and

$$h\big(t, \mathscr{T}(\theta_n(t)) f^n_{\frac{n}{T}\delta_n(t)}(\cdot, v_n(t)), \mathscr{T}(\theta_n(t)) g^n_{\frac{n}{T}\delta_n(t)}(\cdot, u_n(t))\big)$$
$$\leq \gamma(1 + \|u_n(t)\| + \|v_n(t)\|).$$

We claim that $\mathscr{T}(\theta_n(t)) f^n_{\frac{n}{T}\delta_n(t)}(\cdot, v_n(t))$ and $\mathscr{T}(\theta_n(t)) g^n_{\frac{n}{T}\delta_n(t)}(\cdot, u_n(t))$ pointwise converge on $[0,T]$ to $\mathscr{T}(t)v$ and $\mathscr{T}(t)u$ respectively in $\mathscr{C}_0$. The proof is similar to the one given in Theorem 2.1 in [27]. Further, as $\|v_n(t)\| \leq \Upsilon$, $\|u_n(t)\| \leq \Delta$, and $\|\dot{u}_n(t)\| \leq \Theta$, we have

$$\|h_n(t, v_n(t), u_n(t)) + P_n(t, v_n(t), u_n(t))\| \leq (\beta + \gamma)\Big(1 + \|u_n(t)\| + \|v_n(t)\|\Big)$$
$$\leq (\beta + \gamma)\big(1 + \Upsilon + \Delta\big).$$

We can proceed as in Theorem 3.1 to conclude the convergence of (u_n) and (v_n) to the solution of $(\mathscr{P}_\tau)$. □

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