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INFINITELY MANY RADIAL SOLUTIONS FOR A NONLOCAL EQUATION INVOLVING AN ANISOTROPIC OPERATOR

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Abstract. In this paper, we are concerned with a class of fractional scalar equations of Kirchhoff type. By using the variational method and a version of the symmetric mountain pass theorem, we establish the existence of infinitely many radial solutions under certain appropriate assumptions on nonlinearity and Kirchhoff functions.

Keywords. Fractional Kirchhoff equations; BO-ZK operator; Radial solutions; Variational method.

1. Introduction and main results

Consider the following Kirchhoff equation involving BO-ZK operator

$$K\left([u]_s^2\right)((-\Delta_x)^s u - \Delta_y u) + u = f(u),\ (x,y) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m, \tag{1.1}$$

where $[u]_s = \left(\int_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{s}{2}}u|^2 + |\nabla_y u|^2\right)dxdy\right)^{\frac{1}{2}}$, $s \in (0,1)$, $n,m \geq 1$, and $(-\Delta_x)^s$ denotes the fractional Laplacian in x, which is defined via the Fourier transform by $\widehat{(-\Delta_x)^s u}(\xi,\eta) = |\xi|^{2s}\widehat{u}(\xi,\eta)$, where $\widehat{u}$ is the Fourier transform of u. If u is sufficiently smooth, it can be expressed by

$$(-\Delta_x)^s u(x,y) = C_{N,s} P.V. \int_{\mathbb{R}^n} \frac{u(x,y) - u(z,y)}{|x-z|^{n+2s}}dz,\ (x,y) \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalization constant and P.V. stands for the Cauchy principal value, and K, f are a functions satisfying some conditions, which will be specified later.

The presence of the nonlocal term $K\left(\int_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{s}{2}}u|^2 + |\nabla_y u|^2\right)dxdy\right)$ in (1.1) causes some mathematical difficulties and so the study of such a class of equations is of much interest.

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Moreover, equation (1.1) is a version fractional related to the following hyperbolic equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation by considering the changes in the length of the strings produced by transverse vibrations.

On the other hand, when $K \equiv 1$ and $f(u) = u^p$, equation (1.1) appears in the study of solitary waves of the generalized Benjamin-Ono-Zakharov-Kuznetsov equation

$$u_t + \partial_{x_1} \left((-\Delta_x)^s u - \Delta_y u + u^p \right) = 0 \; (x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{1.3}$$

(see [2] for some local and global well-posedness results on this equation when $m = n = 1$). The anisotropic operator $(-\Delta_x)^s u - \Delta_y u$ is observed in the study of toy models parabolic equations for which local diffusions occur only in certain directions and nonlocal diffusions. It models diffusion sensible to the direction in the Brownian and Lévy-Itô processes. For some regularity and rigidity properties of this operator, the readers can refer to [3, 4].

Recently, Esfahani and Esfahani [5] considered the equation (1.1) with $K \equiv 1$. Applying the constrained minimization on the Nehari-manifolds, they show that the equation admits at least a positive ground state radial solution.

It is interesting to note that the fractional Laplacian (isotropic operator) problems have attracted a great attention by various study on the existence of positive, ground state, radial and multiplicity solutions; see, e.g., [6, 7, 8, 9]. This interest comes from its significant applications in several areas such as physics, biology, chemistry and finance; see, e.g., [10, 11, 12, 13, 14]. The fractional Kirchhoff problems were investigated by many researchers, we refer to [15, 16, 17].

Motivated by the above results, in particular, the results presented in [7], our aim in this paper is to prove the existence of infinitely many radial solutions for equation (1.1) without assuming the Ambrosetti-Rabinowitz condition on f. Specifically, we show that the Cerami compactness condition is verified so that infinitely many solutions can be obtained by a variant of the symmetric mountain pass theorem [18, 19]. We require the following conditions on f and K :

- (f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(-t) = -f(t)$ for all $t \in \mathbb{R}$;
- (f_2) there exist $C > 0$ and $2 < p < 2_s^* := \frac{2(n+ms)}{n+(m-2)s}$ such that
$$|f(t)| \le C(1 + |t|^{p-1}) \text{ for all } t \in \mathbb{R};$$
- (f_3) $f(t) = o(t)$ as $t \to 0$;
- (f_4) $\displaystyle\lim_{|t|\to\infty} \frac{F(t)}{|t|^{2\alpha}} = +\infty$ for some $1 < \alpha < \frac{2_s^*}{2}$, where
$$F(t) = \int_0^t f(\tau) d\tau \ge 0 \text{ for all } t \in \mathbb{R};$$
- (f_5) the function $t \mapsto tf(t) - 2F(t)$ is nondecreasing in $(0, +\infty)$;
- (K_1) $K \in C([0, +\infty), [k_0, +\infty))$, where $k_0 > 0$;
- (K_2) $\alpha \widehat{K}(t) \ge tK(t)$ for all $t \ge 0$, where $\widehat{K}(t) = \int_0^t K(\tau) d\tau$;
- (K_3) the function $t \mapsto \widehat{K}(t) - tK(t)$ is nondecreasing in $(0, +\infty)$.

Recall that the fractional Sobolev-Liouville space is defined as

$$\mathscr{H}^s := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}} u|^2 + |\nabla_y u|^2 + u^2 \right) dxdy < \infty \right\}$$

with the norm

$$||u|| = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}} u|^2 + |\nabla_y u|^2 + u^2 \right) dxdy \right)^{\frac{1}{2}}.$$

and the block radial fractional Sobolev-Liouville space

$$\mathscr{H}^s_r := \{ u \in \mathscr{H}^s : u(x,y) = u(|x|,|y|) \text{ for all } (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \}.$$

A function $u \in \mathscr{H}^s_r$ is called a weak solution of (1.1) if

$$K\left([u]_s^2\right) \int_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{s}{2}} u (-\Delta_x)^{\frac{s}{2}} v + \nabla_y u \nabla_y v \right) dxdy + \int_{\mathbb{R}^N} uv dxdy = \int_{\mathbb{R}^N} f(u) v dxdy,$$

for all $v \in \mathscr{H}^s$, where

$$[u]_s = \left(\int_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}} u|^2 + |\nabla_y u|^2 \right) dxdy \right)^{\frac{1}{2}}$$

The main result can be described by the following theorem.

Theorem 1.1. *Assume that $(f_1) - (f_5)$ and $(K_1) - K_3)$ hold. Then equation (1.1) has infinitely many radial solutions.*

2. Auxiliary results and the proof of the main theorem

In this section, we give some preliminary results for the proof of theorem.

Lemma 2.1 ([20, Remark 2.2])**.** *The fractional Sobolev-Liouville space $\mathscr{H}^s$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$, and is compactly embedded into $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$.*

As in [5], using Strauss type inequality, which similar to [21], it can be shown that if $m, n \geq 2$, then $\mathscr{H}^s_r \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in (2, 2_s^*)$. In the case $n = 1$ or $m = 1$, one should consider the space

$$\mathscr{H}^s_r = \{ u \in \mathscr{H}^s : u(x,y) = u(|x|,|y|) \text{ and } u \text{ is decreasing in } x \},$$

and also the embedding $\mathscr{H}^s_r \hookrightarrow L^q(\mathbb{R}^N)$ is compact [22].

Obviously, the energy functional associated to (1.1) defined on $\mathscr{H}^s_r$ by

$$I(u) = \frac{1}{2} \widehat{K}\left([u]_s^2\right) + \frac{1}{2} \int_{\mathbb{R}^N} u^2 dxdy - \int_{\mathbb{R}^N} F(u) dxdy,$$

is of class C^1 and its critical points are solutions of (1.1). In order to establish the existence of solution for (1.1), we need a version of the concentration compactness principle.

Lemma 2.2 ([23, 24])**.** *Assume that $\{u_n\} \subset \mathscr{H}^s$ is bounded verifying*

$$\lim_{n\to\infty} \sup_{z \in \mathbb{R}^N} \int_{B_R(z)} u_n^2 dxdy = 0,$$

for some $R > 0$. Then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$.

We recall that I satisfies the $(Cerami)_c$ compactness condition if any sequence $\{u_n\} \subset \mathscr{H}_r^s$ such that

$$I(u_n) \to c \quad \text{and} \quad (1+||u_n||)||I'(u_n)||_* \to 0 \tag{2.1}$$

possesses a convergent subsequence.

Lemma 2.3. *Assume that $(f_1)-(f_5)$ and $(K_1)-(K_3)$ hold. Then I satisfies the $(Cerami)_c$ compactness condition.*

Proof. Let $\{u_n\} \subset \mathscr{H}_r^s$ be a $(Cerami)_c$ sequence. Then $\{u_n\}$ is bounded. Indeed, suppose by contradiction that $||u_n|| \to \infty$ and let $v_n = \frac{u_n}{||u_n||}$. Therefore v_n is bounded in $\mathscr{H}_r^s$. We may assume that there exists $v \in \mathscr{H}_r^s$ such that $v_n \rightharpoonup v$ in $\mathscr{H}_r^s$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. Define

$$\delta := \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_R(z)} v_n^2 dxdy.$$

If $\delta = 0$, Lemma 2.2 implies

$$v_n \to 0 \text{ in } L^p(\mathbb{R}^N). \tag{2.2}$$

It is easy to see that, for each n, there exists $t_n \in [0,1]$ satisfying

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n).$$

By (2.1), we have $\langle I'(t_n u_n), t_n u_n\rangle = o_n(1)$. Thus, for all $t \in [0,1]$,

$$\begin{aligned} 2I(tu_n) &\le 2I(t_n u_n) - \langle I'(t_n u_n), t_n u_n\rangle + o_n(1) \\ &\le \widehat{K}([t_n u_n]_s^2) - K([t_n u_n]_s^2)[t_n u_n]_s^2 \\ &\quad + \int_{\mathbb{R}^N} (t_n u_n f(t_n u_n) - 2F(t_n u_n))dxdy + o_n(1). \end{aligned}$$

Hence, from (f_1), (f_5) and (K_3), we get

$$\begin{aligned} 2I(tu_n) &\le \widehat{K}([u_n]_s^2) - K([u_n]_s^2)[u_n]_s^2 \\ &\quad + \int_{\mathbb{R}^N} (u_n f(u_n) - 2F(u_n))dxdy + o_n(1) \\ &= 2I(u_n) - \langle I'(u_n), u_n\rangle + o_n(1) \\ &= 2c + o_n(1). \end{aligned} \tag{2.3}$$

Let $\theta > \left(\frac{2c+1}{\min\{1,k_0\}}\right)^{\frac{1}{2}}$. For n large enough, we have $\theta_n := \frac{\theta}{||u_n||} \in [0,1]$. Therefore

$$\begin{aligned} 2I(\theta_n u_n) &= 2I(\theta v_n) \\ &= \widehat{K}([\theta v_n]^2) + \theta^2 \int_{\mathbb{R}^N} v_n^2 dxdy - 2\int_{\mathbb{R}^N} F(\theta v_n)dxdy. \end{aligned} \tag{2.4}$$

By $(f_1)-(f_3)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|tf(t)|, F(t) \le \varepsilon|t|^2 + C_\varepsilon|t|^p \text{ for all } t \in \mathbb{R}. \tag{2.5}$$

This inequality combined with the boundedness of $\{v_n\}$ in $L^2(\mathbb{R}^N)$ and (2.2) imply

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} F(\theta v_n)dxdy = 0.$$

From (K_1) and (2.4), we obtain

$$2I(\theta_n u_n) \geq \min\{1,k_0\}\theta^2 + o_n(1) > 2c+1+o_n(1), \tag{2.6}$$

which contradicts with (2.3). Consequently, $\delta > 0$. Going if necessary to a subsequence, there exists $\{z_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(0)} w_n^2 dxdy = \int_{B_R(z_n)} v_n^2 dxdy > \frac{\delta}{2},$$

where $w_n := v_n(.+z_n)$. Since $||w_n|| = 1$, for some $w \in \mathscr{H}_r^s \backslash \{0\}$, $w_n \rightharpoonup w$ in $\mathscr{H}_r^s$, $w_n \to w$ in $L^2_{loc}(\mathbb{R}^N)$ and $w_n \to w$ a.e. in $\mathbb{R}^N$. Now we define $\widetilde{u}_n = u_n(.+z_n)$. Then $\frac{\widetilde{u}_n}{||\widetilde{u}_n||} = w_n \to w$ a.e. in $\mathbb{R}^N$. Thus, for $(x,y) \in \Omega_0 := \{(x,y) \in \mathbb{R}^N : w(x,y) \neq 0\}$,

$$|\widetilde{u}_n(x,y)| \to \infty. \tag{2.7}$$

By (K_2) and Lemma 2.1, for $||\widetilde{u}_n|| \geq 1$, we have

$$\begin{aligned} c + o_n(1) &= \frac{1}{2}\widehat{K}([u_n]_s^2) + \frac{1}{2}\int_{\mathbb{R}^N} u_n^2 dxdy - \int_{\mathbb{R}^N} F(u_n) dxdy \\ &= \frac{1}{2}\widehat{K}([\widetilde{u}_n]_s^2) + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{u}_n^2 dxdy - \int_{\mathbb{R}^N} F(\widetilde{u}_n) dxdy \\ &\leq \frac{1}{2}\widehat{K}(||\widetilde{u}_n||^2) + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{u}_n^2 dxdy - \int_{\mathbb{R}^N} F(\widetilde{u}_n) dxdy \\ &\leq \frac{\widehat{K}(1)}{2}||\widetilde{u}_n||^{2\alpha} + \frac{C}{2}||\widetilde{u}_n||^2 - \int_{\mathbb{R}^N} F(\widetilde{u}_n) dxdy. \end{aligned}$$

Since $\alpha > 1$, one has

$$\frac{\widehat{K}(1)}{2} + \frac{C}{2||\widetilde{u}_n||^{2(\alpha-1)}} \geq \int_{\mathbb{R}^N} \frac{F(\widetilde{u}_n)}{||\widetilde{u}_n||^{2\alpha}} dxdy + o_n(1). \tag{2.8}$$

In view of Fatou lemma, it follows from (f_4), (2.7) and (2.8) that

$$\begin{aligned} \frac{\widehat{K}(1)}{2} &\geq \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(\widetilde{u}_n)}{||\widetilde{u}_n||^{2\alpha}} dxdy \\ &\geq \int_{\Omega_0} \liminf_{n\to\infty} \frac{F(\widetilde{u}_n)}{|\widetilde{u}_n|^{2\alpha}} |w_n|^{2\alpha} dxdy = +\infty, \end{aligned}$$

which is impossible. Hence $\{u_n\}$ is bounded. Passing to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } \mathscr{H}_r^s, \\ &u_n \to u \text{ a.e. } \mathbb{R}^N, \\ &u_n \to u \text{ in } L^q(\mathbb{R}^N) \text{ for } q \in (2, 2_s^*). \end{aligned} \tag{2.9}$$

For all $v \in \mathscr{H}_r^s$, we set

$$\langle A(u), v \rangle = \int_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{s}{2}} u (-\Delta_x)^{\frac{s}{2}} v + \nabla_y u \nabla_y v \right) dxdy.$$

Since $u_n \rightharpoonup u$, we have $\langle A(u), u_n - u \rangle \to 0$. By use of the boundedness of $\{K([u_n]_s^2)\}$ in $\mathbb{R}$, we deduce

$$\lim_{n\to\infty} \left(K([u_n]_s^2) - K([u]_s^2) \right) \langle A(u), u_n - u \rangle = 0. \tag{2.10}$$

By using (2.5) and (2.9), we see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(f(u_n)-f(u)\right)(u_n-u)dxdy=0. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
o_n(1) &= \langle I'(u_n)-I'(u),u_n-u\rangle \\
&= K([u_n]_s^2)\langle A(u_n-u),u_n-u\rangle+\left(K([u_n]_s^2)-K([u]_s^2)\right)\langle A(u),u_n-u\rangle \\
&\quad +\int_{\mathbb{R}^N}|u_n-u|^2dxdy-\int_{\mathbb{R}^N}\left(f(u_n)-f(u)\right)(u_n-u)dxdy \\
&\geq \min\{k_0,1\}||u_n-u||^2+\left(K([u_n]_s^2)-K([u]_s^2)\right)\langle A(u),u_n-u\rangle \\
&\quad -\int_{\mathbb{R}^N}\left(f(u_n)-f(u)\right)(u_n-u)dxdy
\end{aligned}$$

This combined with (2.10)-(2.11) shows that $u_n\to u$ in $\mathscr{H}_r^s$. □

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