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THE FIRST EIGENVALUE OF $\Delta_p^2 - \Delta_p$ ALONG THE RICCI FLOW

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Abstract. In this paper, we derive the evolution equations for the first eigenvalue of operator $\Delta_p^2 - \Delta_p$ acting on the space of functions on a closed Riemannian manifold along the Ricci flow. We prove that the first nonzero eigenvalue is nondecreasing under the Ricci flow under certain geometric conditions.
Keywords. Eigenvalue; p-biharmonic; Ricci flow; Riemannian surface.

1. Introduction

Let (M,g) be a n-dimensional closed Riemannian manifold. The Laplacian of the metric $g=(g_{ij})$, in local coordinates $\{x^1,x^2,\dots,x^n\}$, is defined by

$$\Delta = g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k\partial_k) = \frac{1}{\sqrt{\det g}}\sum_{i,j}\partial_i(\sqrt{\det g}\, g^{ij}\partial_j),$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $(g^{ij}) = (g_{ij})^{-1}$ and Γ_{ij}^k is the Christoffel coefficient of g. The study of the eigenvalues of the geometric operator dependent of the Laplace operator on a Riemannian manifold has been an active subject in the last 40 years. In particularly, the evolution of the first eigenvalue of geometric operators, under the geometric flows, has been an appealing topic for spectral geometry, which plays an important role in understanding geometry and topology of the manifold itself.

Let (M^n,g_0) be a closed Riemannian manifold. The Ricci flow is the following equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{1.1}$$

with initial condition $g(0)=g_0$, where R_{ij} is the Ricci curvature of metric g. The normalized Ricci flow is

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \frac{2}{n} r g_{ij}, \quad g(0)=g_0, \tag{1.2}$$

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where $r=\frac{\int_M R d\mu}{\int_M d\mu}$, R is the scalar curvature and $d\mu$ is the volume form of metric g. The normalized Ricci flow preserves the volume of the initial Riemannian manifold and the evolution equations (1.1) and (1.2) differ only by a change of scale in spaces and a change of parametrization in time (see [1]). The short time existence and the uniqueness for solutions to the Ricci flow and the normalized Ricci flow was first shown by Hamilton in [1] using the Nash-Moser theorem, and then by De Turk in [2], who substantially simplified the proof. The Ricci flow has been a powerful and important tool in studying the geometry and topology for Riemannian manifolds, in particular, for lower dimensional manifolds.

The main study of the evolution of the eigenvalue of geometric operators began with Perelman's work [3], where he studied the first eigenvalue of the Laplace operator with potential R and proved that the first eigenvalue of $-4\Delta+R$ is nondecreasing under the Ricci flow. Then Cao in [4] derived the evolution formula for the first eigenvalues of $-\Delta+cR$ for $c\geq\frac{1}{4}$ and showed that they are nondecreasing along the Ricci flow. Also, Zeng, He and Chen in [5] investigated this eigenvalue along the Ricci-Bourguignon flow. In [6, 7], Abolarinwa studied the evolution and monotonicity for the first eigenvalue of p-Laplacian and weighted Laplacian along the Ricci-harmonic flow, respectively and the author in [8] studied the evolution for the first eigenvalue of the p-Laplacian under the Yamabe flow. Also, in [9, 10], it was investigated the monotonicity of the eigenvalues of p-Laplacian along the geometric flow and the Ricci flow, respectively. In [11], Fang, Yang and Zhu studied the eigenvalues of geometric operators related to the Witten Laplacian under the Ricci flow. The evolution of the eigenvalue of the biharmonic operator and the p-biharmonic operator was investigated in [12] and [13], respectively.

Motivated by the above results, in this paper, we study the first eigenvalue $\lambda(t)$ of $\Delta_p^2-\Delta_p$ whose metric satisfying the Ricci flow. We prove that if initial scalar curvature is positive and there exists a positive constant β such that $R_{ij}\geq\beta Rg_{ij}$ along the Ricci flow, then, for $p\geq 2$, $\lim_{t\to T}\lambda(t)=\infty$. Also, if the initial scalar curvature is positive and there exists a positive constant $\varepsilon>\frac{1}{p}$ such that $R_{ij}\geq\varepsilon Rg_{ij}$ along the Ricci flow, then the quantity $\lambda(t)e^{-p\varepsilon\alpha t}$ is increasing along the unnormalized Ricci flow (1.1), where $\alpha=R_{\min}(g_0)$. In particular, $\lambda(t)$ is increasing under the Ricci flow (1.1). We show if (M^2,g_0) is a closed Riemannian surface with nonnegative scalar curvature, then $\lambda(t)$ is increasing along the unnormalized Ricci flow (1.1) for $p\geq 2$.

2. PRELIMINARIES

In this section, we first recall some definitions about the p-Laplace, the p-biharmonic operator and give the definition for the first eigenvalue of the operator $\Delta_p^2-\Delta_p$ on a closed Riemannian manifold. We introduce a smooth function along the evolution equations (1.1) and (1.2) on a smooth closed Riemannian manifold, where it is the first nonzero eigenvalue of $\Delta_p^2-\Delta_p$ at $t=t_0$. Let ∇ be the Levi-Civita connection on Riemannian manifold $(M,g(t))$ and $u:M\to\mathbb{R}$. For $p\in[1,+\infty)$, the p-Laplace operator of u is defined as

$$\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)=|\nabla u|^{p-2}\Delta u+(p-2)|\nabla u|^{p-4}(\operatorname{Hess} u)(\nabla u,\nabla u), \tag{2.1}$$

and p-biharmonic operator of u is defined as

$$\Delta_p^2 u=\Delta(|\Delta u|^{p-2}\Delta u), \tag{2.2}$$

where div is the divergence operator, the adjoint of the gradient for the L^2-norm induced by g on the space of differential form and

$$(\operatorname{Hess} u)(X,Y) = \nabla(\nabla u)(X,Y) = X(Yu) - (\nabla_X Y)u \quad X,Y \in \chi(M).$$

The p-biharmonic operator is a generalization of the biharmonic operator.

In this paper, we consider operator $\Delta_p^2 - \Delta_p$. We say that Λ is an eigenvalue of $\Delta_p^2 - \Delta_p$ at time $t \in [0,T)$ whenever for some $u \in W^{1,p}(M)$,

$$\Delta_p^2 u - \Delta_p u = \Lambda |u|^{p-2} u. \tag{2.3}$$

Liu and Su [14] established the existence of at least three weak solutions for the following problem

$$\begin{aligned}
&\Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) \quad \text{in } \Omega,\\
&u = \Delta u = 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a nonempty bounded open set with a sufficient smooth boundary $\partial\Omega$, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. Later, Li and Heidarkhani [15] considered the problem

$$\begin{aligned}
&\Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) + \mu h(x,u) \quad \text{in } \Omega,\\
&u = \Delta u = 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

and showed this problem has three solutions, where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a nonempty bounded open set with a sufficient smooth boundary $\partial\Omega$, $\lambda, \mu > 0$ and $f,h : \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions. Multiplying both sides of (2.3) by u and taking integration, we have

$$\int_M (u\Delta_p^2 u - u\Delta_p u)\, d\mu = \Lambda \int_M |u|^p d\mu, \tag{2.4}$$

or equivalently

$$\int_M (|\Delta u|^p + |\nabla u|^p) d\mu = \Lambda \int_M |u|^p d\mu. \tag{2.5}$$

The above function $u(x,t)$ is called the eigenfunction corresponding to eigenvalue $\Lambda(t)$. The first nonzero eigenvalue $\lambda(t) = \lambda(M, g(t), d\mu)$ of $\Delta_p^2 - \Delta_p$ is defined as follows

$$\lambda(t) = \inf_{0 \neq u} \left\{ \int_M (|\Delta u|^p + |\nabla u|^p) d\mu : u \in C^\infty(M),\ \int_M |u|^p d\mu = 1 \right\}. \tag{2.6}$$

We say that u is a normalized eigenfunction corresponding to eigenvalue λ whenever $\lambda = \int_M (|\Delta u|^p + |\nabla u|^p) d\mu$ and $\int_M |u|^p d\mu = 1$. We do not know whether λ an its corresponding eigenfunction are C^1-differentiable or not along the Ricci flow. For this reason, we use the similar method presented in [16] and define a general smooth function. To this end, at time $t_0 \in [0,T)$, we first let $u_0 = u(t_0)$ be the normalized eigenfunction for the first eigenvalue $\lambda(t_0)$ of $\Delta_p^2 - \Delta_p$. Then, we get $\int_M |u(t_0)|^p d\mu_{g(t_0)} = 1$ and $\Delta^2_{p,g(t_0)} u_0 - \Delta_{p,g(t_0)} u_0 = \lambda(t_0)|u_0|^{p-2}u_0$. We consider the following smooth function

$$v(t) = u_0 \left[\frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))} \right]^{\frac{1}{2(p-1)}} \tag{2.7}$$

along evolution equations (1.1) and (1.2). We assume that

$$u(t)=\frac{v(t)}{(\int_M |v(t)|^p d\mu)^{\frac{1}{p}}} \tag{2.8}$$

where $u(t)$ is a smooth function along the flows (1.1) and (1.2) satisfying $\int_M |u|^p d\mu = 1$, and at time t_0, u is the eigenfunction for $\lambda(t_0)$ of $\Delta^2_{p,g(t_0)} - \Delta_{p,g(t_0)}$. Now, we define a general smooth function by

$$\lambda(t,u(t)) := \int_M (|\Delta u(t)|^p + |\nabla u(t)|^p) d\mu_{g(t)} \tag{2.9}$$

where $\lambda(t_0, u(t_0)) = \lambda(t_0)$, and u is a smooth function satisfying $\int_M |u|^p d\mu = 1$.

3. THE VARIATION OF $\lambda(t)$

In this section, we give some useful evolution formulas for $\lambda(t)$ under the Ricci flow. From [17], we have the following.

Lemma 3.1. *Let $(M, g(t))$, $t\in[0,T)$ be a solution to flow (1.1) or (1.2) on a closed Riemannian manifold. Let $u\in C^\infty(M)$ be on $(M,g(t))$. Then, under Ricci flow equation (1.1),*

(1) $\frac{\partial}{\partial t} g^{ij} = 2R^{ij}$,

(2) $\frac{\partial}{\partial t}(d\mu) = -R d\mu$,

(3) $\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2$,

(4) $\frac{\partial}{\partial t}(\Delta u) = 2R^{ij}\nabla_i\nabla_j u + \Delta u_t$,

and along the normalized Ricci flow equation (1.2), we have

(1) $\frac{\partial}{\partial t} g^{ij} = 2R^{ij} - \frac{2}{n} r g^{ij}$,

(2) $\frac{\partial}{\partial t}(d\mu) = (r-R)d\mu$,

(3) $\frac{\partial}{\partial t}(\Delta u) = 2R^{ij}\nabla_i\nabla_j u - \frac{2}{n} r\Delta u + \Delta u_t$,

where Ric is the Ricci tensor, R is scalar curvature and $u_t = \frac{\partial u}{\partial t}$.

The evolution formula for $\lambda(t)$ under the unnormalized Ricci flow are as follows.

Proposition 3.2. *Let $(M^n, g(t))$ be a solution of the Ricci flow (1.1) on the smooth closed Riemannian manifold (M, g_0). If $\lambda(t)$ denotes the first eigenvalue of $\Delta^2_p - \Delta_p$ under Ricci flow (1.1), then*

$$\begin{aligned} \frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= 2p\int_M (\Delta u)|\Delta u|^{p-2} R^{ij}\nabla_i\nabla_j u\, d\mu + p\int_M |\nabla u|^{p-2} R^{ij}\nabla_i u \nabla_j u\, d\mu \\ &\quad - \int_M R(|\Delta u|^p + |\nabla u|^p) d\mu + \lambda(t_0)\int_M R|u|^p d\mu, \end{aligned} \tag{3.1}$$

where u is the associated normalized evolving function.

Proof. According to the above description, $\lambda(t,u(t))$ is a smooth function with respect to t and $\lambda(t_0,u(t_0)) = \lambda(t_0)$. By derivation of (2.9), we have

$$\frac{d}{dt}\lambda(t,u(t)) = \int_M \frac{\partial(|\Delta u|^p + |\nabla u|^p)}{\partial t} d\mu + \int_M (|\Delta u|^p + |\nabla u|^p)\frac{\partial(d\mu)}{\partial t}. \tag{3.2}$$

From Lemma 3.1, on manifold $(M,g(t))$, we have

$$\frac{\partial(d\mu)}{\partial t} = -Rd\mu, \tag{3.3}$$

and

$$\frac{\partial|\Delta u|^p}{\partial t} = p(\Delta u)|\Delta u|^{p-2}\left(2R^{ij}\nabla_i\nabla_j u + \Delta u_t\right), \tag{3.4}$$

$$\frac{\partial|\nabla u|^p}{\partial t} = p|\nabla u|^{p-2}\left(R^{ij}\nabla_i u\nabla_j u + g^{ij}\nabla_i u\nabla_j u_t\right). \tag{3.5}$$

Substituting (3.3), (3.4) and (3.5) into (3.2) yields

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t)) &= p\int_M(\Delta u)|\Delta u|^{p-2}\left(2R^{ij}\nabla_i\nabla_j u + \Delta u_t\right)d\mu - \int_M(|\Delta u|^p + |\nabla u|^p)Rd\mu \\ &\quad + p\int_M |\nabla u|^{p-2}\left(R^{ij}\nabla_i u\nabla_j u + g^{ij}\nabla_i u\nabla_j u_t\right)d\mu. \end{aligned}\tag{3.6}$$

Taking derivative of both sides $\int_M |u|^p d\mu = 1$ with respect to t, we have

$$\int_M puu_t|u|^{p-2}d\mu = \int_M R|u|^p d\mu. \tag{3.7}$$

We can also write

$$\int_M (|\Delta u|^{p-2}\Delta u\Delta u_t + |\nabla u|^{p-2}\nabla u\nabla u_t)d\mu = \int_M u_t(\Delta_p^2 u - \Delta_p u)d\mu. \tag{3.8}$$

If we multiply both sides of (2.3) on u_t and integrate, then

$$\int_M u_t(\Delta_p^2 u - \Delta_p u)d\mu = \lambda\int_M |u|^{p-2}uu_t d\mu. \tag{3.9}$$

Hence, (3.7) and (3.9) imply that

$$\int_M u_t(\Delta_p^2 u - \Delta_p u)d\mu = \frac{\lambda}{p}\int_M R|u|^p d\mu. \tag{3.10}$$

Therefore, substituting (3.10) into (3.8) implies that

$$\int_M (|\Delta u|^{p-2}\Delta u\Delta u_t + |\nabla u|^{p-2}\nabla u\nabla u_t)d\mu = \frac{\lambda}{p}\int_M R|u|^p d\mu, \tag{3.11}$$

which together with (3.6) obtains (3.1). This completes the proof. □

Now, we give a variation of $\lambda(t)$ under the normalized Ricci flow, which is similar to the previous proposition.

Proposition 3.3. *Let $(M^n, g(t))$ be a solution of the Ricci flow (1.2) on the smooth closed Riemannian manifold (M, g_0). If $\lambda(t)$ denotes the first eigenvalue of $\Delta_p^2 - \Delta_p$ under the Ricci flow (1.2), then*

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= 2p\int_M(\Delta u)|\Delta u|^{p-2}R^{ij}\nabla_i\nabla_j u\,d\mu+\frac{r}{n}\int_M|\nabla u|^p d\mu\\ &+p\int_M|\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu-\int_M R(|\Delta u|^p+|\nabla u|^p)d\mu \qquad (3.12)\\ &+\lambda(t_0)\int_M R|u|^p d\mu-\frac{2pr}{n}\lambda(t_0),\end{aligned}$$

where u is the associated normalized evolving function.

Proof. In the normalized case, we obtain from Lemma 3.1 that

$$\frac{\partial(d\mu)}{\partial t}=(r-R)d\mu \qquad (3.13)$$

and

$$\frac{\partial|\Delta u|^p}{\partial t} = p|\Delta u|^{p-2}\Delta u\left(2R^{ij}\nabla_i\nabla_j u-\frac{2}{n}r\Delta u+\Delta u_t\right), \qquad (3.14)$$

$$\frac{\partial|\nabla u|^p}{\partial t} = p|\nabla u|^{p-2}\left(R^{ij}\nabla_i u\nabla_j u-\frac{r}{n}|\nabla u|^2+g^{ij}\nabla_i u\nabla_j u_t\right). \qquad (3.15)$$

Hence,

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t)) &= 2p\int_M|\Delta u|^{p-2}(\Delta u)R^{ij}\nabla_i\nabla_j u\,d\mu+p\int_M|\Delta u|^{p-2}\Delta u\Delta u_t d\mu\\ &+p\int_M|\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu-\frac{2}{n}r\int_M(|\Delta u|^p+|\nabla u|^p)d\mu\\ &+\frac{r}{n}\int_M|\nabla u|^p d\mu+p\int_M|\nabla u|^{p-2}\nabla u\nabla u_t d\mu \qquad (3.16)\\ &+\int_M(|\Delta u|^p+|\nabla u|^p)(r-R)d\mu.\end{aligned}$$

On the other hand, $\int_M|u|^p d\mu=1$ yields that

$$\begin{aligned}p\int_M(|\Delta u|^{p-2}\Delta u\Delta u_t+|\nabla u|^{p-2}\nabla u\nabla u_t)d\mu &= p\int_M u_t(\Delta_p^2 u-\Delta_p u)d\mu\\ &= p\lambda\int_M|u|^{p-2}uu_t d\mu \qquad (3.17)\\ &= \lambda\int_M\frac{\partial|u|^p}{\partial t}d\mu\\ &= \lambda\int_M|u|^p(R-r)d\mu\\ &= \lambda\int_M R|u|^p d\mu-r\lambda.\end{aligned}$$

Therefore, substituting (3.17) into (3.16), we get (3.12). This completes the proof. □

Theorem 3.4. *Let $g=g(t)$, $[0,T)$ be the evolving metric along the unnormalized Ricci flow (1.1) with $g(0)=g_0$ being the initial metric in M^n. Let $\lambda(t)$ be the first eigenvalue of operator*

$\Delta_p^2 - \Delta_p$ under the Ricci flow. If initial scalar curvature is positive and there is a positive constant β such that $R_{ij} \geq \beta R g_{ij}$ in $M \times [0,T)$, then, for $p \geq 2$, $\lim_{t\to T} \lambda(t) = \infty$.

Proof. On a closed Riemannian manifold M^n, for any smooth function u, we have the Bochner formula as follows

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla\nabla u|^2 + Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle.$$

Since

$$|\nabla\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2,$$

we get that the inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2 + Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle$$

and the condition $R_{ij} - \beta R g_{ij} > 0$ along the Ricci flow results

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2 + \beta R|\nabla u|^2 + \langle \nabla u, \nabla \Delta u\rangle.$$

Now by integration of both sides of the above inequality on closed Riemannian manifold M, we obtain

$$0 \geq \frac{1}{n}\int_M (\Delta u)^2 d\mu + \beta R_{\min}(t)\int_M |\nabla u|^2 d\mu - \int_M (\Delta u)^2 d\mu,$$

or equivalently

$$\beta R_{\min}(t)\int_M |\nabla u|^2 d\mu \leq \frac{n-1}{n}\int_M (\Delta u)^2 d\mu.$$

If $p = 2$, then inequality $(\Delta u)^2 + |\nabla u|^2 \geq (\Delta u)^2$ and above inequality imply

$$\beta R_{\min}(t)\int_M |\nabla u|^2 d\mu \leq \frac{n-1}{n}\int_M [(\Delta u)^2 + |\nabla u|^2] d\mu = \frac{n-1}{n}\lambda(t).$$

If $p > 2$, then the Hölder inequality implies that

$$\begin{aligned}
\beta R_{\min}(t)\int_M |\nabla u|^2 d\mu &\leq \frac{n-1}{n}\left(\int_M |\Delta u|^p d\mu\right)^{\frac{2}{p}}\left(\int_M 1\, d\mu\right)^{\frac{p-2}{p}} \\
&\leq \frac{n-1}{n}\left(\int_M (|\Delta u|^p + |\nabla u|^p) d\mu\right)^{\frac{2}{p}} (Vol(M))^{\frac{p-2}{p}} \\
&\leq \frac{n-1}{n}(\lambda(t))^{\frac{2}{p}}(Vol(M))^{\frac{p-2}{p}}.
\end{aligned}$$

From [1], we have $\lim_{t\to T} R_{\min}(t) = \infty$. Then the last inequality implies that $\lim_{t\to T}\lambda(t) = \infty$. This completes the proof. □

Corollary 3.5. *Let $g = g(t)$ be the evolving metric along the unnormalized Ricci flow (1.1) with $g(0) = g_0$ being the initial metric in M^3. Let $\lambda(t)$ be the first eigenvalue of operator $\Delta_p^2 - \Delta_p$ under the Ricci flow. If initial scalar curvature is positive and there is a positive constant β such that $0 < \beta \leq \frac{1}{3}$ and $R_{ij} \geq \beta R g_{ij}$ in $M \times \{0\}$, then, for $p \geq 2$, $\lim_{t\to T}\lambda(t) = \infty$.*

Proof. From [1], on three dimensional Riemannian manifolds, the inequality $R_{ij} \geq \beta R g_{ij}$ for $0 < \beta \leq \frac{1}{3}$ is preserved by the Ricci flow on $[0,T)$. Therefore, Theorem 3.4 yields that $\lim_{t\to T} \lambda(t) = \infty$. □

Theorem 3.6. *Let $g = g(t)$, $[0,T)$ be the evolving metric along the unnormalized Ricci flow (1.1) with $g(0) = g_0$ being the initial metric in M^n. Let $\lambda(t)$ be the first eigenvalue of operator $\Delta_p^2 - \Delta_p$ under the Ricci flow. If initial scalar curvature is positive and there is a positive constant $\varepsilon > \frac{1}{p}$ such that $R_{ij} \geq \varepsilon R g_{ij}$ in $M \times [0,T)$, then the quantity $\lambda(t)e^{-p\varepsilon\alpha t}$ is increasing along the unnormalized Ricci flow (1.1), where $\alpha = R_{\min}(g_0)$. In particular, $\lambda(t)$ is increasing under Ricci flow (1.1).*

Proof. If $g(t)$ is a solution to the Ricci flow, then we have from [1, 18] that $R_{g(t)} \geq \alpha = R_{\min}(g_0)$. Therefore the inequality $R_{ij} \geq \varepsilon R g_{ij}$ and formula (3.1) of the Proposition 3.2 yield that

$$\begin{aligned} \frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &\geq 2p\varepsilon \int_M R|\Delta u|^P d\mu + p\varepsilon \int_M R|\nabla u|^p d\mu \\ &\quad - \int_M R(|\Delta u|^p + |\nabla u|^p) d\mu + \lambda(t_0) \int_M R|u|^p d\mu. \end{aligned}$$

Now, since the initial scalar curvature is positive, one has that the scalar curvature is positive along the Ricci flow. Hence the quantity $p\varepsilon \int_M R|\Delta u|^P d\mu$ is positive and we can write

$$\begin{aligned} \frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &\geq (p\varepsilon - 1) \int_M R(|\Delta u|^p + |\nabla u|^p) d\mu + \lambda(t_0) \int_M R|u|^p d\mu \\ &\geq (p\varepsilon - 1)\alpha\lambda(t_0) + \alpha\lambda(t_0) = p\varepsilon\alpha\lambda(t_0). \end{aligned}$$

Since $\lambda(t,u(t))$ is a smooth function with respect to time, in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{1}{\lambda(t,u(t))} \frac{d}{dt}\lambda(t,u(t)) \geq p\varepsilon\alpha,$$

which integrating the above inequality with respect to time t on interval $[t_1,t_0] \subset I_0$ leads to

$$\ln \frac{\lambda(t_0,u(t_0))}{\lambda(t_1,u(t_1))} \geq p\varepsilon\alpha(t_0 - t_1).$$

Note that $\lambda(t)$ is the first eigenvalue. Hence, $\lambda(t_0,u(t_0)) = \lambda(t_0)$ and $\lambda(t_1,u(t_1)) \geq \lambda(t_1)$. It follows that

$$\ln \frac{\lambda(t_0)}{\lambda(t_1)} \geq p\varepsilon\alpha(t_0 - t_1),$$

that is,

$$\lambda(t_0)e^{-p\varepsilon\alpha t_0} \geq \lambda(t_1)e^{-p\varepsilon\alpha t_1}.$$

Since t_0 is arbitrary, we have that the quantity $\lambda(t)e^{-p\varepsilon\alpha t}$ is increasing along Ricci flow (1.1). On the other hand, the quantity $e^{-p\varepsilon\alpha t}$ is decreasing along the Ricci flow, therefore, $\lambda(t)$ is increasing under Ricci flow (1.1). □

3.1. **The variation of $\lambda(t)$ on homogeneous manifolds.** In this subsection, we consider the behavior of the eigenvalue when we evolve an initial homogeneous metric.

Proposition 3.7. *Let $(M^n, g(t))$, $t \in [0,T)$ be a solution to the Ricci flow on the smooth closed Riemannian homogeneous manifold (M^n, g_0). If $\lambda(t)$ denote the first eigenvalue of $\Delta_p^2 - \Delta_p$ under the Ricci flow, then*

(1) under the unnormalized Ricci flow (1.1), we have

$$\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} = 2p\int_M (\Delta u)|\Delta u|^{p-2}R^{ij}\nabla_i\nabla_j u\,d\mu + p\int_M |\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu, \tag{3.18}$$

(2) under the normalized Ricci flow (1.2), we have

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= 2p\int_M (\Delta u)|\Delta u|^{p-2}R^{ij}\nabla_i\nabla_j u\,d\mu + \frac{r}{n}\int_M |\nabla u|^p d\mu \\ &\quad + p\int_M |\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu - \frac{2pr}{n}\lambda(t_0).\end{aligned} \tag{3.19}$$

Proof. Since the evolving metric remains homogeneous and a homogeneous manifold has constant scalar curvature. Therefore, (3.1) implies that

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= 2p\int_M (\Delta u)|\Delta u|^{p-2}R^{ij}\nabla_i\nabla_j u\,d\mu + p\int_M |\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu \\ &\quad - R\int_M (|\Delta u|^p + |\nabla u|^p)d\mu + R\lambda(t_0)\int_M |u|^p d\mu \\ &= 2p\int_M (\Delta u)|\Delta u|^{p-2}R^{ij}\nabla_i\nabla_j u\,d\mu + p\int_M |\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu.\end{aligned}$$

Similarly, (3.12) yields (3.19). This completes the proof. □

Now we investigate the evolution of the first eigenvalue on Bianchi classes. Let (M,g) be a locally homogeneous closed 3-manifold, there are nine classes of such manifolds. They are classification into two groups, the first consists of $H(3)$, $H(2)\times\mathbb{R}^1$ and $So(3)\times\mathbb{R}^1$ and the other one includes $\mathbb{R}^3$, $SU(2)$, $SL(2,\mathbb{R})$, $Heisenberg$, $E(1,1)$ and $E(2)$ which are called Bianchi classes. Milnor [19] provided a frame $\{X_i\}_{i=1}^3$ where both the metric and Ricci tensors are diagonalized and this property is preserved by the Ricci flow. Now let $\{\theta\}_{i=1}^3$ be a dual to Milnor's frame. We consider the metric $g(t)$ as

$$g(t) = A(t)(\theta_1)^2 + B(t)(\theta_2)^2 + C(t)(\theta_3)^2.$$

Then the Ricci flow becomes a system of ODE with three variables $\{A(t), B(t), C(t)\}$.

Now we study the behavior of the first eigenvalue of operator $\Delta_p^2 - \Delta_p$ in each classes separately.

Case 1: $\mathbb{R}^3$

In this case, all metrics are flat. For all $t \geq 0$, we have $g(t) = g_0$, where g_0 is initial metric, therefore, $\lambda(t)$ is constant.

Case 2: Heisenberg

This class is isomorphic to the set of upper-triangular 3×3 matrices endowed with the usual matrix multiplication. Under the metric g_0, we choose a frame $\{X_i\}_{i=1}^3$ in which

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$

also under the normalization $A_0B_0C_0=1$ we have

$$R_{11}=\frac{1}{2}A^3,\quad R_{22}=-\frac{1}{2}A^2B,\quad R_{33}=-\frac{1}{2}A^2C,\quad R=-\frac{1}{2}A^2.$$

Under Proposition 3.7 and Ricci components in *Hiesenberg* case, we get

$$\begin{aligned}
&\frac{d}{dt}\lambda(t,u(t))|_{t=t_0}\\
=\ & 2p\int(\Delta u)|\Delta u|^{p-2}\big[\frac{1}{2}A^3\nabla^1\nabla^1u-\frac{1}{2}A^2B\nabla^2\nabla^2u-\frac{1}{2}A^2C\nabla^3\nabla^3u\big]d\mu\\
&+p\int|\nabla u|^{p-2}\big[\frac{1}{2}A^3\nabla^1u\nabla^1u-\frac{1}{2}A^2B\nabla^2u\nabla^2u-\frac{1}{2}A^2C\nabla^3u\nabla^3u\big]d\mu\\
\le\ & pA^2\int(\Delta u)|\Delta u|^{p-2}\big[g_{11}\nabla^1\nabla^1u+g_{22}\nabla^2\nabla^2u+g_{33}\nabla^3\nabla^3u\big]d\mu\\
&+\frac{1}{2}pA^2\int|\nabla u|^{p-2}\big[g_{11}\nabla^1u\nabla^1u+g_{22}\nabla^2u\nabla^2u+g_{33}\nabla^3u\nabla^3u\big]d\mu\\
=\ & pA^2\int|\Delta u|^pd\mu+\frac{1}{2}pA^2\int|\nabla u|^pd\mu\le pA^2\lambda(t_0).
\end{aligned}\tag{3.20}$$

Since $\lambda(t,u(t))$ is a smooth function with respect to time t. Hence, in any sufficiently small neighborhood of t_0 as $[t_0,t_1]$, we have

$$\frac{d}{dt}\lambda(t,u(t))\le pA^2\lambda(t,u(t)).\tag{3.21}$$

Hence, by integration on $[t_0,t_1]$ of both hand sides of above inequality, we obtain

$$\lambda(t_1)e^{-p\int_0^{t_1}A^2(t)dt}\le\lambda(t_0)e^{-p\int_0^{t_0}A^2(t)dt}\tag{3.22}$$

Since t_0 is arbitrary, for any $t\in[0,T)$, the quantity $\lambda(t)e^{-p\int_0^tA^2(\tau)d\tau}$ is nonincreasing. Also in a similar way, we have $\frac{d}{dt}\lambda(t,u(t))|_{t=t_0}\ge -pA^2\lambda(t_0)$, which implies that the quantity $\lambda(t)e^{p\int_0^tA^2(\tau)d\tau}$ is nondecreasing. So, from the above results, we get the following corollary.

Corollary 3.8. *Let $\lambda(t)$ be the first eigenvalue of operator $\Delta_p^2-\Delta_p$ on Heisenberg Riemannian manifold $(\mathscr{H}^3,g_0)$. Then the quantities $\lambda(t)e^{-p\int_0^tA^2(\tau)d\tau}$ and $\lambda(t)e^{p\int_0^tA^2(\tau)d\tau}$ are nonincreasing and nondecreasing along the Ricci flow, respectively.*

Case 3: E(2)
Manifold E(2) is the group of isometries of the Euclidian plane. Dependent to the metric g_0, we choose the frame $\{X_i\}_{i=0}^3$ such that

$$[X_2,X_3]=X_1,\quad [X_3,X_1]=X_2,\quad [X_1,X_2]=0,$$

In this case, under the normalization $A_0B_0C_0=1$, we have

$$R_{11}=\frac{1}{2}A\left(A^2-B^2\right),\quad R_{22}=\frac{1}{2}B\left(B^2-A^2\right),\quad R_{33}=-\frac{1}{2}C(A-B)^2,\quad R=-\frac{1}{2}(A-B)^2.$$

Cao and Saloff-Coste in [20] proved that, for initial tensors A_0 and B_0,

- If $A_0=B_0$ then $A=B$, in this case $g(t)=g_0$ where g_0 is constant.
- If $A_0>B_0$ then $A>B$ in this case we have $\frac{d}{dt}\lambda(t,u(t))|_{t=t_0}\ge -p(A^2-B^2)\lambda(t_0)$ and $\frac{d}{dt}\lambda(t,u(t))|_{t=t_0}\le p\left(A^2-B^2\right)\lambda(t_0)$.

Case 4: E(1,1)
Manifold E(1,1) is the group of the isometries of the plane with flat Lorentz metric. For a given metric g_0, by a frame $\{X_i\}_{i=0}^3$, we have

$$[X_1, X_2] = 0, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2.$$

Also under the normalization $A_0 B_0 C_0 = 1$, we obtain

$$R_{11} = \frac{1}{2} A\left(A^2 - C^2\right), \quad R_{22} = -\frac{1}{2} B\left(A + C\right)^2, \quad R_{33} = \frac{1}{2} C\left(C^2 - A^2\right), \quad R = -\frac{1}{2}\left(A + C\right)^2.$$

- If $A_0 = C_0$, then along the Ricci flow

$$-p\left(A + C\right)^2 \lambda(t_0) \leq \frac{d}{dt}\lambda(t, u(t))|_{t=t_0} \leq p\left(A + C\right)^2 \lambda(t_0).$$

- If $A_0 > C_0$, then along the Ricci flow

$$-p\left(\left(A^2 - C^2\right) - \frac{1}{3}\left(A + C\right)^2\right)\lambda(t_0) \leq \frac{d}{dt}\lambda(t, u(t))|_{t=t_0} \leq p\left(A + C\right)^2 \lambda(t_0).$$

Case 5: SU(2)
Similarly, by the frame $\{X_i\}_{i=0}^3$, we have

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

In this case, under the normalization $A_0 B_0 C_0 = 1$, we have

$$R_{11} = \frac{1}{2} A[A^2 - (B - C)^2], \quad R_{22} = \frac{1}{2} B[B^2 - (A - C)^2], \quad R_{33} = \frac{1}{2} C[C^2 - (A - B)^2].$$

- If $A_0 = B_0 = C_0$, then $\lambda(t) = \lambda(0)$.
- If $A_0 = B_0 > C_0$, then

$$pC^2 \lambda(t_0) \leq \frac{d}{dt}\lambda(t, u(t))|_{t=t_0} \leq pA^2 \lambda(t_0).$$

Case 6: SL(2,$\mathbb{R}$)
On $SL(2, \mathbb{R})$, there is no Einstein metric. From the frame $\{X_i\}_{i=0}^3$, we get

$$[X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

In this case, we also have

$$R_{11} = \frac{1}{2} A[A^2 - (B - C)^2], \quad R_{22} = \frac{1}{2} B[B^2 - (A + C)^2], \quad R_{33} = \frac{1}{2} C[C^2 - (A + B)^2].$$

- If $A > B = C$, then

$$p\left(B^2 - (A + B)^2\right)\lambda(t_0) \leq \frac{d}{dt}\lambda(t, u(t))|_{t=t_0} \leq pA^2 \lambda(t_0).$$

- If $A \leq B - C$, then

$$p\left(A^2 - (A + B)^2\right)\lambda(t_0) \leq \frac{d}{dt}\lambda(t, u(t))|_{t=t_0} \leq p\left(B^2 - (A + C)^2\right)\lambda(t_0).$$

3.2. **The variation of $\lambda(t)$ on surfaces.** In this subsection, we write Propositions 3.2 and 3.3 in some remarkable particular cases and by these we prove Theorem 3.10.

Corollary 3.9. *Let $(M^2, g(t))$ be a solution of the Ricci flow on the smooth closed Riemannian surface (M, g_0). If $\lambda(t)$ denotes the first eigenvalue of operator $\Delta_p^2 - \Delta_p$ under the Ricci flow, then*

(1) under the unnormalized Ricci flow (1.1), we have

$$\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} = (p-1)\int_M R|\Delta u|^p\,d\mu + \frac{p-2}{2}\int_M R|\nabla u|^p\,d\mu + \lambda(t_0)\int_M R|u|^p d\mu, \tag{3.23}$$

(2) under the normalized Ricci flow (1.2), we have

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= (p-1)\int_M R|\Delta u|^p\,d\mu + \frac{p-2}{2}\int_M R|\nabla u|^p\,d\mu + \frac{r}{2}\int_M |\nabla u|^p d\mu \\ &\quad + \lambda(t_0)\int_M R|u|^p d\mu - pr\lambda(t_0),\end{aligned} \tag{3.24}$$

where u is the associated normalized evolving function.

Proof. On a surface, we have $R_{ij} = \frac{1}{2}Rg_{ij}$. Then (3.1) yields that

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= 2p\int_M (\Delta u)|\Delta u|^{p-2}R^{ij}\nabla_i\nabla_j u\,d\mu + p\int_M |\nabla u|^{p-2}R^{ij}\nabla_i u\nabla_j u\,d\mu \\ &\quad - \int_M R(|\Delta u|^p + |\nabla u|^p)d\mu + \lambda(t_0)\int_M R|u|^p d\mu \\ &= p\int_M R|\Delta u|^p\,d\mu + \frac{p}{2}\int_M R|\nabla u|^p\,d\mu - \int_M R(|\Delta u|^p + |\nabla u|^p)d\mu \\ &\quad + \lambda(t_0)\int_M R|u|^p d\mu \\ &= (p-1)\int_M R|\Delta u|^p\,d\mu + \frac{p-2}{2}\int_M R|\nabla u|^p\,d\mu + \lambda(t_0)\int_M R|u|^p d\mu.\end{aligned}$$

Similarly, using (3.12), we can conclude (3.24). □

Theorem 3.10. *Let (M^2, g_0) be a closed Riemannian surface with nonnegative scalar curvature. Then the first eigenvalue of $\Delta_p^2 - \Delta_p$ is increasing along the unnormalized Ricci flow (1.1) for $p \geq 2$.*

Proof. From lemma 3.1, along the unnormalized Ricci flow on a surface, we have

$$\frac{\partial R}{\partial t} = \Delta R + R^2.$$

Since the solution to the corresponding ODE $\frac{\partial y}{\partial t} = y^2$ with initial condition $y(0) := y_0 = \inf_M R(0)$ is $y(t) = \frac{1}{y_0 - t}$ on $[0, T')$, where $T' = \min\{y_0, T\}$. By use of the scalar maximum principle, the nonnegative scalar curvature holds on $[0, T')$ along the flow (1.1). Then evolution equation (3.23) of Corollary 3.9 implies that $\frac{d}{dt}\lambda(u,t) \geq 0$ in any sufficiently small neighborhood of t_0. Therefore, in any enough small neighborhood of t_0, $\lambda(t)$ is increasing. Since t_0 is arbitrary, then $\lambda(t)$ is increasing along Ricci flow (1.1). □

4. Example

Example 4.1. Let (M^n, g_0) be an Einstein manifold, that is, there exists a constant c such that $Ric(g_0) = cg_0$. The solution to Ricci flow (1.1) on (M^n, g_0) is $g(t) = (1-2ct)g_0$. Therefore $Ric(g(t)) = \frac{c}{1-2ct}g(t)$ and $R_{g(t)} = \frac{cn}{1-2ct}$. Substituting these into (3.1), we get

$$\begin{aligned}
\frac{d}{dt}\lambda(t,u(t))|_{t=t_0} &= \frac{2pc}{1-2ct_0}\int_M |\Delta u|^p\, d\mu + \frac{pc}{1-2ct_0}\int_M |\nabla u|^p\, d\mu \\
&\quad -\frac{cn}{1-2ct_0}\int_M (|\Delta u|^p + |\nabla u|^p) d\mu + \lambda(t_0)\frac{cn}{1-2ct_0}\int_M |u|^p d\mu \\
&= \frac{2pc}{1-2ct_0}\lambda(t_0) - \frac{pc}{1-2ct_0}\int_M |\nabla u|^p\, d\mu \\
&\leq \frac{2pc}{1-2ct_0}\lambda(t_0).
\end{aligned}$$

For any t sufficiently close to t_0, we have

$$\frac{1}{\lambda(t,u(t))}\frac{d\lambda(t,u(t))}{dt} \leq \frac{2pc}{1-2ct}.$$

Therefore, for any t_1 sufficiently close to t_0, where $t_1 > t_0$, we get

$$\ln\frac{\lambda(t_1)}{\lambda(t_0)} \leq \ln\frac{\lambda(t_1,u(t_1))}{\lambda(t_0,u(t_0))} \leq \ln(\frac{1-2ct_0}{1-2ct_1})^p,$$

equivalently

$$\lambda(t_0)(1-2ct_0)^p \geq \lambda(t_1)(1-2ct_1)^p.$$

Since t_0 is arbitrary, then $\lambda(t)(1-2ct)^p$ is decreasing along unnormalized Ricci flow (1.1).

5. Concluding remarks

The problem of evolution and monotonicity of the first eigenvalue of a geometric operator along the geometric flows is a known problem and is a important tool in understanding geometry of the manifold. In this paper, we considered the evolution and monotonicity of the first nonzero eigenvalue of geometric operator $\Delta_p^2 - \Delta_p$, which it acts on the space of functions on a smooth closed oriented Riemannian manifold along the Ricci flow and normalized Ricci flow. We established that the first eigenvalue $\lambda(t)$ of the operator $\Delta_p^2 - \Delta_p$ diverges as t approaches to maximal existence time. We gave evolution formulas for $\lambda(t)$ along the Ricci flow and normalized Ricci flow and using them we showed if the initial scalar curvature is positive and there is a positive constant $\varepsilon > \frac{1}{p}$ such that $R_{ij} \geq \varepsilon R g_{ij}$ along the unnormalized Ricci flow, then $\lambda(t)$ is increasing. Also, if the scalar curvature of (M^2, g_0) is nonnegative, then $\lambda(t)$ is increasing along the Ricci flow for $p \geq 2$.

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