J. Nonlinear Funct. Anal. 2020 (2020), Article ID 31 https://doi.org/10.23952/jnfa.2020.31

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

BLOW-UP OF A NONLINEAR VISCOELASTIC WAVE EQUATION WITH DISTRIBUTED DELAY COMBINED WITH STRONG DAMPING AND SOURCE TERMS

ABDELBAKI CHOUCHA[1], DJAMEL OUCHENANE[2], SALAH BOULAARAS[3,4,*]

[1]Laboratory of Operator Theory and PDEs: Foundations and Applications,
Department of Mathematics, Faculty of Exact Sciences, University of El Oued, El Oued, Algeria
[2]Laboratory of pure and applied Mathematics, Amar Teledji Laghouat University, Algeria
[3]Department of Mathematics, College of Sciences and Arts, Al-Rass,
Qassim University, Kingdom of Saudi Arabia
[4]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO),
University of Oran 1, Ahmed Benbella, Oran, Algeria

Abstract. In this paper, we are concerned with the problem of a nonlinear viscoelastic wave equation with distributed delay, strong damping and source terms. We obtain a blow-up result of solutions under suitable conditions.

Keywords. Blow-up; Nonlinear viscoelastic wave equation; Distributed delay; Strong damping.

1. INTRODUCTION

In this paper, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| u_t(x,t-\rho)d\rho = b|u|^{p-2}.u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \; x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), \quad (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \; x \in \Omega, \end{cases} \tag{1.1}$$

where $\omega, b, \mu_1 > 0$, $p > 2$ and τ_1, τ_2 are the time delay with $0 \leq \tau_1 < \tau_2$, μ_2 is an L^∞ function, and g is a differentiable function under the assumptions (A_1), (A_2), and (A_3).

*Corresponding author.
E-mail addresses: abdelbaki.choucha@gmail.com (A. Choucha), ouchenanedjamel@gmail.com (D. Ouchenane), saleh_boulaares@yahoo.fr, s.boularas@qu.edu.sa (S. Boulaaras)

Received May 19, 2020; Accepted July 9, 2020.

Viscous materials are the opposite of elastic materials that possess the ability to store and dissipate the mechanical energy. The mechanical properties of these viscous substances are of great importance when they appear in many applications of natural sciences. Many authors investigated this problem after the beginning of the new millennium (see, for example, [1, 2, 3]).

If $w = 0$, that is, the absence of Δu_t, and $\mu_1 = \mu_2 = 0$, problem (1.1) was studied by Berrimi and Messaoudi [4]. By using the Galerkin method, they established the local existence result. Also, they showed the local solution is global in time under a suitable conditions, and with the same rate of decaying (polynomial or exponential) of the kernel g. They also proved that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution. Their results were also obtained under weaker conditions than those used by Cavalcanti, Cavalcanti and Ferreira [5].

In [6], Cavalcanti *et al.* considered the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^{\gamma} u = 0. \tag{1.2}$$

They obtained an exponential decay result. The later result was improved by Berrimi and Messaoudi [4], and they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate. There are many results on this field under the assumptions of the kernel g. For the problem (1.1) and with $\mu_1 \neq 0$, Kafini and Messaoudi [7] proved a blow up result for the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{\infty} g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}.u, \ \ (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{cases} \tag{1.3}$$

where g satisfies $\int_0^{\infty} g(s)ds < (2p-4)/(2p-3)$. Initial data are supported with negative energy like that $\int_{\Omega} u_0(x)u_1(x)dx > 0$. If $(w > 0)$, Song and Xue [8] considered with the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{\infty} g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \ \ (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x). \end{cases} \tag{1.4}$$

Under suitable assumptions on g that there are solutions of (1.4) with the initial energy, they showed the blow up in a finite time. For the same problem, Song and Zhang [9] proved that there are solutions of (1.4) with the positive initial energy that blow up in finite time. In [10], Zennir studied the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ +a|u_t|^{m-2}.u_t = |u|^{p-2}.u, \ \ (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega. \end{cases} \tag{1.5}$$

He proved the exponential growth result under suitable assumptions. In [11], Guo, Yuan and Lin considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(t-s)ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \\ u(x,t) = 0, \ \ x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \ \ x \in \Omega. \end{cases} \tag{1.6}$$

They showed a blow-up result if $p > m$, and established the global existence.

In this paper, we studied problem (1.1). All damping mechanisms are considered at the same time (i.e., $w > 0$, $g \neq 0$, and $\mu_1 > 0, \mu_2 \in L^\infty$). These assumptions make our problem different from the ones previously studied. In particular, the blow-up of solutions. Our goal is to extend the results of the blow-up results to our strong damping for a viscoelastic problem with distributed delay. We prove the blow-up result under the following suitable assumptions.

(**A1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and decreasing function such that

$$g(t) \geq 0, \quad 1 - \int_0^\infty g(s)\,ds = l > 0. \tag{1.7}$$

(**A2**) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t), \quad t \geq 0. \tag{1.8}$$

(**A3**) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞ function such that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| d\rho \leq \mu_1, \quad \delta > \frac{1}{2}. \tag{1.9}$$

2. Blow-up solutions

In this section, we prove the blow-up result of solutions of problem (1.1). First, as in [12], we introduce the new variable

$$y(x, \rho, \rho, t) = u_t(x, t - \rho\rho),$$

then we obtain

$$\begin{cases} \rho y_t(x, \rho, \rho, t) + y_\rho(x, \rho, \rho, t) = 0, \\ y(x, 0, \rho, t) = u_t(x, t). \end{cases} \tag{2.1}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds. \tag{2.2}$$

Therefore, problem (1.1) takes the form

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ +\mu_1 u_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y(x, 1, \rho, t) d\rho = b|u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ \rho y_t(x, \rho, \rho, t) + y_\rho(x, \rho, \rho, t) = 0 \end{cases} \tag{2.3}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \\ y(x, \rho, \rho, 0) = f_0(x, \rho\rho), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \end{cases} \tag{2.4}$$

where

$$(x, \rho, \rho, t) \in \Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$$

Theorem 2.1. *Assume that (1.7), (1.8), and (1.9) hold. Let*

$$\begin{cases} 2 < p < \dfrac{2n-2}{n-2}, \quad n \geq 3, \\ p \geq 2, \qquad n = 1, 2. \end{cases} \tag{2.5}$$

Then for any initial data

$$(u_0, u_1, f_0) \in \mathscr{H} \ / \ \mathscr{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)),$$

problem (2.4) has a unique solution $u \in C([0,T]; H_0^1)$ *for some* $T > 0$.

We define the following energy functional.

Lemma 2.2. *Assume that (1.7), (1.8), (1.9) and (2.5) hold. Let* $u(x,t)$ *be a solution of (2.3). Then* $E(t)$ *is non-increasing, that is,*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) \\ &\quad + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx - \frac{b}{p}\|u\|_p^p, \end{aligned} \tag{2.6}$$

which satisfies

$$E(t) \leq -c_1(\|u_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx) \tag{2.7}$$

Proof. Multiplying the equation $(2.3)_1$ by u_t and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) - \frac{b}{p}\|u\|_p^p\} \\ &= -\mu_1\|u_t\|_2^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y(x,1,\rho,t)d\rho dx \\ &+ \frac{1}{2}(g'o\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx \\ &= -\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} 2|\mu_2(\rho)|yy_\rho d\rho d\rho dx \\ &= \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,0,\rho,t)d\rho dx - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx \\ &= \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)d\rho)\|u_t\|_2^2 - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx. \end{aligned} \tag{2.9}$$

It follows that

$$\begin{aligned} \frac{d}{dt}E(t) &= -\mu_1\|u_t\|_2^2 - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|u_t y(x,1,\rho,t)d\rho dx + \frac{1}{2}(g'o\nabla u) \\ &\quad - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2 + \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)d\rho)\|u_t\|_2^2 \\ &\quad - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx. \end{aligned} \tag{2.10}$$

By use of (2.8) and (2.9), we get (2.6) immediately. From Young's inequality, (1.7), (1.8), (1.9) and (2.10), we obtain (2.7). This completes the proof. □

Lemma 2.3. *There exists $c>0$, depending on Ω only, such that*

$$(\int_\Omega |u|^p dx)^{s/p} \leq c[\|\nabla u\|_2^2 + \|u\|_p^p], \tag{2.11}$$

for all $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$.

Using the fact that $\|u\|_2^2 \leq c\|u\|_p^2 \leq c(\|u\|_p^p)^{2/p}$, we have the following.

Corollary 2.4. *There exists $C>0$, depending on Ω, such that*

$$\|u\|_2^2 \leq c[\|\nabla u\|_2^{4/p} + (\|u\|_p^p)^{2/p}]. \tag{2.12}$$

Lemma 2.5. *There exists $C>0$, depending on Ω, such that*

$$\|u\|_p^s \leq C[\|\nabla u\|_2^2 + \|u\|_p^p] \tag{2.13}$$

for all $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$.

Proof. If $\|u\|_p \geq 1$, then

$$\|u\|_p^s \leq \|u\|_p^p.$$

If $\|u\|_p \leq 1$, then $\|u\|_p^s \leq \|u\|_p^2$. Using Sobolev embedding theorems, we have

$$\|u\|_p^s \leq \|u\|_p^2 \leq c\|\nabla u\|_2^2.$$

This completes the proof. □

Now we define the functional

$$\begin{aligned} \mathbb{H}(t) = -E(t) &= \frac{b}{p}\|u\|_p^p - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 \\ &\quad -\frac{1}{2}(go\nabla u) - \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx. \end{aligned} \tag{2.14}$$

Theorem 2.6. *Assume that (1.7)-(1.9), and (2.5) hold, and $E(0)<0$. Then the solution of problem (2.3) blows up in finite time.*

Proof. From (2.6), we have

$$E(t) \leq E(0) \leq 0. \tag{2.15}$$

Therefore

$$\begin{aligned} \mathbb{H}'(t) = -E'(t) &\geq c_1(\|u_t\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx) \\ &\geq c_1\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx \geq 0 \end{aligned} \tag{2.16}$$

and

$$0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{2.17}$$

We set

$$\mathscr{K}(t) = \mathbb{H}^{1-\alpha} + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx + \frac{\varepsilon\omega}{2}\int_\Omega (\nabla u)^2 dx. \tag{2.18}$$

where $\varepsilon > 0$ is to be assigned later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{2.19}$$

Multiplying $(2.3)_1$ by u and with a derivative of (2.18), we get

$$\begin{aligned} \mathscr{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s)dsdx \\ &\quad -\varepsilon\|\nabla u\|_2^2 + \varepsilon b\int_\Omega |u|^p dx - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|uy(x,1,\rho,t)d\rho dx. \end{aligned} \tag{2.20}$$

Using

$$\begin{aligned} \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|uy(x,1,\rho,t)d\rho dx &\leq \varepsilon\{\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)\|u\|_2^2 \\ &\quad + \frac{1}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx\}. \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.\nabla u(s)dxds &= \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.(\nabla u(s)-\nabla u(t))dxds \\ &\quad + \varepsilon\int_0^t g(s)ds\|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2}\int_0^t g(s)ds\|\nabla u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{2.22}$$

we obtain from (2.20) that

$$\begin{aligned} \mathscr{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon(1-\frac{1}{2}\int_0^t g(s)ds)\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ &\quad -\varepsilon\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)\|u\|_2^2 - \frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|y^2(x,1,\rho,t)d\rho dx \\ &\quad +\frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{2.23}$$

Using (2.16) and setting δ_1 such that $\dfrac{1}{4\delta_1 c_1} = \kappa\mathbb{H}^{-\alpha}(t)$, we get

$$\begin{aligned} \mathscr{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 \\ &\quad -\varepsilon[(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ &\quad -\varepsilon\frac{\mathbb{H}^\alpha(t)}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)\|u\|_2^2 + \frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{2.24}$$

For $0<a<1$, it follows from (2.14) that

$$\begin{aligned} \varepsilon b\|u\|_p^p &= \varepsilon p(1-a)\mathbb{H}(t)+\frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2+\varepsilon ba\|u\|_p^p \\ &+\frac{\varepsilon p(1-a)}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2+\frac{\varepsilon}{2}p(1-a)(go\nabla u) \\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx, \end{aligned} \tag{2.25}$$

which together with (2.24) implies that

$$\begin{aligned} \mathscr{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2 \\ &+\varepsilon[(\frac{p(1-a)}{2})(1-\int_0^t g(s)ds)-(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 \\ &-\varepsilon\frac{H^\alpha(t)}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)\|u\|_2^2+\varepsilon p(1-a)\mathbb{H}(t)+\varepsilon ba\|u\|_p^p \\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx \\ &+\frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u). \end{aligned} \tag{2.26}$$

Using (2.12), (2.17) and Young's inequality, we get

$$\begin{aligned} \mathbb{H}^\alpha(t)\|u\|_2^2 &\leq (b\int_\Omega|u|^p dx)^\alpha\|u\|_2^2 \\ &\leq c\{(\int_\Omega|u|^p dx)^{\alpha+2/p}+(\int_\Omega|u|^p dx)^\alpha\|\nabla u\|_2^{4/p}\} \\ &\leq c\{(\int_\Omega|u|^p dx)^{(p\alpha+2)/p}+\|\nabla u\|_2^2+(\int_\Omega|u|^p dx)^{p\alpha/(p-2)}\}. \end{aligned} \tag{2.27}$$

Exploiting (2.19), we have

$$2<p\alpha+2\leq p, \quad and \quad 2<\frac{\alpha p^2}{p-2}\leq p.$$

Consequently, it follows from Lemma 2.2 that

$$\mathbb{H}^\alpha(t)\|u\|_2^2\leq c\{\|u\|_p^p dx+\|\nabla u\|_2^2\}. \tag{2.28}$$

Combining (2.26) and (2.28), we obtain

$$\begin{aligned}\mathscr{K}'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon[\frac{p(1-a)}{2}+1]\|u_t\|_2^2\\ &+\frac{\varepsilon}{2}(p(1-a)+1)(go\nabla u)\\ &+\varepsilon\{(\frac{p(1-a)}{2}-1)-\int_0^t g(s)ds(\frac{p(1-a)-1}{2})\\ &-\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)\}\|\nabla u\|_2^2\\ &+\varepsilon[ab-\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)]\|u\|_p^p+\varepsilon p(1-a)\mathbb{H}(t)\\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx. \qquad (2.29)\end{aligned}$$

We take $a>0$ small enough such that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

and assume

$$\int_0^\infty g(s)ds<\frac{\frac{p(1-a)}{2}-1}{(\frac{p(1-a)}{2}-\frac{1}{2})}=\frac{2\alpha_1}{2\alpha_1+1}. \qquad (2.30)$$

Then, we choose κ such that

$$\alpha_2=(\frac{p(1-a)}{2}-1)-\int_0^t g(s)ds(\frac{p(1-a)-1}{2})-\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)>0$$

and

$$\alpha_3=ab-\frac{c}{4c_1\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho)>0.$$

Fixing κ and a, we have ε small enough such that

$$\alpha_4=(1-\alpha)-\varepsilon\kappa>0.$$

Thus, for some $\beta>0$, (2.29) becomes

$$\begin{aligned}\mathscr{K}'(t) \geq\ & \beta\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+(go\nabla u)+\|u\|_p^p\\ &+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_2(\rho)|y^2(x,\rho,\rho,t)d\rho d\rho dx\}. \qquad (2.31)\end{aligned}$$

It follows that

$$\mathscr{K}(t)\geq\mathscr{K}(0)>0,\quad t>0. \qquad (2.32)$$

Next, using Holder's and Young's inequalities, we have

$$\|u\|_2=(\int_\Omega u^2dx)^{\frac{1}{2}}\leq[(\int_\Omega(|u|^2)^{p/2}dx)^{\frac{2}{p}}.(\int_\Omega 1dx)^{1-\frac{2}{p}}]^{\frac{1}{2}}\leq c\|u\|_p \qquad (2.33)$$

and

$$|\int_\Omega uu_tdx|\leq\|u_t\|_2.\|u\|_2\leq c\|u_t\|_2.\|u\|_p.$$

It follows that

$$\begin{aligned} |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} &\geq c\|u_t\|_2^{\frac{1}{1-\alpha}}.\|u\|_p^{\frac{1}{1-\alpha}} \\ &\leq c[\|u_t\|_2^{\frac{\theta}{1-\alpha}} + \|u\|_p^{\frac{\mu}{1-\alpha}}], \end{aligned} \tag{2.34}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta=2(1-\alpha)$ to get

$$\frac{\mu}{1-\alpha}=\frac{2}{1-2\alpha}\leq p$$

Subsequently, for $s=\frac{2}{(1-2\alpha)}$, we obtain

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} \leq c[\|u_t\|_2^2+\|u\|_p^s].$$

Therefore, Lemma 2.3 gives that

$$\begin{aligned} |\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} &\geq c[\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2] \\ &\leq c[\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2+(go\nabla u)]. \end{aligned} \tag{2.35}$$

Subsequently,

$$\begin{aligned} \mathscr{K}^{\frac{1}{1-\alpha}}(t) &= (\mathbb{H}^{1-\alpha}+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx+\frac{\varepsilon\omega}{2}\int_\Omega \nabla u^2 dx)^{\frac{1}{1-\alpha}} \\ &\leq c[\mathbb{H}(t)+|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}+\|u\|_2^{\frac{2}{1-\alpha}}+\|\nabla u\|_2^{\frac{2}{1-\alpha}}] \\ &\leq c[\mathbb{H}(t)+\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2+(go\nabla u)]. \end{aligned} \tag{2.36}$$

From (2.31) and (2.36), we have

$$\mathscr{K}'(t)\geq \lambda \mathscr{K}^{\frac{1}{1-\alpha}}(t), \tag{2.37}$$

where $\lambda>0$, which depends only on β and c. By integration of (2.37), we obtain

$$\mathscr{K}^{\frac{\alpha}{1-\alpha}}(t)\geq \frac{1}{\mathscr{K}^{\frac{-\alpha}{1-\alpha}}(0)-\lambda\frac{\alpha}{(1-\alpha)}t}.$$

Hence, $\mathscr{K}(t)$ blows up in time

$$T\leq T^*=\frac{1-\alpha}{\lambda\alpha\mathscr{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

Acknowledgements

The authors would like to thank the anonymous referees and the handling editor for their careful reading and for relevant remarks/suggestions which helped them to improve the paper.

REFERENCES

[1] S. Boulaaras, A. Choucha, D. Ouchenane, B. Cherif, Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms, Adv. Differ. Equ. 2020 (2020), 310.

[2] A. Rahmoune, D. Ouchenane, S. Boulaaras, B. Cherif, Growth of solutions for a coupled nonlinear Klein-Gordon system with strong damping, source, and distributed delay terms, Adv. Differ. Equ. 2020, (2020), 335.

[3] A. Zarai, A. Draifia, S. Boulaaras, Blow-up of solutions for a system of nonlocal singular viscoelastic equations, Appl. Anal. 97 (2018) 2231-2245.

[4] S. Berrimi, S. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, Nonlinear Anal. 64 (2006), 2314-2331.

[5] M. M. Cavalcanti, D. Cavalcanti, J. Ferreira, Existence and uniform decay for nonlinear viscoelastic equation with strong damping, Math. Meth. Appl. Sci. 24 (2001), 1043-1053.

[6] M. M. Cavalcanti, D. Cavalcanti, P. J. S. Filho, J. A. Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, Differential and Integral Equations, 14 (2001), 85-116.

[7] M. Kafini, S.A. Messaoudi, A blow-up result in a Cauchy viscoelastic problem, Appl. Math. Lett. 21 (2008), 549-553.

[8] H.T. Song, D.S. Xue, Blow up in a nonlinear viscoelastic wave equation with strong damping, Nonlinear Anal. 109 (2014), 245-251.

[9] H.T. Song, C.K. Zhong, Blow-up of solutions of a nonlinear viscoelastic wave equation, Nonlinear Anal. 11 (2010), 3877-3883.

[10] K. Zennir, Exponential growth of solutions with L_p-norm of a nonlinear viscoelastic hyperbolic equation, J. Nonlinear Sci. Appl. 6 (2013), 252-262.

[11] L. Guo, Z. Yuan, G. Lin, Blow up and global existence for a nonlinear viscoelastic wave equation with strong damping and nonlinear damping and source terms, J. Appl. Math. 6 (2015), 806-816.

[12] S. Nicaise, C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Differential Integral Equ. 21 (2008), 935-958.