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ON A COUPLED SYSTEM OF KIRCHHOFF BEAM EQUATIONS WITH NONLINEAR FRICTIONAL DAMPING AND SOURCE TERM

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Abstract. Whenever an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time and introduces changes of the tension in the string. This motivated Kirchhoff to propose a nonlinear correction of the classical D'Alembert equation. In this paper, we prove the existence of the solution to the mixed problem for the coupled beam system of Kirchhoff type with nonlinear frictional damping and source terms. Moreover, by Nakao method we prove the uniform decay of the solution.

Keywords. Coupled beam system; Nonlinear frictional damping; Source term.

1. INTRODUCTION

Many structures in mechanical engineering, electrical engineering, civil engineering and aerospace engineering are formed by a single beam or more than one. One of the first mathematical analysis for the Kirchhoff beam equation for $\Omega = (0,L) \subset \mathbb{R}$, $L > 0$,

$$u_{tt} + u_{xxxx} - M\left(\int_0^L |u_x|^2\, dx\right) u_{xx} = 0 \tag{1.1}$$

was done by Ball (1973) [1] . Tucsnak (1996) [2] extended for the beam equation in $\Omega \subset \mathbb{R}^n$

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) = 0, \tag{1.2}$$

where

$$|\nabla u|^2 =: \int_\Omega |\nabla u|^2\, dx.$$

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For the nonlinear and damped extensible beam equation, we cite Cavalcanti et al. [3] with references therein

$$u_{tt}+\Delta^2 u+\alpha u+M\left(|\nabla u|^2\right)(-\Delta u)+f(u)+g(u_t)=0 \text{ in } \Omega\times(0,\infty), \tag{1.3}$$

where Ω is any bounded or unbounded open set of $\mathbb{R}^n, \alpha>0$ and f,g are power like functions. The existence of global solutions was proved by means of fixed point theorems and continuity arguments. In fact, $f(u)$ and $g(u_t)$ are external source and nonlinear damping, respectively.

Coupled systems appear in different contexts representing different phenomena, for instance, robot arms, rotor turbine and helicopter blades, turbomachineries, electronic equipment, antennas, missiles, panels, pipelines, buildings, bridges, etc. From mathematical point of view, this structures is presented as vibrations of a beam, laminated beam, the transmission problem, thermoelasticity and so on. In this paper, we deal with the existence and the energy decay estimate of global solutions for the coupled beams system of Kirchhoff type with nonlinear frictional damping and source terms, given by

$$u_{tt}+\Delta^2 u+M(|\nabla u|^2+|\nabla v|^2)(-\Delta u)+|u_t|^{p-1}u_t=|u|^{q-1}u \text{ in } \Omega\times(0,T), \tag{1.4}$$

$$v_{tt}+\Delta^2 v+M(|\nabla u|^2+|\nabla v|^2)(-\Delta v)+|v_t|^{p-1}v_t=|v|^{q-1}v \text{ in } \Omega\times(0,T), \tag{1.5}$$

$$(u(x,0),v(x,0))=(u_0,v_0),\ x\in\Omega, \tag{1.6}$$

$$(u_t(x,0),v_t(x,0))=(u_1,v_1),\ x\in\Omega, \tag{1.7}$$

$$u(x,t)=\frac{\partial u}{\partial\eta}(x,t)=v(x,t)=\frac{\partial v}{\partial\eta}(x,t)=0,\ x\in\partial\Omega, t\geq 0, \tag{1.8}$$

where $p\geq 1$, $q>1$ are real number, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $M(s)$ is a continuous function on $[0,+\infty)$.

Timoshenko [4] came up with a new theory for a coupled system, which is better suited for engineering practice and is nowadays widely used for moderately thick beams. In his theory, it was also assumed that the plane cross-sections u is perpendicular to the beam centerline and remain plane. Then, an additional kinematics variable represents the rotation angle of a filament of the beam φ was added in the displacement assumptions. The pioneer linear model was given by two coupled partial differential equations

$$\rho u_{tt}(x,t)-K(u_x-K\varphi)_x=0, \quad x\in(0,L),\ t>0,$$
$$I_\rho\varphi_{tt}(x,t)-EI\varphi_{xx}(x,t)-K(u_x-K\varphi)=0, \quad x\in(0,L),\ t\geq 0,$$

where t is the time variable and x is the space coordinate along the beam, the length of which is L, in its equilibrium position. The coefficients ρ, I_ρ, E, I and K are the mass per unit length, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus, respectively.

Hansen and Spies [5] derived from Timoshenko's theory, a coupled model for a structure of two identical beams of uniform thickness with an adhesive layer (of negligible thickness and mass) bonding the two adjoining surfaces in such a way that a slip is allowed while they are continuously in contact with each other. The adhesive layer produces a restoring force which is assumed to be proportional to the amount of slip. The model derived is called laminated beams and is given by

$$\begin{aligned}
&\rho w_{tt} + G(\psi - w_x)_x = 0,\\
&I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0,\\
&3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0,
\end{aligned}$$

where $w = w(x,t)$ is the transverse displacement, $\psi = \psi(x,t)$ is the rotation angle, $s = s(x,t)$ is proportional to the amount of slip along the interface. The positive parameters ρ, I_ρ, G, D, and γ, are the density, mass moment of inertia, shear stiffness, flexural rigidity, and adhesive stiffness, respectively. The non-negative parameter β is called the adhesive damping, and s_t is a structural damping of the system. These structures have gained tremendous popularity and are of considerable importance in both science and engineering fields.

The transmission problem consists of a coupled system of wave propagation over bodies with two physically different types of materials. One component is a simple elastic part while the other is a component endowed with stabilization property. The transmission problem to hyperbolic equations was first studied by Dautray and Lions [6]. In [7], Liu considered a transmission problem in a bounded domain with a distributed delay given by

$$\begin{aligned}
&u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)\,ds = 0, \quad x \in \Omega,\ t > 0,\\
&v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0,
\end{aligned}$$

under the boundary and the transmission conditions

$$\begin{aligned}
&u(0,t) = u(L_3,t) = 0,\\
&u(L_i,t) = v(L_i,t), \quad i = 1,2,\\
&au_x(L_i,t) = bv_x(L_i,t), \quad i = 1,2,
\end{aligned}$$

and the initial conditions

$$\begin{aligned}
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,\\
&v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1, L_2),\\
&u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0, \tau_2),
\end{aligned}$$

where $0 < L_1 < L_2 < L_3$, $\Omega = (0, L_1) \cup (L_2, L_3)$, a, b, μ_1 are positive constants, and the initial data $(u_0, u_1, v_0, v_1, f_0)$ belongs to suitable space. Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real number satisfying $0 \leq \tau_1 < \tau_2$. Using a semigroup theorem, the author proved the existence and uniqueness of global solutions under suitable assumptions on the weight of damping and the weight of distributed delay, and established the exponential stability of the solution by introducing a suitable Lyapunov functional.

The one dimensional thermoelastic system for a body of length L is given by the coupled system

$$\begin{aligned}
&u_{tt} - u_{xx} + \alpha\theta_x = 0, \quad x \in (0,L),\ t > 0,\\
&\theta_t - \theta_{xx} + \beta u_{xt} = 0, \quad x \in (0,L),\ t > 0,
\end{aligned} \tag{1.9}$$

where u, θ represent the small transversal vibrations and temperature, respectively. The existence, uniqueness and regularity of the solution of (1.9) is well known by now and was due to the pioneer work of Dafermos [8]. Thermoelastic system type p-Laplacian was presented by

Raposo et al. in [9]

$$\begin{aligned} &u_{tt} - \Delta_p u + \theta = |u|^{r-1}u, \\ &\theta_t - \Delta\theta = u_t, \end{aligned}$$

where $\Delta_p u$ is the nonlinear p-Laplacian operator, $2 \leq p < \infty$. The authors applied the potential well theory and the global solution was constructed by means of the Faedo-Galerkin approximations by taking into account the fact that the initial data is in appropriated set of stability created from the Nehari manifold.

Regarding the model (1.4)–(1.5), Park and Bae [10] proved the existence of the solution to the mixed problem for the coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory source term of the form

$$\begin{aligned} &u_{tt} + M(||\nabla u||^2 + ||\nabla v||^2)(-\Delta u) - \Delta u_t = 0, \text{ in } \Omega \times (0,\infty), \\ &v_{tt} + M(||\nabla u||^2 + ||\nabla v||^2)(-\Delta v) - \Delta v_t = 0, \text{ in } \Omega \times (0,\infty), \\ &u = v = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} 0, \text{ on } \partial\Omega \times (0,\infty), \\ &M(||\nabla u||^2 + ||\nabla v||^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} + u + u_t + g(t)|u_t|^\rho u_t = g * |u|^\gamma u \text{ on } \Gamma_0 \times (0,\infty), \\ &M(||\nabla u||^2 + ||\nabla v||^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} + v + v_t + g(t)|v_t|^\rho v_t = g * |v|^\gamma v = 0, \text{ on } \Gamma_0 \times (0,\infty), \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with C^2 boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1, \bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 have positive measures, $M(s)$ is in C^1 class like as $1+s$,

$$[g * u](t) = \int_0^t g(t-r)u(r)\,dr$$

and ν denotes the unit outer normal vector pointing towards Ω.

The global existence of global solutions and asymptotic behaviour of the energy for the wave equations of Kirchhoff type with nonlocal boundary condition was proved by Ferreira et al. [11]. They took into account the system bellow

$$\begin{aligned} &u_{tt} + M(||\nabla u||^2 + ||\nabla v||^2)(-\Delta u) - \Delta u_t + f_1(u) = 0, \text{ in } \Omega \times (0,\infty), \\ &v_{tt} + M(||\nabla u||^2 + ||\nabla v||^2)(-\Delta v) - \Delta v_t + f_2(v) = 0, \text{ in } \Omega \times (0,\infty), \\ &u = v = 0, \text{ on } \Gamma_0 \times (0,\infty), \\ &u + \int_0^t g_1(t-s)M(||\nabla u||^2 + ||\nabla v||^2)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_t}{\partial \nu}(s)\,ds = 0, \text{ in } \Gamma_1 \times (0,\infty), \\ &v + \int_0^t g_2(t-s)M(||\nabla u||^2 + ||\nabla v||^2)\frac{\partial v}{\partial \nu}(s) + \frac{\partial v_t}{\partial \nu}(s)\,ds = 0, \text{ in } \Gamma_1 \times (0,\infty), \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0, Γ_1 closed, disjoint, $\Gamma_0 \neq \emptyset$ and ν is the unit normal vector pointing towards the exterior of Ω and g_i are positive and non decreasing, while the functions $f_i \in C^1(\mathbb{R}), i = 1,2$, satisfies $f_i(s)s \geq 0, \forall s \in \mathbb{R}$.

The following system was considered by Ye [12]

$$\begin{aligned} &u_{tt} + M(||\nabla u||^2 + ||\nabla v||^2)(-\Delta u) - a|u_t|^{q-2}u_t = b|u|^{p-2}u|v|^p \text{ in } \Omega \times (0,\infty), \\ &v_{tt} + M(||\nabla u||^2 + ||\nabla v||^2)(-\Delta v) - a|v_t|^{q-2}v_t = b|v|^{p-2}v|u|^p \text{ in } \Omega \times (0,\infty). \end{aligned} \tag{1.10}$$

The author proved the global existence and energy decay results. The model (1.10) has its origin in the nonlinear vibrations of an elastic string introduced by Narasimha [13].

The existence, uniqueness, and uniform decay rates of the energy of solution for a nonlinear degenerate coupled beams system with weak damping was presented by Lobato et al. [14] in

$$\begin{aligned}
&K_1(x,t)u_{tt}+\Delta^2 u+M(||u||^2+||v||^2)(-\Delta u)+u_t=f_1(u,v), \ \text{ in } \Omega\times(0,T),\\
&K_2(x,t)v_{tt}+\Delta^2 v+M(||u||^2+||v||^2)(-\Delta v)+v_t=f_2(u,v) \ \text{ in } \Omega\times(0,T),\\
&u=v=\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 \ \text{ on } \partial\Omega\times(0,T),
\end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n\geq 1$, with smooth boundary Γ, $T>0$ is a real arbitrary number, ν is the unit normal at $\Gamma\times(0,T)$ direct towards the exterior of $\Omega\times(0,T)$, with $M\in C^1([0,\infty))$ and

$$K_i\in C^1([0,T]:H_0^1(\Omega)\cap L^\infty(\Omega)),\, i=1,2.$$

Recently, the nonexistence of global solutions for positive initial energy was proved by Pişkin and Ekinci [15] for the coupled nonlinear Kirchhoff type equation with degenerate damping and source terms. They considered the following system

$$\begin{aligned}
&|u_t|^j u_{tt}+M(||\nabla u||^2+||\nabla v||^2)(-\Delta u)-(|u|^k+|v|^l)u_t^{p-1}=f_1(u,v), \ \text{ in } \Omega\times(0,\infty),\\
&|v_t|^j v_{tt}+M(||\nabla u||^2+||\nabla v||^2)(-\Delta v)-(|v|^\theta+|u|^\rho)v_t^{q-1}=f_2(u,v), \ \text{ in } \Omega\times(0,\infty),
\end{aligned}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $(n=1;2;3)$, $p,q\geq 1$, $j,k,l,q,\theta,\rho\geq 0$, $M(s)$ is a nonnegative locally Lipschitz function and

$$\begin{aligned}
f_1(u,v)&=a|u+v|^{2(r+1)}(u+v)+b|u|^r u|v|^{r+2},\\
f_2(u,v)&=a|u+v|^{2(r+1)}(u+v)+b|v|^r v|u|^{r+2},
\end{aligned}$$

where $a,b>0$ are constants with $-1<r$ if $n=1,2$ and $-1<r<1$ when $n=3$.

The abstract formulation of the coupled system was given by Pereira et al. [16] where they studied the existence of solutions by Faedo-Galerkin's method and proved the exponential decay by using Nakao's theorem for the following problem

$$\begin{aligned}
&K_1u''+A_1^2u+M\left(\left|A_1^{1/2}u\right|^2+\left|A_2^{1/2}v\right|^2\right)A_1u+\delta_1u'=0,\\
&K_2v''+A_2^2v+M\left(\left|A_1^{1/2}u\right|^2+\left|A_2^{1/2}v\right|^2\right)A_2v+\delta_2v'=0,\\
&(K_1u')(0)=K_1^{1/2}u_1, \qquad (K_2v')(0)=K_2^{1/2}v_1,
\end{aligned}$$

for $\delta_i>0$, $i=1,2$, where $M\in C^0[0,+\infty)$ with $M(\lambda)\geq 0\ \forall\,\lambda>0$ and K_i, $i=1,2$ are symmetrical linear operators in H with $(K_iw,w)>0$, $\forall\ w\in H$, A_i, $i=1,2$ are self-adjoint and positive linear operator, with domain $\mathscr{D}(A_i)$ dense in H, that is, there exist positive constants m_i, $i=1,2$, such that $(A_1v,v)\geq m_1|v|^2$, $\forall\, v\in\mathscr{D}(A_1)$, $(A_2w,w)\geq m_2|w|^2$, $\forall\, w\in\mathscr{D}(A_2)$.

The Kirchhoff's function $M(\cdot)$ considered in this paper is more general than in those ones in the previous results. The outline of the paper is as follows. In Section 2, we introduce some notations, and present some hypothesis and lemmas, which are needed in the proof of our results. In Section 3, we introduce the stability set, associated with the nonlinear source term, created from the Nehari Manifold. In Section 4, we prove the existence of solutions by using the

Faedo-Galerkin method. In Section 5, the last section, the energy is given based on the result of Nakao.

2. Preliminaires

In this section, we present some hyphotesis and lemmas needed in the proof our results. For simplicity of notations, hereafter, we denote $|\cdot|$ the Lebesgue Space $L^2(\Omega)$ norm, and $|\cdot|_p$ we denote the space $L^p(\Omega)$ norm.

(H.1): We suppose that

$$1 \leq p < \infty \text{ if } n \leq 2 \text{ and } 1 \leq p < \frac{n+2}{n-2} \text{ if } n > 2, \tag{2.1}$$

$$1 < q < \infty \text{ if } n \leq 2 \text{ and } 1 < q < \frac{n}{n-2} \text{ if } n > 2. \tag{2.2}$$

(H.2): $M \in C([0,\infty),\mathbb{R})$ with $M(\lambda) \geq -\beta$, $\forall\, \lambda \geq 0$, $0 < \beta < \lambda_1$, λ_1 the first eigenvalue of the problem $\Delta^2 u - \lambda(-\Delta u) = 0$.

Remark 2.1. If λ_1 is a first engevalue of $\Delta^2 u - \lambda(-\Delta u) = 0$, with campled boundary conditions

$$u\big|_{\partial\Omega} = \frac{\partial u}{\partial \eta}\bigg|_{\partial\Omega} = 0,$$

then (see Miklin [17])

$$\lambda_1 = \inf_{u \in H_0^2(\Omega)} \frac{|\Delta u|^2}{|\nabla u|^2} > 0 \text{ and } |\nabla u|^2 \leq \frac{1}{\lambda_1}|\Delta u|^2. \tag{2.3}$$

Lemma 2.2 (Sobolev-Poincaré Inequality, [18]). *Let p be a real number with $2 < p < \infty$ if $n = 1,2$ or $2 \leq p \leq \frac{2n}{n-2}$ when $n \geq 3$, then there exists a constant $C > 0$ such that*

$$|u|_p \leq C|\nabla u|, \; \forall\, u \in H_0^1(\Omega).$$

Lemma 2.3 (Nakao's Lemma, [19]). *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$ satisfying*

$$\operatorname*{supess}_{t \leq s \leq t+1} \phi^{1+\alpha}(s) \leq C_0[\phi(t) - \phi(t+1)]$$

for $t \geq 0$, where C_0 and α are positive constants. Then, we have, for each $t \geq 0$,

- *$\phi(t) \leq Ce^{-kt}$, if $\alpha = 0$.*
- *$\phi(t) \leq C(1+t)^{-\frac{1}{\alpha}}$, if $\alpha > 0$,*

where C and k are positive constants.

3. Potential well

In this section, we use the potential well theory, a powerful tool in the study of the global existence of solutions to partial differential equations first developed by Payne and Sattingen [20]. It is known that the energy of a PDE system, in some sense, splits into the kinetic and potential energy. By the idea of Ye [21], we are able to construct a set of stability corresponding to the source term created from the Nehari manifold. We will prove that there is a valley or a well of the depth d created in the potential energy. If d is strictly positive, then we find that, for solutions with the initial data in the good part of the potential well, the potential energy of the solution can never escape the potential well. In general, it is possible that the energy from the source term to cause the blow-up in a finite time. However, in the good part of the potential well, it remains bounded. As a result, the total energy of the solution remains finite on any time interval $[0,T)$ providing the global existence of the solution. We proceed by defining the functional $J:\left(H_0^2(\Omega)\right)^2\to\mathbb{R}$ by

$$J(u,v)\overset{\text{def}}{=}\frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)-\frac{1}{q+1}\left(|u|_{q+1}^{q+1}+|v|_{q+1}^{q+1}\right). \tag{3.1}$$

For $(u,v)\in\left(H_0^2(\Omega)\right)^2$, we have

$$J(\lambda u,\lambda v)=\frac{\lambda^2}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)-\frac{\lambda^{q+1}}{q+1}\left(|u|_{q+1}^{q+1}+|v|_{q+1}^{q+1}\right),\ \lambda\geq 0. \tag{3.2}$$

Associated with J, we have the well-know Nehari manifold

$$\mathscr{N}\overset{\text{def}}{=}\left\{(u,v)\in\left(H_0^2(\Omega)\right)^2\setminus\{(0,0)\};\left[\frac{d}{dt}J(\lambda u,\lambda v)\right]_{\lambda=1}=0\right\}. \tag{3.3}$$

Equivalently,

$$\mathscr{N}=\left\{(u,v)\in\left(H_0^2(\Omega)\right)^2\setminus\{(0,0)\};\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)=\left(|u|_{q+1}^{q+1}+|v|_{q+1}^{q+1}\right)\right\}. \tag{3.4}$$

We define as in the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [22],

$$0<d\overset{\text{def}}{=}\inf_{(u,v)\in(H_0^2(\Omega))^2\setminus\{(0,0)\}}\sup_{\lambda\geq 0}J(\lambda u,\lambda v). \tag{3.5}$$

Similar to the results in [23], one has

$$0<d=\inf_{(u,v)\in\mathscr{N}}J(u,v). \tag{3.6}$$

We now introduce

$$W=\left\{(u,v)\in\left(H_0^2(\Omega)\right)^2;J(u,v)<d\right\}\cup\{(0,0)\}, \tag{3.7}$$

and partition it into two sets as follows

$$W_1=\left\{(u,v)\in(H_0^2(\Omega))^2;\frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)>\frac{1}{q+1}\left(|u|_{q+1}^{q+1}+|v|_{q+1}^{q+1}\right)\right\}\cup\{(0,0)\}, \tag{3.8}$$

$$W_2=\left\{(u,v)\in(H_0^2(\Omega))^2;\frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)<\frac{1}{q+1}\left(|u|_{q+1}^{q+1}+|v|_{q+1}^{q+1}\right)\right\}. \tag{3.9}$$

So, we define by W_1 the set of stability for problem (1.4)–(1.8).

4. Existence of global solutions

In this section, we prove the existence of global weak solutions of system (1.4)–(1.8).

Theorem 4.1. *Let* $(u_0,v_0)\in[W_1]^2$, $J(u_0,v_0)<d$ *and* $(u_1,v_1)\in[L^2(\Omega)]^2$. *If the hypothesis (H.1) and (H.2) holds, then there exists functions* $u,v:[0,T)\to L^2(\Omega)$ *in the class*

$$(u,v)\in\left[L^\infty(0,T;H_0^2(\Omega))\cap L^\infty(0,T;L^{q+1}(\Omega))\right]^2, \tag{4.1}$$

$$(u_t,v_t)\in\left[L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;L^{p+1}(\Omega))\right]^2, \tag{4.2}$$

such that, for all $(w,z)\in H_0^2(\Omega)\times H_0^2(\Omega)$,

$$\frac{d}{dt}(u_t(t),w)+(\Delta u(t),\Delta w)+M\left(|\nabla u(t)|^2+|\nabla v(t)|^2\right)(-\Delta u(t),w) \\ +(|u_t(t)|^{p-1}u_t(t),w)-(|u(t)|^{q-1}u(t),w)=0 \ \text{ in } \mathscr{D}'(0,T), \tag{4.3}$$

$$\frac{d}{dt}(v_t(t),z)+(\Delta v(t),\Delta z)+M\left(|\nabla u(t)|^2+|\nabla v(t)|^2\right)(-\Delta v(t),z) \\ +(|v_t(t)|^{p-1}v_t(t),z)-(|v(t)|^{q-1}v(t),z)=0 \ \text{ in } \mathscr{D}'(0,T), \tag{4.4}$$

$$(u(0),v(0))=(u_0,v_0),\ (u_t(0),v_t(0))=(u_1,v_1). \tag{4.5}$$

4.1. **Approximate problem.** Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^2(\Omega)$ by the eigenvectors of the operator $-\Delta$, that is, $-\Delta w_\nu=\lambda_\nu w_\nu$, with $\lambda_\nu\to\infty$ and $w_\nu\big|_{\partial\Omega}=\dfrac{\partial w_\nu}{\partial\eta}\Big|_{\partial\Omega}=0$.

Let $V_m=\text{Span}[w_1,w_2,\dots,w_m]$. For all $w,z\in V_m$, let

$$u^m(t)=\sum_{j=1}^m g_{jm}(t)w_j,\ v^m(t)=\sum_{j=1}^m h_{jm}(t)w_j,$$

be a solution of the approximate system

$$(u_{tt}^m(t),w)+(\Delta u^m(t),\Delta w)+M\left(|\nabla u^m(t)|^2+|\nabla v^m(t)|^2\right)(-\Delta u^m(t),w) \\ +(|u_t^m(t)|^{p-1}u_t^m(t),w)-(|u^m(t)|^{q-1}u^m(t),w)=0, \tag{4.6}$$

$$(v_{tt}^m(t),z)+(\Delta v^m(t),\Delta z)+M\left(|\nabla u^m(t)|^2+|\nabla v^m(t)|^2\right)(-\Delta v^m(t),z) \\ +(|v_t^m(t)|^{p-1}v_t^m(t),z)-(|v^m(t)|^{q-1}v^m(t),z)=0, \tag{4.7}$$

$$(u^m(0),v^m(0))=(u_{0m},v_{0m})\to(u_0,v_0) \text{ strongly in } [H_0^2(\Omega)]^2, \tag{4.8}$$

$$(u_t^m(0),v_t^m(0))=(u_{1m},v_{1m})\to(u_1,v_1) \text{ strongly in } [L^2(\Omega)]^2. \tag{4.9}$$

The system (4.6)–(4.9) has a local solution in $[0,t_m)$, $0<t_m\le T$, by virtue of Carathéodory's Theorem, see [24]. The following priori estimates allow us to extend this solution to the interval $[0,T)$.

4.2. **Priori estimates.** Let $w=u_t^m(t)$ and $z=v_t^m(t)$ in (4.6) and (4.7), respectively. Integrating from 0 to t, $0\le t\le t_m$, summing up the results and taking into account that

$$\int_0^t\frac{1}{2}M\left(|\nabla u^m(s)|^2+|\nabla v^m(s)|^2\right)\frac{d}{ds}\left(|\nabla u^m(s)|^2+|\nabla v^m(s)|^2\right)=\frac{1}{2}\widehat{M}\left(|\nabla u^m(s)|^2+|\nabla v^m(s)|^2\right)\Big|_0^t$$

where we denote the primitive function of $M(s)$ by $\widehat{M}(s)$, we obtain

$$\frac{1}{2}\left[|u_t^m(t)|^2+|v_t^m(t)|^2+|\Delta u^m(t)|^2+|\Delta v^m(t)|^2+\widehat{M}\left(|\nabla u^m(t)|^2+|\nabla v^m(t)|^2\right)\right]- \frac{1}{q+1}\left(|u^m(t)|_{q+1}^{q+1}+|v^m(t)|_{q+1}^{q+1}\right)+\int_0^t\left(|u_t^m(s)|_{p+1}^{p+1}+|v_t^m(s)|_{p+1}^{p+1}\right)ds= \frac{1}{2}\left[|u_{1m}|^2+|v_{1m}|^2+|\Delta u_{0m}|^2+|\Delta v_{0m}|^2+\widehat{M}\left(|\nabla u_{0m}|^2+|\nabla v_{0m}|^2\right)\right]- \frac{1}{q+1}\left(|u_{0m}|_{q+1}^{q+1}+|v_{0m}|_{q+1}^{q+1}\right). \tag{4.10}$$

Let

$$E_m(t)=\frac{1}{2}\left[|u_t^m(t)|^2+||v_t^m(t)|^2+|\Delta u^m(t)|^2+|\Delta v^m(t)|^2+\widehat{M}\left(|\nabla u^m(t)|^2+|\nabla v^m(t)|^2\right)\right] -\frac{1}{q+1}\left(|u^m(t)|_{q+1}^{q+1}+|v^m(t)|_{q+1}^{q+1}\right). \tag{4.11}$$

From (4.10), we have

$$E_m(t)+\int_0^t\left(|u_t^m(t)|^2+|v_t^m(t)|^2\right)ds=E_m(0). \tag{4.12}$$

Now, by (H.2) and (2.3), it follows that

$$\widehat{M}\left(|\nabla u^m(t)|^2+|\nabla v^m(t)|^2\right)\geq-\frac{\beta}{\lambda_1}\left(|\Delta u^m(t)|^2+|\Delta v^m(t)|^2\right) \tag{4.13}$$

and

$$\widehat{M}\left(|\nabla u_{0m}|^2+|\nabla v_{0m}|^2\right)\leq\frac{m_0}{\lambda_1}\left(|\Delta u_{0m}|^2+|\Delta v_{0m}|^2\right), \tag{4.14}$$

where

$$m_0=\max_{0\leq s\leq|\nabla u_{0m}|^2+|\nabla v_{0m}|^2\leq C} M(s).$$

By use of (4.10) and (4.14), we get

$$\frac{1}{2}\left[|u_t^m(t)|^2+|v_t^m(t)|^2+\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u^m(t)|^2+|\Delta v^m(t)|^2\right)\right] -\frac{1}{q+1}\left(|u^m(t)|_{q+1}^{q+1}+|v^m(t)|_{q+1}^{q+1}\right)\leq E_m(t)\leq\frac{1}{2}\left(|u_{1m}|^2+|v_{1m}|^2\right)+C_1J(u_{0m},v_{0m}), \tag{4.15}$$

where $C_1=C_1(m_0,\lambda_1,\beta)>0$ is a constant independent of m and t. We have $J(u_{0m},v_{0m})<d$. From (4.9), we obtain

$$\frac{1}{2}\left[|u_t^m(t)|^2+|v_t^m(t)|^2+\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u^m(t)|^2+|\Delta v^m(t)|^2\right)\right] -\frac{1}{q+1}\left(|u^m(t)|_{q+1}^{q+1}+|v^m(t)|_{q+1}^{q+1}\right)\leq C_2, \tag{4.16}$$

where $C_2 > 0$ is a constant independent of m and t. We can extend the approximate solution $(u^m(t), v^m(t))$ to the interval $[0,T)$, $T > 0$. By use of (4.12) and (4.16), we have

$$(u^m, v^m) \text{ are bounded in } \left[L^\infty\left(0,T;H_0^2(\Omega)\right) \cap L^\infty\left(0,T;L^{q+1}(\Omega)\right)\right]^2, \tag{4.17}$$

$$(u_t^m, v_t^m) \text{ are bounded in } \left[L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;L^{p+1}(\Omega)\right)\right]^2. \tag{4.18}$$

Now by standard projection arguments as described in Lions [25], we have from the approximate equations and the estimates (4.17) and (4.18) that

$$(u_{tt}^m, v_{tt}^m) \text{ are bounded in } \left[L^2\left(0,T;H^{-2}(\Omega)\right)\right]^2. \tag{4.19}$$

4.3. **Passage to the limit.** From estimates (4.17) and (4.19), there exist subsequences of $(u^m)_{m\in\mathbb{N}}$ and $(v^m)_{m\in\mathbb{N}}$ such that, as $m \to \infty$,

$$(u^m, v^m) \overset{*}{\rightharpoonup} (u,v) \text{ weakly star in } \left[L^\infty(0,T;H_0^2(\Omega))\right]^2, \tag{4.20}$$

$$(u_t^m, v_t^m) \overset{*}{\rightharpoonup} (u_t,v_t) \text{ weakly star in } \left[L^\infty(0,T;L^2(\Omega))\right]^2, \tag{4.21}$$

$$(u_{tt}^m, v_{tt}^m) \overset{*}{\rightharpoonup} (u_{tt},v_{tt}) \text{ weakly star in } \left[L^2(0,T;L^2(\Omega))\right]^2. \tag{4.22}$$

Applying the Aubin-Lions Lemma ([25] Theorem 5.1), we get from (4.20) and (4.22) that

$$(u^m, v^m) \to (u,v) \text{ strongly in } \left[L^2(0,T;H_0^1(\Omega))\right]^2, \tag{4.23}$$

$$(u_t^m, v_t^m) \to (u_, v_t) \text{ strongly in } \left[L^2(0,T;L^2(\Omega))\right]^2. \tag{4.24}$$

In particular,

$$(u^m, v^m) \to (u,v) \text{ strongly in } \left[L^2(0,T;L^2(\Omega))\right]^2. \tag{4.25}$$

Then, by (4.24) and (4.25), we have

$$(u^m, v^m) \to (u,v) \text{ a.e. in } \Omega \times (0,T), \tag{4.26}$$

$$(u_t^m, v_t^m) \to (u_t,v_t) \text{ a.e. in } \Omega \times (0,T). \tag{4.27}$$

Since M is continuos, it follows that

$$M(|\nabla u^m|^2 + |\nabla v^m|^2) \to M(|\nabla u|^2 + |\nabla v|^2) \text{ strongly in } L^2(0,T). \tag{4.28}$$

Therefore, by use of (4.26) and (4.28), we get

$$M\left(|\nabla u^m|^2 + |\nabla v^m|^2\right)\left(-\Delta u^m, -\Delta v^m\right) \rightharpoonup M\left(|\nabla u|^2 + |\nabla v|^2\right)\left(-\Delta u, -\Delta v\right) \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{4.29}$$

Now, we observe that

$$\int_0^T \left(\left| |u^m(t)|^{q-1} u^m(t) \right|^{\frac{q+1}{q}} + \left| |v^m(t)|^{q-1} v^m(t) \right|^{\frac{q+1}{q}} \right) dt = \int_0^T \left(\left|u^m(t)\right|^{q+1} + \left|v^m(t)\right|^{q+1} \right) dt \le C_3$$

and

$$\int_0^T \left(\left| |u_t^m(t)|^{p-1} u_t^m(t) \right|^{\frac{p+1}{p}} + \left| |v_t^m(t)|^{p-1} v_t^m(t) \right|^{\frac{p+1}{p}} \right) dt = \int_0^T \left(\left|u_t^m(t)\right|^{p+1} + \left|v_t^m(t)\right|^{p+1} \right) dt \le C_4,$$

where C_3 and C_4 are positive constants independent of m and t. So,

$$\left(|u^m|^{q-1}u^m, |v^m|^{q-1}v^m\right) \text{ is bounded in } \left[L^{\frac{q+1}{q}}\left(0,T;L^{\frac{q+1}{q}}(\Omega)\right)\right]^2,$$
$$\left(|u_t^m|^{p-1}u_t^m, |v_t^m|^{p-1}v_t^m\right) \text{ is bounded in } \left[L^{\frac{p+1}{p}}\left(0,T;L^{\frac{p+1}{p}}(\Omega)\right)\right]^2.$$

Therefore

$$\left(|u^m|^{q-1}u^m, |v^m|^{q-1}v^m\right) \to (|u|^{q-1}u, |v|^{q-1}v) \text{ a.e. in } \Omega\times(0,T),$$
$$\left(|u_t^m|^{p-1}u_t^m, |v_t^m|^{p-1}v_t^m\right) \to (|u_t|^{p-1}u_t, |v_t|^{p-1}v_t) \text{ a.e. in } \Omega\times(0,T).$$

So, by use of Lions's Lemma [25], we have

$$\left(|u^m|^{q-1}u^m, |v^m|^{q-1}v^m\right) \rightharpoonup \left(|u|^{q-1}u, |v|^{q-1}v\right) \text{ weakly in } \left[L^{\frac{q+1}{q}}\left(0,T;L^{\frac{q+1}{q}}(\Omega)\right)\right]^2, \quad (4.30)$$

$$\left(|u_t^m|^{p-1}u_t^m, |v_t^m|^{p-1}v_t^m\right) \rightharpoonup \left(|u_t|^{p-1}u_t, |v_t|^{p-1}v_t\right) \text{ weakly in } \left[L^{\frac{p+1}{p}}\left(0,T;L^{\frac{p+1}{p}}(\Omega)\right)\right]^2. \quad (4.31)$$

From the convergence (4.20), (4.21), (4.29)–(4.31), we can pass to the limit approximate equations (4.6) and (4.7) and obtain, for all $w,z\in V_m$, in $\mathscr{D}'(0,T)$

$$\begin{aligned}\frac{d}{dt}(u_t(t),w) + (\Delta u(t),\Delta w) + M\left(|\nabla u(t)|^2+|\nabla v(t)|^2\right)(-\Delta u(t),w) +&\\ (|u_t(t)|^{p-1}u_t(t),w) - (|u(t)|^{q-1}u(t),w) = 0&\end{aligned} \quad (4.32)$$

and

$$\begin{aligned}\frac{d}{dt}(v_t(t),z) + (\Delta v(t),\Delta z) + M\left(|\nabla u(t)|^2+|\nabla v(t)|^2\right)(-\Delta v(t),z) +&\\ (|v_t(t)|^{p-1}v_t(t),z) - (|v(t)|^{q-1}v(t),z) = 0.&\end{aligned} \quad (4.33)$$

As V_m is dense in $H_0^2(\Omega)$, the equations (4.32) and (4.33) are valid for all $w,z\in H_0^2(\Omega)$. The verification of the initial data is obtained in a standard way. The proof of Theorem 4.1 is complete.

5. Asymptotic behaviour

In this section, we study the asymptotic behavior of solutions. By using the Nakao's method, we show that the energy associated to the system (1.4)–(1.8) is exponentially stable for $p=1$ and polynomially stable when $p>1$. The main result of this paper is given by the following theorem.

Theorem 5.1. *Under the hypothesis of Theorem 4.1, the energy associated to system (1.4)–(1.8) satisfies:*

$$\textit{If } p=1, \textit{ then } E(t)\le Ce^{-kt}, \textit{ for all } t\ge 0,$$
$$\textit{when } p>1, \textit{ then } E(t)\le C(1+t)^{-\frac{2}{p-1}}, \textit{ for all } t\ge 0,$$

where C and k are positive constants.

Proof. Let $w = u_t(t)$ and $z = v_t(t)$ in the equations (4.32) and (4.33), respectively. Summing up the results, we obtain

$$\frac{d}{dt}\Big(|u_t(t)|^2 + |v_t(t)|^2 + \frac{1}{2}|\Delta u(t)|^2 + \frac{1}{2}|\Delta v(t)|^2 + \frac{1}{2}\widehat{M}(|\nabla u(t)|^2 + |\nabla v(t)|^2) - \frac{1}{q+1}(|u(t)|_{q+1}^{q+1} + |v(t)|_{q+1}^{q+1})\Big) + |u_t(t)|^{p+1} + |v_t(t)|^{p+1} = 0.$$

That is,

$$\frac{d}{dt}E(t) + |u_t(t)|_{p+1}^{p+1} + |v_t(t)|_{p+1}^{p+1} \le 0, \tag{5.1}$$

where

$$E(t) = \frac{1}{2}[|u_t(t)|^2 + |v_t(t)|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 + \widehat{M}(|\nabla u(t)|^2 + |\nabla v(t)|^2) - \frac{1}{q+1}(|u(t)|_{q+1}^{q+1} + |v(t)|_{q+1}^{q+1})].$$

Integrating from t to $t+1$, $t \ge 0$, we have

$$\int_t^{t+1} \left(|u_t(s)|_{p+1}^{p+1} + |v_t(s)|_{p+1}^{p+1}\right) ds \le E(t) - E(t+1). \tag{5.2}$$

Let

$$F^{p+1}(t) = \int_t^{t+1} \left(|u_t(s)|_{p+1}^{p+1} + |v_t(s)|_{p+1}^{p+1}\right) ds. \tag{5.3}$$

Now,

$$\begin{aligned} F^2(t) &= \int_t^{t+1} \left(|u_t(s)|^2 + |v_t(s)|^2\right) ds \le C \int_t^{t+1} \left(|u_t(s)|_{p+1}^{p+1} + |v_t(s)|_{p+1}^{p+1}\right) ds \\ &\le C[E(t) - E(t+1)], \end{aligned} \tag{5.4}$$

where $C > 0$ is a constant. Thus, there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$, and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$|u_t(t_i)| + |v_t(t_i)| \le 4F(t), \; i = 1, 2. \tag{5.5}$$

Let $w = u(t)$ and $z = v(t)$ in the equations (4.32) and (4.33), respectively. Integrating from t_1 to t_2, summing up the results and using (H.2), we have

$$\begin{aligned} &\int_{t_1}^{t_2} \left[\left(1 - \frac{\beta}{\lambda_1}\right)\left(|\Delta u(s)|^2 + |\Delta v(s)|^2\right) - |u(s)|_{q+1}^{q+1} - |v(s)|_{q+1}^{q+1}\right] ds \le \\ &|u_t(t_1)||u(t_1)| + |u_t(t_2)||u(t_2)| + |v_t(t_1)||v(t_1)| + |v_t(t_2)||v(t_2)| + \\ &\int_{t_1}^{t_2}\int_\Omega \left(|u_t(s)|^p|u(s)| + |v_t(s)|^p|v(s)|\right) dxds. \end{aligned} \tag{5.6}$$

Now, our goal is to estimate the last term in the right-hand of inequality (5.6). By using Hölder inequality and applying the Sobolev-Poincaré inequality, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega}|u_t(s)|^p|u(s)|dxds &\leq \int_{t_1}^{t_2}|u_t(s)|_{p+1}^p|u(s)|_{p+1}ds \leq C_1\int_{t_1}^{t_2}|u_t(s)|_{p+1}^p|\nabla u(s)|ds\\ &\leq C_2\int_{t_1}^{t_2}|u_t(s)|_{p+1}^p|\Delta u(s)|ds \leq C_2 \operatorname*{supess}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s)\int_{t_1}^{t_2}|u_t(s)|_{p+1}^p ds\\ &\leq C_2F^p(t)\operatorname*{supess}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s). \qquad (5.7)\end{aligned}$$

Similarly,

$$\int_{t_1}^{t_2}\int_{\Omega}|v_t(s)|^p|v(s)|dxds \leq C_2F^p(t)\operatorname*{supess}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s). \qquad (5.8)$$

From (5.5) and Sobolev-Poincaré inequality, we have

$$|u_t(t_i)||u(t_i)|+|v_t(t_i)||v(t_i)| \leq C_3F(t)\operatorname*{supess}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s),\ i=1,2. \qquad (5.9)$$

By using estimates (5.7)–(5.9) in (5.6), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\left[\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u(s)|^2+|\Delta v(s)|^2\right)-|u(s)|_{q+1}^{q+1}-|v(s)|_{q+1}^{q+1}\right]ds \leq\\ [2C_3F(t)+2C_2F^p(t)]\operatorname*{supess}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s) \overset{\text{def}}{=} G^2(t). \qquad (5.10)\end{aligned}$$

From (5.4) and (5.10), it follows that

$$\int_{t_1}^{t_2} E(s)ds \leq C_4[F^2(t)+G^2(t)].$$

Then, there exists $t^*\in[t_1,t_2]$ such that $E(t^*)\leq C_5[F^2(t)+G^2(t)]$. From (5.1), we have

$$\begin{aligned}\operatorname*{supess}_{t\leq s\leq t+1} E(s) &\leq E(t^*)+\int_{t_1}^{t_2}\left(|u_t(t)|_{p+1}^{p+1}+|v_t(t)|_{p+1}^{p+1}\right)ds \leq C_6[F^2(t)+G^2(t)]\\ &\leq C_7[F^2(t)+F^{2p}(t)]+\frac{1}{2}\operatorname*{supess}_{t\leq s\leq t+1} E(s)\end{aligned}$$

Then,

$$\operatorname*{supess}_{t\leq s\leq t+1} E(s) \leq C_8F^2(t)[1+F^{2(p-1)}(t)]. \qquad (5.11)$$

We have $E(t) \leq E(0)$, for all $t \geq 0$. It follows that

$$F^{2(p-1)}(t) = \int_t^{t+1} \left(|u_t(s)|_{p+1}^{2(p-1)} + |v_t(s)|_{p+1}^{2(p-1)} \right) ds$$
$$\leq \left(\int_t^{t+1} \left(|u_t(s)|_{p+1}^{p+1} + |v_t(s)|_{p+1}^{p+1} \right) ds \right)^{\frac{p+1}{2(p-1)}}$$
$$= \left(E(0) - E(t+1)\right)^{\frac{p+1}{2(p-1)}} \leq E(0)^{\frac{p+1}{2(p-1)}} = C_0. \tag{5.12}$$

Therefore,

$$\operatorname*{supess}_{t \leq s \leq t+1} E(s) \leq C_9 F^2(t) = C_9 [F^{p+1}(t)]^{\frac{2}{p+1}}$$

and then

$$\operatorname*{supess}_{t \leq s \leq t+1} E^{\frac{p+1}{2}}(s) \leq \tilde{C}_0 [E(t) - E(t+1)], \tag{5.13}$$

where $\tilde{C}_0 > 0$ is constant depending of $E(0)$ and C_1, $i = 1, \ldots, 9$ are positive constants.
Case 1. $p = 1$. From (5.13), we obtain

$$\operatorname*{supess}_{t \leq s \leq t+1} E(s) \leq \tilde{C}_0 [E(t) - E(t+1)].$$

By use of Lemma 2.3, we get

$$E(t) \leq Ce^{-kt}, \text{ for all } t \geq 0,$$

where C and k are positive constants.
Case 2. $p > 1$. We have $\dfrac{p+1}{2} = 1 + \dfrac{p-1}{2}$ whence by (5.13)

$$\operatorname*{supess}_{t \leq s \leq t+1} E^{1+\dfrac{p-1}{2}}(s) \leq \tilde{C}_0 [E(t) - E(t+1)]. \tag{5.14}$$

Then, from (5.14) and Lemma 2.3, we have

$$E(t) \leq C(1+t)^{-\frac{2}{p-1}}, \text{ for all } t \geq 0,$$

where C is a positive constant. □

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REFERENCES

[1] J.M. Ball, Initial boundary value problem for an extensible beam, J. Math. Anal. Appl. 42 (1973), 61-90.
[2] M. Tucsnak, Semi-internal stabilization for a nonliner Euler-Bernoulli equation, Math. Method. Appl. Sci. 19 (1996), 897-907.
[3] M.M.Cavalcanti, V.N.Domingos Cavalcanti, J.A.Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, Commun. Contemp. Math. 6 (2004), 705-731.

[4] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, Lond. Edinb. Dubl. Phil. Mag. 41 (1921), 744-746.
[5] S.W. Hansen, R.D. Spies, Structural damping in laminated beams due to interfacial slip, J. Sound Vib. 204 (1997), 183-202.
[6] R. Dautray, J.L. Lions, Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques, Vol. 1, Masson, Paris, 1985.
[7] G. Liu, Well-posedness and exponential decay of solutions for a transmission problem with distributed delay, Electron. J. Differential Equations 2017 (2017), 174.
[8] C.M. Dafermos, On the existence and the asymptotic stability of solution to the equations of linear thermoelasticity, Arch. Rational Mech. Anal. 29 (1968), 241-271.
[9] C. A. Raposo, Adriano P. Cattai, Joilson O. Ribeiro. Global solution for a thermoelastic system with p - Laplacian, Appl. Math. Lett. 86 (2018), 119-125.
[10] J.Y. Park, J.J. Bae, On coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term, Appl. Math. Comput. 129 (2002), 87-105.
[11] J Ferreira, D.C. Pereira, M.L. Santos, Stability for a coupled system of wave equations of Kirchhoff type with nonlocal boundary conditions, Electron. J. Differential Equations 2003 (2003), 85.
[12] Y. Ye, Global existence and energy decay for a coupled system of kirchhoff type equations with damping and source terms, Acta Math. Appl. Sci. 32 (2016), 731-738.
[13] R. Narasimha, Non-linear vibration of an elastic string, J. Sound Vib. 8 (1968), 134-146.
[14] R.F.C. Lobato, D.C. Pereira, M.L. Santos, Exponential decay to the degenerate nonlinear coupled beams system with weak damping, ISRN Math. Phys. 2012 (2012), Article ID 659289
[15] E. Pişkin, F. Ekinci, Nonexistence of global solutions for coupled kirchhoff-type equations with degenerate damping terms, J. Nonlinear Funct. Anal. 2018 (2018), Article ID 48.
[16] D.C. Pereira, R.F.C. Lobato, C.A. Raposo, Energy decay to an abstract coupled system of extensible beams models, Appl. Math. Inf. Sci. 6 (2012) 447-452.
[17] S.G. Miklin, Variational methods in mathematical physics, Pergamon Press, Oxford, 1964.
[18] L.C. Evans, Partial Differential Equations, American Mathematical Society, 1998.
[19] M. Nakao, Decay of solutions for some nonlinear evolution equations, J. Math. Anal. Appl. 60 (1977), 542-549.
[20] L.E. Payne, D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations. Israel J. Math. 22 (1975), 273-303
[21] Y. Ye, Global existence and asymptotic behavior of solutions for a class of nonlinear degenerate wave equations, Differ. Equ. Nonlinear Mech. (2007), Article ID 019685
[22] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973), 349-381.
[23] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications 24, Birkhouser Boston Inc., Boston, MA, 1996.
[24] E.A. Coddington, N. Levinson, Theory of ordinary differential equations, McGraw-Hill Inc., New York, 1955.
[25] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod-Gauthier Villars, Paris, 1969.