J. Nonlinear Funct. Anal. 2020 (2020), Article ID 33 https://doi.org/10.23952/jnfa.2020.33

MATHRES Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

SOME FIXED POINT THEOREMS OF λ-CONTRACTIVE MAPPINGS IN MENGER *PSM*-SPACES

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Abstract. In this paper, on the basis of the probabilistic metric space and the S-metric space, we introduce a new concept of the probabilistic metric space, which is called the Menger probabilistic S-metric space, and investigate some topological properties of this space and their related examples. We also prove some fixed point theorems. Finally, an example is provided to support our new results.

Keywords. Menger *PSM*-space; Contraction mapping; Fixed point.

1. Introduction

In 1942, Menger [1] used distribution functions as the values of the metric instead of nonnegative real numbers, to develop metric space theorems and introduced the Menger probabilistic metric space (briefly, Menger *PM*-space). In 1960, Schweizer and Sklar [2, 3] introduced t-norm into the *PM*-space, and studied some properties of this space. In 1994, Chang, Cho and Kang [4] further summarized some important properties of the space. In 2006, Mustafa and Sims [5] introduced a new concept of metric spaces, which is called the G-metric space as a generalization of the metric space. After that, Segehi, Rao, and Shobe [6] investigated the concept of the D-metric space proposed by Dhage [7], meanwhile they also introduced the D^*-metric space. On the basis of the results, Sedghi and Aliouche [8] defined the notion of a generalized mertic space or a S-metric space as a generalization of the G-metric space and the D^*-metric space.

In 2014, Zhou *et al*. [9] defined a generalized metric space, which is now called the Menger probabilistic G-metric space (briefly, Menger *PGM*-space) as a generalized of the *PM*-space and the G-metric space, and they also obtained some fixed point theorems. After that, Zhu, Tu and Wu [10, 11] obtained some common fixed point theorems under the condition of the ϕ-contraction and weak compatible mappings. In 2015, Hasanvand and Khanehgir [12] introduced

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Received October 15, 2019; Accepted June 17, 2020.

the Menger probabilistic b-metric space (briefly, Menger PbM-space), and established some interesting fixed point theorems.

Inspired by the results mentioned above, on the basis of the probabilistic metric space and the S-metric space, we introduce a new metric space, which is called the Menger probabilistic S-metric space (briefly, Menger PSM-space), and investigate some properties of this space. We define a λ-contractive mapping, and prove some fixed point theorems of the mapping. An example is also provided to support our new results.

2. Preliminaries

In order to prove our main results, we need the following notions and lemmas.

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+$ be the set of all positive integers, and $\mathbb{N}$ be the set of all natural numbers, respectively.

Definition 2.1. ([4]) A mapping $F : R \to R^+$ is called a distribution function if it is nondecreasing and left-continuous with $\sup_{t\in R} F(t) = 1$ and $\inf_{t\in R} F(t) = 0$.

We next denote by $\mathfrak{D}$ the set of all distribution functions while H is used to denote the specific distribution function defined by

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Definition 2.2. ([2]) A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied, for all $a,b,c,d \in [0,1]$,

(Δ-1) $\Delta(a,1) = a$;

(Δ-2) $\Delta(a,b) = \Delta(b,a)$;

(Δ-3) $a \geq c, b \geq d \Rightarrow \Delta(a,b) \geq \Delta(c,d)$;

(Δ-4) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.

Three typical examples of t-norm are:

$$\Delta_1(a,b) = \min\{a+b-1, 0\}, \ \Delta_2(a,b) = ab, \ \Delta_m(a,b) = \min\{a,b\}, \forall a,b \in [0,1].$$

The above three t-norm examples satisfy that $\Delta_1 \leq \Delta_2 \leq \Delta_m$.

Definition 2.3. ([4]) A mapping $\Delta : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a quadrilateral norm (for short, a t-norm) if the following conditions are satisfied, for all $a,b,c,d,e,f \in [0,1]$,

(Δ-1) $\Delta(a,1,1) = a$;

(Δ-2) $\Delta(a,b,c) = \Delta(a,c,b) = \Delta(b,a,c) = \Delta(b,c,a) = \Delta(c,a,b) = \Delta(c,b,a)$;

(Δ-3) $a \geq d, b \geq e, c \geq f \Rightarrow \Delta(a,b,c) \geq \Delta(d,e,f)$;

(Δ-4) $\Delta(\Delta(a,b,c),d,e) = \Delta(a,\Delta(b,c,d),e) = \Delta(a,b,\Delta(c,d,e))$.

Three typical examples of t-norm are:

$$\Delta_1(a,b,c) = \min\{a+b+c-2, 0\}, \ \Delta_2(a,b,c) = abc, \ \Delta_m(a,b,c) = \min\{a,b,c\}$$

for all $a,b,c \in [0,1]$.

The above three t-norm examples satisfy that $\Delta_1 \leq \Delta_2 \leq \Delta_m$.

Definition 2.4. ([4]) A Menger *PM*-space is a triple (X,F,Δ), where X is a nonempty set, Δ is a continuous t-norm, and F is a mapping from $X\times X\to\mathscr{D}$ ($F_{x,y}$ denotes the value of at the pair (x,y)) satisfying the following conditions:

(PM-1) $\quad F_{x,y}(t)=1$ for all $x,y\in X$, and $t>0$, if and only if $x=y$;

(PM-2) $\quad F_{x,y}(t)=F_{y,x}(t)$ for all $x,y\in X$, and $t>0$;

(PM-3) $\quad F_{x,z}(s+t)\geq\Delta(F_{x,y}(s),F_{y,z}(t))$ for all $x,y,z\in X$, and $s,t>0$.

Definition 2.5. ([8]) Let X be a nonempty set and let $S:X\times X\times X\to\mathbb{R}^+$ be a function satisfying the following axioms:

($S1$) $\quad S(x,y,z)=0\Leftrightarrow x=y=z$;

($S2$) $\quad S(x,y,z)\leq S(x,x,a)+S(y,y,a)+S(z,z,a),\ \forall\, x,y,z,a\in X$.

Then the function S is called a generalized metric or a S-metric on X, and the pair (X,S) is called a S-metric space.

In this paper, we introduce the concept of the Menger probabilistic S-metric space as follows:

Definition 2.6. A Menger probabilistic S-metric space (for short, Menger *PSM*-space) is a triple (X,S^*,Δ), where X is a nonempty set, Δ is a continuous t-norm, and S^* is a mapping from $X\times X\times X\to\mathfrak{D}$ ($S^*_{x,y,z}$ denotes the value of S^* at the pair (x,y,z)) satisfying the following conditions:

(PSM-1) $\quad S^*_{x,y,z}(t)=H(t)$ for all $x,y,z\in X$ and $t>0$ if and only if $x=y=z$;

(PSM-2) $\quad S^*_{x,y,z}(r+s+t)\geq\Delta(S^*_{x,x,a}(r),S^*_{y,y,a}(s),S^*_{z,z,a}(t))$ for all $x,y,z,a\in X$ and $r,s,t\geq0$.

Example 2.7. Let $X=R$, $S(x,y,z)=|x-z|+|y-z|$. Then (X,S) is a S-metric space. Define S^* by

$$S^*_{x,y,z}=\frac{t}{t+S(x,y,z)}.$$

for all $x,y,z\in X$, and $t>0$, $\Delta=\Delta_m$. Then (X,S^*,Δ) is a Menger *PSM*-space.

Proof. It is obvious that S^* satisfies (PSM-1). Next, we prove that S^* satisfies $(PSM-2)$.

In fact, we need to show that, for all $x,y,z,a\in X$ and $r,s,t>0$,

$$\begin{aligned}S^*_{x,y,z}(r+s+t)&=\frac{r+s+t}{r+s+t+S(x,y,z)}\\&\geq\min\left\{\frac{r}{r+S(x,x,a)},\frac{s}{s+S(y,y,a)},\frac{t}{t+S(z,z,a)}\right\}\\&=\Delta\left(\frac{r}{r+S(x,x,a)},\frac{s}{s+S(y,y,a)},\frac{t}{t+S(z,z,a)}\right).\end{aligned}$$

Since $S(x,y,z)\leq S(x,x,a)+S(y,y,a)+S(z,z,a)$, we have

$$r+s+t+S(x,y,z)\leq r+s+t+S(x,x,a)+S(y,y,a)+S(z,z,a).$$

Then,

$$\frac{r+s+t}{r+s+t+S(x,y,z)}\geq\frac{r+s+t}{r+s+t+S(x,x,a)+S(y,y,a)+S(z,z,a)}.$$

Furthermore,

$$\begin{aligned}&\max\left\{\frac{r}{r+S(x,x,a,)},\frac{s}{s+S(y,y,a,)},\frac{t}{t+S(z,z,a,)}\right\}\\&\geq\frac{r+s+t}{r+s+t+S(x,x,a)+S(y,y,a)+S(z,z,a)}\\&\geq\min\left\{\frac{r}{r+S(x,x,a,)},\frac{s}{s+S(y,y,a,)},\frac{t}{t+S(z,z,a,)}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}S^*_{x,y,z}(r+s+t)&=\frac{r+s+t}{r+s+t+S(x,y,z)}\\&\geq\min\left\{\frac{r}{r+S(x,x,a,)},\frac{s}{s+S(y,y,a,)},\frac{t}{t+S(z,z,a,)}\right\}\\&=\Delta\left(S^*_{x,x,a}(r),S^*_{y,y,a}(s),S^*_{z,z,a}(t)\right).\end{aligned}$$

So S^* satisfies (PSM-2), i.e., (X,S^*,Δ) is a Menger PSM-space. □

Lemma 2.8. *Let (X,S^*,Δ) be a Menger PSM-space. Then $S^*_{x,x,y}(t)=S^*_{y,y,x}(t)$, for all $t>0$.*

Proof. For any $t>0$ and $0<\delta<t$, we have

$$\begin{aligned}S^*_{x,x,y}(t)&\geq\Delta\left(S^*_{x,x,x}\left(\frac{\delta}{2}\right),S^*_{x,x,x}\left(\frac{\delta}{2}\right),S^*_{y,y,x}(t-\delta)\right)\\&=\Delta\left(1,1,S^*_{y,y,x}(t-\delta)\right)=S^*_{y,y,x}(t-\delta)).\end{aligned}$$

Since S^* is left-continuous, letting $n\to\infty$, we have $S^*_{x,x,y}(t)\geq S^*_{y,y,x}(t)$.

In the same way, we can get $S^*_{x,x,y}(t)\leq S^*_{y,y,x}(t)$. Therefore, $S^*_{x,x,y}(t)=S^*_{y,y,x}(t)$, for all $t>0$. □

Definition 2.9. Let (X,S^*,Δ) be a Menger PSM-space and let x_0 be any point in X. For any $\varepsilon>0$ and δ with $0<\delta<1$, and (ε,δ)-neighborhood of x_0 is the set of all points y in X for which $S^*_{x_0,x_0,y}(\varepsilon)>1-\delta$. We write

$$N_{x_0}(\varepsilon,\delta)=\left\{y\in X:S^*_{x_0,x_0,y}(\varepsilon)>1-\delta\right\}.$$

Lemma 2.10. *If $0<\varepsilon_1\leq\varepsilon_2$ and $0<\delta_1\leq\delta_2<1$, then*

$$N_{x_0}(\varepsilon_1,\delta_1)\subset N_{x_0}(\varepsilon_2,\delta_2).$$

Proof. Suppose that $z\in N_{x_0}(\varepsilon_1,\delta_1)$. For any $\varepsilon_1>0$ and $0<\delta_1<1$, we get $S^*_{x_0,x_0,z}(\varepsilon_1)>1-\delta_1$. Since S^* is monotone, we have

$$S^*_{x_0,x_0,z}(\varepsilon_2)\geq S^*_{x_0,x_0,z}(\varepsilon_1)>1-\delta_1\geq1-\delta_2.$$

Therefore, from Definition 2.9, one has $z\in N_{x_0}(\varepsilon_2,\delta_2)$. This completes the proof. □

Theorem 2.11. *Let (X,S^*,Δ) be a Menger PSM-space. Then (X,S^*,Δ) is a Hausdorff space in the topology induced by the family of $\{N_{x_0}(\varepsilon,\delta)\}$ of (ε,δ)-neighborhoods.*

Proof. We show that the following properties are satisfied.

(A) For any $x_0 \in X$, there exists at least one neighborhood N_{x_0} of x_0 and every neighbor hood of x_0 contains x_0.

(B) If $N^1_{x_0}$ and $N^2_{x_0}$ are neighborhood of x_0, then there exists a neighborhood of x_0, $N^3_{x_0}$, which satisfies $N^3_{x_0} \subset N^1_{x_0} \cap N^2_{x_0}$.

(C) If N_{x_0} is a neighborhood of x_0 and $y_0 \in N_{x_0}$, then there exists a neighborhood of y_0, N_{y_0}, which satisfies $N_{y_0} \subset N_{x_0}$.

(D) If $x_0 \neq y_0$, then there exist disjoint neighborhood N_{x_0} and N_{y_0}, which satisfy $x_0 \in N_{x_0}$ and $y_0 \in N_{y_0}$.

Now we prove that $(A)-(D)$ hold.

(A) For any $\varepsilon > 0$ and $0 < \delta < 1$, $x_0 \in X$, since $S^*_{x_0,x_0,x_0}(\varepsilon) = 1 > 1-\delta$ for any $\varepsilon > 0$, we have $x_0 \in N_{x_0}(\varepsilon,\delta)$.

(B) For any $\varepsilon_1, \varepsilon_2 > 0$ and $0 < \delta_1, \delta_2 < 1$, let

$$N^1_{x_0}(\varepsilon_1,\delta_1) = \left\{ y \in X : S^*_{x_0,x_0,y}(\varepsilon_1) > 1-\delta_1 \right\}.$$

and

$$N^2_{x_0}(\varepsilon_2,\delta_2) = \left\{ y \in X : S^*_{x_0,x_0,y}(\varepsilon_2) > 1-\delta_2 \right\}.$$

be the neighborhood of x_0. Consider

$$N^3_{x_0} = \left\{ y \in X : S^*_{x_0,x_0,y}(\min\{\varepsilon_1,\varepsilon_2\}) > 1-\min\{\delta_1,\delta_2\} \right\}.$$

Clearly, $x_0 \in N^3_{x_0}$ and, since $\min\{\varepsilon_1,\varepsilon_2\} \le \varepsilon_1$ and $\min\{\delta_1,\delta_2\} \le \delta_1$, we have $N^3_{x_0} \subset N^1_{x_0}(\varepsilon_1,\delta_1)$. In the same way, we can get $N^3_{x_0} \subset N^2_{x_0}(\varepsilon_2,\delta_2)$. It follows that

$$N^3_{x_0} \subset N^1_{x_0} \cap N^2_{x_0}.$$

(C) Let $N_{x_0} = \left\{ z \in X, S^*_{x_0,x_0,z}(\varepsilon_1) > 1-\delta_1 \right\}$ be the neighborhood of x_0. Since $y_0 \in N_{x_0}$, then $S^*_{x_0,x_0,y_0}(\varepsilon_1) > 1-\delta_1$.

Now, $S^*_{x_0,x_0,y_0}(t)$ is left-continuous at ε_1, so there exist $\varepsilon_0 < \varepsilon_1$ and $\delta_0 < \delta_1$ such that

$$S^*_{x_0,x_0,y_0}(\varepsilon_0) > 1-\delta_0 > 1-\delta_1.$$

Let $N_{y_0} = \left\{ z \in X, S^*_{y_0,y_0,z}(\varepsilon_2) > 1-\delta_2 \right\}$, where $0 < \varepsilon_2 < \frac{\varepsilon_1-\varepsilon_0}{2}$ and δ_2 is chosen such that

$$\Delta(1-\delta_0, 1-\delta_2, 1-\delta_2) > 1-\delta_1.$$

Such a δ_2 exists since Δ is continuous, and $\Delta(1-\delta_0,1,1) = 1-\delta_0 > 1-\delta_1$.

Now, we suppose that $u \in N_{y_0}$, so that $S^*_{y_0,y_0,u}(\varepsilon_2) > 1-\delta_2$. Then, since S^* is monotone, it follows from $(PSM\text{-}2)$ that

$$\begin{aligned} S^*_{x_0,x_0,u}(\varepsilon_1) &= S^*_{u,u,x_0}(\varepsilon_1) \\ &\ge \Delta\left(S^*_{u,u,y}\left(\frac{\varepsilon_1-\varepsilon_0}{2}\right), S^*_{u,u,y}\left(\frac{\varepsilon_1-\varepsilon_0}{2}\right), S^*_{x_0,x_0,y}(\varepsilon_0) \right) \\ &\ge \Delta(1-\delta_2, 1-\delta_2, 1-\delta_0) = \Delta(1-\delta_0, 1-\delta_2, 1-\delta_2) > 1-\delta_1. \end{aligned}$$

This shows $u \in N_{x_0}$. Hence, $N_{y_0} \subset N_{x_0}$.

(D) Let $y_0 \neq x_0$, Then there exist $\varepsilon > 0$ and $0 \le a < 1$ such that $S^*_{x_0,x_0,y}(\varepsilon) = a$.

Since Δ is continuous and $\Delta(1,1,1)=1>a$, then there exist $0\le b_1,b_2<1$ such that $\Delta(b_1,b_1,b_2)>a$. Let

$$N_{x_0}=\left\{z\in X: S^*_{x_0,x_0,z}\left(\frac{\varepsilon}{3}\right)>b_1\right\}.$$

and

$$N_{y_0}=\left\{z\in X: S^*_{y_0,y_0,z}\left(\frac{\varepsilon}{3}\right)>b_2\right\}.$$

Now, we suppose that there exists a point $v\in N_{x_0}\cap N_{y_0}$ such that

$$S^*_{x_0,x_0,v}\left(\frac{\varepsilon}{3}\right)>b_1,\ S^*_{y_0,y_0,v}\left(\frac{\varepsilon}{3})\right)>b_2.$$

Then, by (PSM-2), we have

$$a=S^*_{x_0,x_0,v}(\varepsilon)\ge\Delta\left(S^*_{x_0,x_0,v}\left(\frac{\varepsilon}{3}\right),S^*_{x_0,x_0,v}\left(\frac{\varepsilon}{3}\right),S^*_{y_0,y_0,v}\left(\frac{\varepsilon}{3}\right)\right)\ge\Delta(b_1,b_1,b_2)>a.$$

which is a contradiction. Therefore, $N_{x_0}\bigcap N_{y_0}=\emptyset$. This completes the proof. □

Next, we give the definition of convergence of sequences in PSM-spaces.

Definition 2.12. Let (X,S^*,Δ) be a Menger PSM-space, and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point $x\in X$ $(x_n\to x)$, if, for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{(\varepsilon,\delta)}$ such that $x_n\in N_x(\varepsilon,\delta)$ whenever $n>M_{(\varepsilon,\delta)}$.

(2) $\{x_n\}$ is called a Cauchy sequence if, for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{(\varepsilon,\delta)}$ such that $S^*_{x_n,x_m,x_l}(\varepsilon)>1-\delta$ whenever $n,m,l>M_{(\varepsilon,\delta)}$.

(3) (X,S^*,Δ) is said to be complete if every Cauchy sequence in X converges to a point in X.

Lemma 2.13. *Let (X,S^*,Δ) be a Menger PSM-space, and $\{x_n\}$ be a sequence in X. Then the followings assertions are equivalent:*

(1) *$\{x_n\}$ is convergent to $x\in X$;*

(2) *$S^*_{x_n,x_n,x}(t)\to 1$ as $n\to\infty$, for all $t>0$;*

(3) *$S^*_{x,x,x_n}(t)\to 1$ as $n\to\infty$, for all $t>0$.*

Proof. $(1)\Rightarrow(2)$. Since $x_n\to x(n\to\infty)$, then, for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{(\varepsilon,\delta)}$ such that $x_n\in N_x(\varepsilon,\delta)$ whenever $n>M_{(\varepsilon,\delta)}$. It follows that $S^*_{x_n,x_n,x}(\varepsilon)>1-\delta$, i.e., $S^*_{x_n,x_n,x}(\varepsilon)-1>-\delta$.

Furthermore, $S^*_{x_n,x_n,x}(\varepsilon)<1+\delta$. Hence, $|S^*_{x_n,x_n,x}(\varepsilon)-1|<\delta$ whenever $n>M_{(\varepsilon,\delta)}$. Therefore, $S^*_{x_n,x_n,x}(t)\to 1$ as $n\to\infty$, for all $t>0$.

$(2)\Rightarrow(1)$. Since $S^*_{x_n,x_n,x}(t)\to 1$ as $n\to\infty$, for all $t>0$, then, for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{(\varepsilon,\delta)}$ such that $|S^*_{x_n,x_n,x}(\varepsilon)-1|<\delta$. It follows that $S^*_{x_n,x_n,x}(\varepsilon)>1-\delta$, i.e., $x_n\in N_x(\varepsilon,\delta)$ for all $n>M_{(\varepsilon,\delta)}$. From Definition 2.12, we have that $\{x_n\}$ is convergent to $x\in X$.

$(2)\Leftrightarrow(3)$. By using Lemma 2.8, we have $S^*_{x_n,x_n,x}(t)=S^*_{x,x,x_n}(t)$. Hence, $(2)\Leftrightarrow(3)$. □

Lemma 2.14. *Let (X,S^*,Δ) be a Menger PSM-space, and $\{x_n\}$ be a sequence in X. Then the followings are equivalent:*

(1) *the sequence $\{x_n\}$ is a Cauchy sequence;*

(2) *for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{(\varepsilon,\delta)}$ such that $S^*_{x_n,x_n,x_m}(\varepsilon)>1-\delta$ whenever $n,m>M_{(\varepsilon,\delta)}$.*

Proof. $(1) \Rightarrow (2)$. It is obvious from the Definition 2.12.

$(2) \Rightarrow (1)$. Since Δ is continuous and $\Delta(1,1,1) = 1$, then for any $\varepsilon > 0$ and $0 < \delta < 1$, there exists $\delta_0 \in [0,1]$ such that

$$\Delta(1, 1-\delta_0, 1-\delta_0) > 1-\delta.$$

By (2), there exists a positive integer $M_{(\varepsilon,\delta)}$ such that when $n,m,l > M_{(\varepsilon,\delta)}$

$$S^*_{x_m,x_m,x_n}\left(\frac{\varepsilon}{3}\right) = S^*_{x_n,x_n,x_m}\left(\frac{\varepsilon}{3}\right) > 1-\delta_0,\ S^*_{x_l,x_l,x_n}\left(\frac{\varepsilon}{3}\right) = S^*_{x_n,x_n,x_l}\left(\frac{\varepsilon}{3}\right) > 1-\delta_0.$$

It follows that

$$S^*_{x_n,x_m,x_l}(\varepsilon) \geq \Delta\left(S^*_{x_n,x_n,x_n}\left(\frac{\varepsilon}{3}\right), S^*_{x_m,x_m,x_n}\left(\frac{\varepsilon}{3}\right), S^*_{x_l,x_l,x_n}\left(\frac{\varepsilon}{3}\right)\right) > \Delta(1, 1-\delta_0, 1-\delta_0) > 1-\delta.$$

Hence, it follows from Definition 2.12 that $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.15. *Let (X, S^*, Δ) be a Menger PSM-space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, and let x,y be two vectors in X. If $x_n \to x$, $y_n \to y$ as $n \to \infty$, then, for all $t > 0$, $S^*_{x_n,x_n,y_n}(t) \to S^*_{x,x,y}(t)$, as $n \to \infty$.*

Proof. For any $t > 0$, there exists $\delta > 0$ such that $t > 2\delta$. It follow from (PSM-2) that

$$\begin{aligned}
&S^*_{x_n,x_n,y_n}(t) \\
&\geq S^*_{x_n,x_n,y_n}(t-\delta) \\
&\geq \Delta\left(S^*_{x_n,x_n,x}\left(\frac{\delta}{4}\right), S^*_{x_n,x_n,x}\left(\frac{\delta}{4}\right), S^*_{y_n,y_n,x}\left(t-\frac{3\delta}{2}\right)\right) \\
&\geq \Delta\left(S^*_{x_n,x_n,x}\left(\frac{\delta}{4}\right), S^*_{x_n,x_n,x}\left(\frac{\delta}{4}\right), \Delta\left(S^*_{y_n,y_n,y}\left(\frac{\delta}{4}\right), S^*_{y_n,y_n,y}\left(\frac{\delta}{4}\right), S^*_{x,x,y}(t-2\delta)\right)\right) \\
&= \Delta\left(S^*_{x,x,x_n}\left(\frac{\delta}{4}\right), S^*_{x,x,x_n}\left(\frac{\delta}{4}\right), \Delta\left(S^*_{y,y,y_n}\left(\frac{\delta}{4}\right), S^*_{y,y,y_n}\left(\frac{\delta}{4}\right), S^*_{y,y,x}(t-2\delta)\right)\right).
\end{aligned}$$

and

$$\begin{aligned}
&S^*_{x,x,y}(t) \\
&\geq S^*_{x,x,y}(t-\delta) \\
&\geq \Delta\left(S^*_{x,x,x_n}\left(\frac{\delta}{4}\right), S^*_{x,x,x_n}\left(\frac{\delta}{4}\right), S^*_{y,y,x_n}\left(t-\frac{3\delta}{2}\right)\right) \\
&\geq \Delta\left(S^*_{x,x,x_n}\left(\frac{\delta}{4}\right), S^*_{x,x,x_n}\left(\frac{\delta}{4}\right), \Delta\left(S^*_{y,y,y_n}\left(\frac{\delta}{4}\right), S^*_{y,y,y_n}\left(\frac{\delta}{4}\right), S^*_{x_n,x_n,y_n}(t-2\delta)\right)\right).
\end{aligned}$$

Letting $n \to \infty$ in the above two inequalities and noting that Δ is continuous, we have

$$\lim_{n\to\infty} S^*_{x_n,x_n,y_n}(t) \geq S^*_{x,x,y}(t-2\delta).$$

and

$$S^*_{x,x,y}(t) \geq \lim_{n\to\infty} S^*_{x_n,x_n,y_n}(t-2\delta).$$

Letting $\delta \to 0$ in the above two inequalities, since S^* is left- continuous, we conclude that

$$\lim_{n\to\infty} S^*_{x_n,x_n,y_n}(t) = S^*_{x,x,y}(t),$$

for any $t > 0$. This completes the proof. □

3. Main results

In this section, we will establish some new fixed point theorems in Menger *PSM*-spaces.

Definition 3.1. Let (X, S^*, Δ) be a Menger *PSM*-space. A mapping $f : X \to X$ is said to be contractive if there exists a constant $\lambda \in (0,1)$ such that

$$S^*_{fx,fy,fz}(t) \geq S^*_{x,y,z}(t/\lambda), \tag{3.1}$$

for all $x, y, z \in X$ and $t > 0$.

The mapping f satisfying condition (3.1) is called a λ-contraction.

Let Δ be a given t-norm. Then (by associativity) a family of mappings $\Delta^n : [0,1] \to [0,1]$ for each $n \geq 1$ is defined as follows:

$$\Delta^1(t) = \Delta(t,t,t),\ \Delta^2(t) = \Delta\left(t,t,\Delta^1(t)\right),\ \cdots \Delta^n(t) = \Delta\left(t,t,\Delta^{n-1}(t)\right),\ \cdots$$

for any $t \in [0,1]$.

Definition 3.2. ([13]) A t-norm Δ is said to be Hadzić-type if the family of functions $\{\Delta^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t = 1$, that is, for any $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that

$$t > 1 - \delta \Rightarrow \Delta^n(t) > 1 - \varepsilon$$

for each $n \geq 1$.

A typical examples of the t-norm of Hadzić-type is $\Delta_m = \min\{a,b,c\}$, where $a, b, c \in [0,1]$.

Lemma 3.3. *Let (X, S^*, Δ) be a Menger PSM-space with Δ of Hadzić-type and $\{x_n\}$ be a sequence in X. Suppose that there exists $\lambda \in (0,1)$ satisfying*

$$S^*_{x_n,x_n,x_{n+1}}(t) \geq S^*_{x_{n-1},x_{n-1},x_n}(t/\lambda),$$

for any $n \geq 1$ and $t > 0$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Since $S^*_{x_n,x_n,x_{n+1}}(t) \geq S^*_{x_{n-1},x_{n-1},x_n}(t/\lambda)$, by induction, we have

$$S^*_{x_n,x_n,x_{n+1}}(t) \geq S^*_{x_0,x_0,x_1}(t/\lambda^n).$$

Since X is a Menger *PSM*-space, we have $S^*_{x_0,x_0,x_1}(t/\lambda^n) \to 1$, as $n \to \infty$. It follows that

$$\lim_{n\to\infty} S^*_{x_n,x_n,x_{n+1}}(t) = 1, \tag{3.2}$$

for any $t > 0$.

Now, let $n \geq 1$, and $t > 0$. We show, by induction, that, for any $k \geq 0$,

$$S^*_{x_n,x_n,x_{n+k}}(t) \geq \Delta^k\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right)\right). \tag{3.3}$$

For $k = 0$, since $\Delta(a,b,c)$ is a real number, $\Delta^0(a,b,c) = 1$, for all $a, b, c \in [0,1]$. Hence,

$$S^*_{x_n,x_n,x_n}(t) = 1 \geq \Delta^0\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right)\right).$$

Assume that (3.3) holds for some $k \geq 1$. Since Δ is monotone, it follows from (PSM-2) that

$$\begin{aligned}
&S^*_{x_n,x_n,x_{n+k+1}}(t)\\
&= S^*_{x_n,x_n,x_{n+k+1}}\left(\frac{t-\lambda t}{2}+\frac{t-\lambda t}{2}+\lambda t\right)\\
&\geq \Delta\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_{n+k+1},x_{n+k+1},x_{n+1}}(\lambda t)\right)\\
&= \Delta\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_{n+1},x_{n+1},x_{n+k+1}}(\lambda t)\right)\\
&\geq \Delta\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_n,x_n,x_{n+k}}(t)\right)\\
&\geq \Delta\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right), \Delta^k\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right)\right)\right)\\
&= \Delta^{k+1}\left(S^*_{x_n,x_n,x_{n+1}}\left(\frac{t-\lambda t}{2}\right)\right).
\end{aligned}$$

So we have the conclusion.

Now we show that $\{x_n\}$ is a Cauchy sequence in X, *i.e.*, $\lim_{n,m\to\infty} S^*_{x_n,x_n,x_m}(t) = 1$ for any $t > 0$. Let $t > 0$, and $\varepsilon > 0$ be given. By hypothesis, $\Delta^n : n \geq 1$ is equicontinuous at 1, and $\Delta^n(1) = 1$. So, there exists $\delta > 0$ such that

$$\Delta^n(a) > 1-\varepsilon, \forall\, a \in (1-\delta, 1]. \tag{3.4}$$

for all $n \geq 1$. It follows from (3.2) that $\lim_{n\to\infty} S^*_{x_n,x_n,x_{n+1}}(t-\lambda t) = 1$. Hence, there exists $n_0 \in N$ such that $S^*_{x_n,x_n,x_{n+1}}(t-\lambda t) \in (1-\delta, 1]$ for any $n \geq n_0$. Therefore, by use of (3.3) and (3.4), we conclude that $S^*_{x_n,x_n,x_{n+k}}(t) > 1-\varepsilon$ for any $k \geq 0$. This shows $\lim_{n,m\to\infty} S^*_{x_n,x_n,x_m}(t) = 1$ for any $t > 0$. From Lemma 2.14, we conclude that $\{x_n\}$ is a Cauchy sequence in X. This completes the proof. □

Theorem 3.4. *Let (X, S^*, Δ) be a complete Menger PSM-space with Δ of Hadzić-type. Let $\lambda \in (0,1)$ and $f : X \to X$ be a λ-contraction. Then, for any $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to a unique fixed point of f.*

Proof. Take an arbitrary point x_0 in X. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n x_0$ for all $n \geq 0$. By (3.1), for any $t > 0$, we have

$$S^*_{x_n,x_n,x_{n+1}}(t) = S^*_{fx_{n-1},fx_{n-1},fx_n}(t) \geq S^*_{x_{n-1},x_{n-1},x_n}(t/\lambda).$$

By Lemma 3.3, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. By (3.1), we have

$$S^*_{fx,fx,fx_n}(t) \geq S^*_{x,x,x_n}(t/\lambda),$$

for all $t > 0$. Let $n \to \infty$. Since $x_n \to x, fx_n \to x (n \to \infty)$, we have

$$S^*_{fx,fx,x}(t) = 1,$$

for all $t > 0$. Hence $fx = x$.

Next, we show the uniqueness. Suppose that u is another fixed point of f. It follows from (3.1) that

$$S^*_{x,x,u}(t) = S^*_{fx,fx,fu}(t) \geq S^*_{x,x,u}(t/\lambda) \geq \cdots \geq S^*_{x,x,u}(t/\lambda^n).$$

Let $n \to \infty$. Since X is a Menger PSM-space, $S^*_{x,x,u}(t/\lambda^n) \to 1$, as $n \to \infty$, then

$$S^*_{x,x,u}(t) = 1$$

for any $t > 0$, which implies that $x = u$. Therefore, f has a unique fixed point in X. This completes the proof. □

Theorem 3.5. *Let (X, S^*, Δ) be a complete Menger PSM-space with Δ of Hadzić-type. Let $f : X \to X$ be a mapping satisfying*

$$S^*_{fx,fy,fz}(\lambda t) \geq \frac{1}{3}\left(S^*_{x,x,fx}(t) + S^*_{y,y,fy}(t) + S^*_{z,z,fz}(t)\right). \tag{3.5}$$

for all $x, y, z \in X$, where $\lambda \in (0,1)$. Then, for any $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to a unique fixed point of f.

Proof. Take an arbitrary point x_0 in X. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n x_0$ for all $n \geq 0$. By (3.5), for any $t > 0$, we have

$$\begin{aligned} S^*_{x_n,x_n,x_{n+1}}(\lambda t) &= S^*_{fx_{n-1},fx_{n-1},fx_n}(\lambda t) \\ &\geq \frac{1}{3}\left(2S^*_{x_{n-1},x_{n-1},fx_{n-1}}(t) + S^*_{x_n,x_n,fx_n}(t)\right) \\ &= \frac{1}{3}\left(2S^*_{x_{n-1},x_{n-1},x_n}(t) + S^*_{x_n,x_n,x_{n+1}}(\lambda t)\right), \end{aligned}$$

which implies that

$$S^*_{x_n,x_n,x_{n+1}}(t) \geq S^*_{x_{n-1},x_{n-1},x_n}(t/\lambda).$$

By Lemma 3.3, we have that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. By (3.5), we have

$$S^*_{fx,fx,fx_n}(t) \geq \frac{1}{3}\left(2S^*_{x,x,fx}(t/\lambda) + S^*_{x_n,x_n,fx_n}(t/\lambda)\right).$$

Let $n \to \infty$. Since $x_n \to x, fx_n \to x(n \to \infty)$, we have, for any $t > 0$,

$$\begin{aligned} S^*_{x,x,fx}(t) &= S^*_{fx,fx,x}(t) \\ &= \lim_{n\to\infty} S^*_{fx,fx,fx_n}(t) \\ &\geq \frac{1}{3}\left(2S^*_{x,x,fx}(t/\lambda) + S^*_{x,x,x}(t/\lambda)\right) \\ &\geq \frac{1}{3}\left(2S^*_{x,x,fx}(t) + S^*_{x,x,x}(t/\lambda)\right). \end{aligned}$$

This implies that $S^*_{x,x,fx}(t) \geq S^*_{x,x,x}(t/\lambda) = 1$. So, $S^*_{x,x,fx}(t) = 1$. It follows that $fx = x$.

Next, we show the uniqueness. Suppose that u is another fixed point of f. By use of (**??**, for any $t > 0$, we have

$$S^*_{x,x,u}(t) = S^*_{fx,fx,fu}(t) \geq \frac{1}{3}\left(2S^*_{x,x,fx}(t/\lambda) + S^*_{u,u,fu}(t/\lambda)\right) = 1.$$

This shows that $x = u$. Therefore, f has a unique fixed point in X. This complete the proof. □

Now we give an example to support our new result.

Example 3.6. Let $X=[0,+\infty)$, and define a function $S^*: X^3\times[0,+\infty)\to[0,1]$ by

$$S^*_{x,y,z}(t)=\frac{t}{t+S(x,y,z)}.$$

for all $x,y,z\in X$, and $t>0$, where $S(x,y,z)=|x-z|+|y-z|$. From Example 2.7, we know that (X,S^*,Δ_m) is a Menger PSM-space.

(1) Let $\lambda\in(0,1)$. Define a mapping $f:X\to X$ by $fx=\lambda x$ for all $x\in X$. For any $t>0$, and all $x,y,z\in X$, we have

$$S^*_{fx,fy,fz}(t)=\frac{t}{t+\lambda(|x-z|+|y-z|)}.$$

and

$$S^*_{x,y,z}(t)=\frac{t/\lambda}{t/\lambda+(|x-z|+|y-z|)}$$

Therefore, we conclude that f is a λ-contraction and f has a fixed point in X by Theorem 3.4. In fact, the fixed point is $x=0$.

(2) Let $\lambda\in(0,1)$. Define a mapping $f:X\to X$ by $fx=1$ for all $x\in X$. For any $t>0$, and all $x,y,z\in X$, since

$$S^*_{fx,fy,fz}(t)=S^*_{1,1,1}(t)=1$$

and

$$\frac{1}{3}\left(S^*_{x,x,fx}(t/\lambda)+S^*_{y,y,fy}(t/\lambda)+S^*_{z,z,fz}(t/\lambda)\right)\leq 1,$$

we conclude that

$$S^*_{fx,fy,fz}(t)\geq\frac{1}{3}\left(S^*_{x,x,fx}(t/\lambda)+S^*_{y,y,fy}(t/\lambda)+S^*_{z,z,fz}(t/\lambda)\right).$$

It is clear that all the conditions of Theorem 3.5 are satisfied. Thus f has a fixed point in X. In fact, the fixed point is $x=1$.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (under Grant No. 11071169, 11271105) and the Natural Science Foundation of Zhejiang Province (under Grant No. Y6110287).

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