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GLOBAL ASYMPTOTIC STABILITY OF AN SEIRS MODEL AND ITS SOLUTIONS BY NONSTANDARD FINITE DIFFERENCE SCHEMES

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Abstract. In this paper, we present a mathematically rigorous analysis for the global asymptotic stability of a recognized SEIRS model with nonlinear incidence and vertical transmission. Our main objective is to re-establish the local asymptotic stability of the disease endemic equilibrium point without the assumption proposed in [L. Qi, J. Cui, The stability of an SEIRS model with nonlinear incidence, vertical transmission and time delay, Appl. Math. Comput. 221 (2013), 360-366] and prove that this equilibrium point is not only locally asymptotically stable but also globally asymptotically stable if it exists. Therefore, the global stability of the model is established rigorously. Additionally, we design nonstandard finite difference (NSFD) schemes that preserve essential properties of the SEIRS model using the Mickens methodology. A set of numerical experiments are performed to support the validity of the theoretical results as well as to show advantages and efficiency of the NSFD schemes over the standard finite difference schemes.
Keywords. Asymptotic stability; Nonlinear incidence; Nonstandard finite difference schemes; SEIRS model; Vertical transmission.

1. Introduction

The mathematical modeling of epidemics has been acknowledged as an effective approach to study and analyze the dynamic transmission of infectious diseases (see, for example, [2, 4]). In particular, epidemic models are often used to specify mechanism and characteristics of the spread. Based on this basis, effective strategies to control and prevent diseases can be proposed [2, 4, 13, 19, 20]. In many SIERS models, the total population is divided into four classes, which are susceptible class (S), exposed class (E), infectious class(I), and recovered class (R). It should be emphasized that SEIRS models in particular and epidemic models in general have had an important role both in theory and practice [2, 4, 14, 17, 18, 21, 22, 23, 28] and attracted

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the attention of many researchers on several aspects (see [2, 4, 14, 17, 18, 21, 22, 23, 28] and references therein). The study of SEIRS models have many important applications in the real world applications.

Recently, Qi and Cui [28] proposed and analyzed an SEIRS model with nonlinear incidence, vertical transmission and time delay of the form:

$$\begin{aligned}
\frac{dS}{dt} &= \mu - \mu S(t) - (1-p)\mu I(t) - \beta h(S(t))I(t) + \gamma e^{-\mu\tau} I(t-\tau),\\
\frac{dE}{dt} &= \beta h(S(t))I(t) - (\varepsilon+\mu)E(t) + (1-p)\mu I(t),\\
\frac{dI}{dt} &= \varepsilon E(t) - (\mu+\gamma)I(t),\\
\frac{dR}{dt} &= \gamma I(t) - \mu R(t) - \gamma e^{-\mu\tau} I(t-\tau),
\end{aligned} \tag{1.1}$$

where the total population is divided into four classes, namely, S, I, E and R classes. Note that $S(t)+I(t)+R(t)+E(t) \equiv 1$. The parameters of the model are all positive, and the function h is assumed that $h(0) = 0$ and $h'(S) > 0$ for all $S \in [0,1]$. We refer the readers to [28] for more details of the model.

We now consider an essential particular case of model (1.1), that is, $\tau = 0$. Then, we have the model without delay of the form:

$$\begin{aligned}
\frac{dS}{dt} &= \mu - \mu S(t) - (1-p)\mu I(t) - \beta h(S(t))I(t) + \gamma I(t),\\
\frac{dE}{dt} &= \beta h(S(t))I(t) - (\varepsilon+\mu)E(t) + (1-p)\mu I(t),\\
\frac{dI}{dt} &= \varepsilon E(t) - (\mu+\gamma)I(t),\\
\frac{dR}{dt} &= -\mu R(t).
\end{aligned} \tag{1.2}$$

From the fourth equation of model (1.2), we obtain $R(t) = R(0)e^{-\mu t}$. Consequently, we only need to consider the following system, which depends on S, I and E only:

$$\begin{aligned}
\frac{dS}{dt} &= \mu - \mu S(t) - (1-p)\mu I(t) - \beta h(S(t))I(t) + \gamma I(t),\\
\frac{dE}{dt} &= \beta h(S(t))I(t) - (\varepsilon+\mu)E(t) + (1-p)\mu I(t),\\
\frac{dI}{dt} &= \varepsilon E(t) - (\mu+\gamma)I(t).
\end{aligned} \tag{1.3}$$

We recall from [28] some dynamic properties of system (1.3). Consider model (1.3) in the feasible region

$$\Omega = \Big\{(S,I,E) | S,I,E \geq 0,\ S+I+E \leq 1\Big\}. \tag{1.4}$$

Then, the Ω is a positively invariant set of system (1.3). The reproduction number of the model (1.3) is given by

$$\mathscr{R}_0 = \frac{\varepsilon\big[(1-p)\mu + \beta h(1)\big]}{(\varepsilon+\mu)(\mu+\gamma)}.$$

System (1.3) always possesses the disease free equilibrium point

$$E_0 = (1, 0, 0),$$

while the unique endemic equilibrium $E^* = (S^*, E_0^*, I^*)$ exists if and only if $\mathscr{R}_0 > 1$, where S^*, E_0^* and I^*, are determined by the system

$$\begin{aligned} \beta h(S^*) &= \frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon} - (1-p)\mu, \\ I^* &= \frac{1-S^*}{\dfrac{\mu+\gamma}{\varepsilon}+1}, \\ E_0^* &= \frac{\mu+\gamma}{\varepsilon} I^*. \end{aligned} \tag{1.5}$$

It was proved that (see [28]):
(i) the disease free equilibrium E_0 of system (1.3) is globally asymptotically stable if $\mathscr{R}_0 < 1$;
(ii) if $\mathscr{R}_0 > 1$ and $\beta h'(S^*)I^* \geq \dfrac{\mu(\gamma+\varepsilon+\mu)}{\gamma+\varepsilon}(H_1)$, the endemic equilibrium E^* of system (1.3) is locally asymptotically stable.

It is important to note that the necessary and sufficient condition for the existence of E^* is $\mathscr{R}_0 > 1$ only. On the other hand, it is known that in many mathematical models including epidemic models, if unique positive equilibrium points exist, then they are globally asymptotically stable (see, for instance, [2, 4, 23]). In other words, there is a steady growth for the models. This fact allows us to predict that the proposed hypothesis (H_1) is unique the sufficient condition for the local stability of E^* and if it exists, then it is not only locally asymptotically stable but also globally asymptotically stable. Motivated by this, we will prove that if $\mathscr{R}_0 > 1$, then the endemic equilibrium E^* of system (1.3) is globally asymptotically stable.

To prove the above statement, first, we apply the well-known result constructed in [29] to reduce system (1.3) to a sub-system depending on two components S and I only. Next, the Lyapunov indirect method is used to study the local stability of system (1.3). Last, we prove the global stability of the equilibrium point E^* based on the Poincare-Bendixson theorem in combination with the Bendixson-Dulac criterion. It should be emphasized that the reduction of the dimension of system (1.3) helps us study the stability more easily. The main result shows that if the endemic equilibrium point E^* exists, then it is globally asymptotically stable. This result is not only an important improvement of the results constructed in [28] but also valuable in many contexts of the real world applications because it confirms that the model has a steady growth. After completing the analysis of the stability, we design nonstandard finite difference (NSFD) schemes preserving the positivity and boundedness of model (1.3). Then, NSFD schemes are used to simulate model (1.3) and to confirm the validity of the theoretical results. It is important to note that the concept of NSFD schemes was first proposed by Mickens [24, 25, 26, 27] to compensate for the weaknesses of the standard finite difference (SFD) methods. Nowadays, NSFD methods have been widely applied to solve numerous real-life problems governed by differential equations associated with the mathematical modelling of phenomena arising in the natural and engineering sciences (see [1, 11, 12, 24, 25, 26, 27] and references therein). Recently, some results on NSFD schemes for the systems of differential equations describing important phenomena and processes in the real-world applications have been constructed [5, 6, 7, 8, 9, 10, 15, 16]. All the results confirm that NSFD methods not only overcome drawbacks of the standard methods but also provide reliable numerical solutions. That is why we propose and use NSFD schemes to solve numerically the SEIRS model and support theoretical results. As will be seen later, the global stability of system (1.3) is convincingly confirmed by numerical simulations generated NSFD schemes, and the advantages and efficiency of NSFD schemes over standard ones are shown clearly. The paper is organized as follows. In Section 2, we prove some auxiliary results and main results on the global stability of system (1.3). Numerical simulations are performed in Section 3. Some conclusions are provided in the last section.

2. The results

2.1. **Auxiliary results.**

Lemma 2.1. *The dynamics of system (1.3) is qualitatively equivalent to the dynamics of the following sub-system*

$$\begin{aligned} \frac{dS}{dt} &= \mu - \mu S - (1-p)\mu I - \beta h(S) I + \gamma I, \\ \frac{dI}{dt} &= \varepsilon\big(1 - S - I\big) - (\mu+\gamma) I. \end{aligned} \tag{2.1}$$

Proof. Let $N(t) = S(t) + I(t) + E(t)$. From system (1.3), we have

$$\frac{dN}{dt} = \mu(1-N).$$

This implies that $\lim_{t\to\infty} N(t) = 1$. The well-known results in [29] imply that the dynamics of system (1.3) is qualitatively equivalent to the dynamics of limit system (2.1). Note that we have used the representation $E = 1 - S - I$ to obtain system (2.1). The proof is complete. □

Remark 2.2. System (2.1) has a unique positive equilibrium point $\widehat{E}^* = (S^*, I^*)$ if and only if $\mathscr{R}_0 > 1$. Clearly, the stability of the endemic equilibrium E^* of system (1.3) is equivalent to the stability of the equilibrium point $\widehat{E}^* = (S^*, I^*)$ of system (2.1). Moreover, system (2.1) always has the boundary equilibrium point $\widehat{E}^0 = (1, 0)$ for all the values of the parameters. The local stability of the model when $\mathscr{R}_0 > 1$ is presented in Lemma 2.3.

Lemma 2.3. *The boundary equilibrium point $\widehat{E}^0 = (1,0)$ of system (2.1) is unstable when $\mathscr{R}_0 > 1$.*

Proof. Computing the Jacobian matrix of the system (2.1) at $\widehat{E}^0$ we obtain

$$J_0 = \begin{pmatrix} -\mu & -(1-p)\mu - \beta h(1) + \gamma \\ -\varepsilon & -(\varepsilon+\mu+\gamma) \end{pmatrix}.$$

Clearly, $Trace(J_0) < 0$. It is easy to verify that

$$\det(J_0) = (\varepsilon+\mu)(\mu+\gamma)(1-\mathscr{R}_0) < 0.$$

Therefore, if λ_1 and λ_2 are the eigenvalues of the matrix J_0, then

$$Re(\lambda_1) Re(\lambda_2) < 0.$$

Hence, $\widehat{E}_0$ is unstable. The proof is complete. □

2.2. **Main results.**

Theorem 2.4. *If $\mathscr{R}_0 > 1$, then the endemic equilibrium E^* of system (1.3) is locally asymptotically stable.*

Proof. By use of Lemma 2.1, we only need to consider the local stability of $\widehat{E}^*$ of system (2.1). Note that E^* and $\widehat{E^*}$ exist if and only of $\mathscr{R}_0 > 1$. Computing the Jacobian matrix of system (2.1) at $\widehat{E}^*$, we obtain

$$J = \begin{pmatrix} -\mu - \beta h'(S^*)I^* & -(1-p)\mu - \beta h(S^*) + \gamma \\ -\varepsilon & -(\varepsilon + \mu + \gamma) \end{pmatrix}.$$

The Routh-Hurwitz criteria [2, Theorem 4.4] reveals that $\widehat{E}^*$ is locally asymptotically stable if

$$Trace(J) < 0, \qquad \det(J) > 0. \tag{2.2}$$

Obviously, $Trace(J) < 0$.

On the other hand,

$$\det(J) = \mu(\varepsilon + \mu + \gamma) + \beta h'(S^*)I^*(\varepsilon + \mu + \gamma) - \varepsilon\Big[(1-p)\mu + \beta h(S^*)\Big] + \gamma\varepsilon. \tag{2.3}$$

Using the first equation of the system (1.5), we get

$$\varepsilon\Big[(1-p)\mu + \beta h(S^*)\Big] = (\varepsilon + \mu)(\mu + \gamma). \tag{2.4}$$

Combining (2.4) and (2.3), we have

$$\det(J) = \beta h'(S^*)I^*(\varepsilon + \mu + \gamma) > 0.$$

Thus, condition (2.2) is satisfied, and we conclude that $\widehat{E}^*$ is locally asymptotically stable of model (2.1). Hence, the proof is complete. □

Theorem 2.5. *If $\mathscr{R}_0 > 1$, then the endemic equilibrium E^* of system (1.3) is not only locally asymptotically stable but also globally asymptotically stable in the interior $int(\Omega)$ of Ω.*

Proof. By use of Lemma 2.1, it is sufficient to prove that the equilibrium point $\widehat{E}^*$ of system (2.1) is globally asymptotically stable in the interior of Ω. Indeed, let $F(S,I)$ and $G(S,I)$ denote the right-hand side of system (2.1). We use a Dulac function defined by

$$D(S,I) \equiv 1.$$

Clearly, D is continuously differentiable. Note that

$$\frac{\partial(DF)}{\partial S} + \frac{\partial(DG)}{\partial I} = -\mu - \beta h'(S)I - (\varepsilon + \mu + \gamma) < 0,$$

for all $(S,I) \in int(\Omega)$. Therefore, the system has no periodic orbits or graphics in $int(\Omega)$.

On the other hand, all of the solutions of system (2.1) are bounded. By use of Theorem 2.4 and Lemma 2.3, we have if $\mathscr{R}_0 > 1$, then $\widehat{E}^*$ is locally asymptotically stable and $\widehat{E}^0$ is unstable. Therefore, applying the classical Poincare-Bendixson theorem [2, 23] we easily conclude that $\widehat{E}^*$ is globally asymptotically stable of the system (2.1). This proof is completed. □

Remark 2.6. In Theorem 2.4, the assumption (H_1) was not used. Meanwhile, Theorem 2.5 affirms that if the endemic equilibrium point E^* exists ($\mathscr{R}_0 > 1$), then it is not only locally asymptotically stable but also globally asymptotically stable. Therefore, Theorems 2.4 and 2.5 extend and improve the constructed results in [28].

3. NSFD SCHEMES FOR MODEL (1.3) AND NUMERICAL SIMULATIONS

In this section, numerical simulations are performed to validate the theoretical results. Before performing numerical simulations, we construct NSFD schemes that preserve the positivity and boundedness of the system (1.3).

3.1. NSFD schemes for System (1.3).

First, from the assumption $h(0) = 0$ and $h'(0) > 0$ for all $S \in [0,1]$, we can represent the function $h(S)$ in the form

$$h(S) = Sf(S), \tag{3.1}$$

where f is a function well defined on the close interval $[0,1]$. Then, we propose the following NSFD scheme

$$\begin{aligned}
\frac{S_{k+1}-S_k}{\phi(h)} &= \mu - \mu S_{k+1} - (1-p)\mu I_k - \beta S_{k+1} f(S_k) I_k + \gamma I_k,\\
\frac{E_{k+1}-E_k}{\phi(h)} &= \beta S_{k+1} f(S_k) I_k - (\varepsilon+\mu)E_{k+1} + (1-p)\mu I_k,\\
\frac{I_{k+1}-I_k}{\phi(h)} &= \varepsilon E_{k+1} - \mu I_{k+1} - \gamma I_k,
\end{aligned} \tag{3.2}$$

where $\phi(h) = h + \mathscr{O}(h^2)$ as $h \to 0$ and the function f is defined by (3.1).

We introduce the following assumption to establish the positivity and the boundedness of the NSFD scheme (3.2)

$$\phi(h) < \gamma^{-1}, \quad \forall h > 0. \tag{3.3}$$

Remark 3.1. There are many ways to choose denominator functions $\phi(h)$ satisfying (3.3), for instance (see [24, 25, 26, 27]),

$$\phi(h) = \frac{1-e^{-\tau h}}{\tau}, \quad \tau \geq \gamma.$$

General classes of nonstandard denominator functions can be found in [10].

Theorem 3.2. *Assume that $\phi(h)$ is a function satisfying condition (3.3). Then, the set Ω defined by (1.4) is a positively invariant set of system (3.2), i.e, if $(S_0, E_0, I_0) \in \Omega$, then $(S_k, E_k, I_k) \in \Omega$ for all $k > 0$. In other words, the NSFD scheme (3.2) preserves the positivity and the boundedness of system (1.3) for any finite step size $h > 0$ under condition (3.3).*

Proof. We prove this theorem by the mathematical induction. Assume $(S_k, E_k, I_k) \in \Omega$ and ket $N_k = S_k + E_k + I_k$. Note that $N_k \leq 1$. From system (3.2), we obtain

$$N_{k+1} = \frac{N_k + \phi\mu}{1+\phi\mu} \leq \frac{1+\phi\mu}{1+\phi\mu} = 1,$$

which implies that $S_{k+1}+E_{k+1}+I_{k+1} \leq 1$. We transform scheme (3.2) to the explicit form as follows

$$\begin{aligned} S_{k+1} &= \frac{S_k+\phi\mu\left[1-(1-p)I_k\right]+\phi\gamma I_k}{1+\phi\mu+\phi\beta f(S_k)I_k}, \\ E_{k+1} &= \frac{E_k+\phi\beta S_{k+1}f(S_k)I_k+\phi(1-p)\mu I_k}{1+\phi(\varepsilon+\mu)}, \\ I_{k+1} &= \frac{(1-\phi\gamma)I_k+\phi\varepsilon E_{k+1}}{1+\phi\mu}. \end{aligned} \tag{3.4}$$

From $0 \leq I_k \leq 1$, we get $1-(1-p)I_k \geq p \geq 0$. On the other hand, $1-\phi\gamma > 0$ since $\phi < \gamma^{-1}$. Therefore, we deduce that $S_{k+1}, E_{k+1}, I_{k+1} \geq 0$ if $S_k, E_k, I_k \geq 0$. Consequently, the proof is completed. □

Remark 3.3. For the NSFD scheme (3.2), we have

$$N_{k+1} = \frac{N_k+\phi\mu}{1+\phi\mu},$$

which implies that $\lim_{k\to\infty} N_k = 1$. Meanwhile, for the ODE model (1.3), we have $\lim_{t\to\infty} N(t) = 1$. Hence, the NSFD scheme preserves the convergence of the total population of the continuous model.

3.2. **Numerical examples.** In Examples 3.4 and 3.5, we use NSFD scheme (3.2) with the step size $h = 10^{-3}$ to confirm the global stability of the endemic equilibrium E^* when $\mathscr{R}_0 > 1$.

Example 3.4 (When the condition (H_1) is not satisfied)**.** Consider the model (1.3) with the data [28]

$$h(S) = \frac{S}{1+\alpha S}, \quad \alpha = 0.1, \quad \beta = 0.9, \quad \varepsilon = 0.6, \quad p = 0.3, \quad \gamma = 0.1, \quad \mu = 0.2.$$

In this case, $\mathscr{R}_0 = 2.395 > 1$ and $E^* = (0.2975, 0.2342, 0.4683)$. It is easy to check that

$$\beta h'(S^*)I^* < \frac{\mu(\gamma+\varepsilon+\mu)}{\gamma+\varepsilon},$$

which implies that condition (H_1) is not satisfied. However, numerical solutions to system (1.3) depicted in Figure 1 indicate that E^* is globally asymptotically stable. Hence, the condition (H_1) is only a technical one.

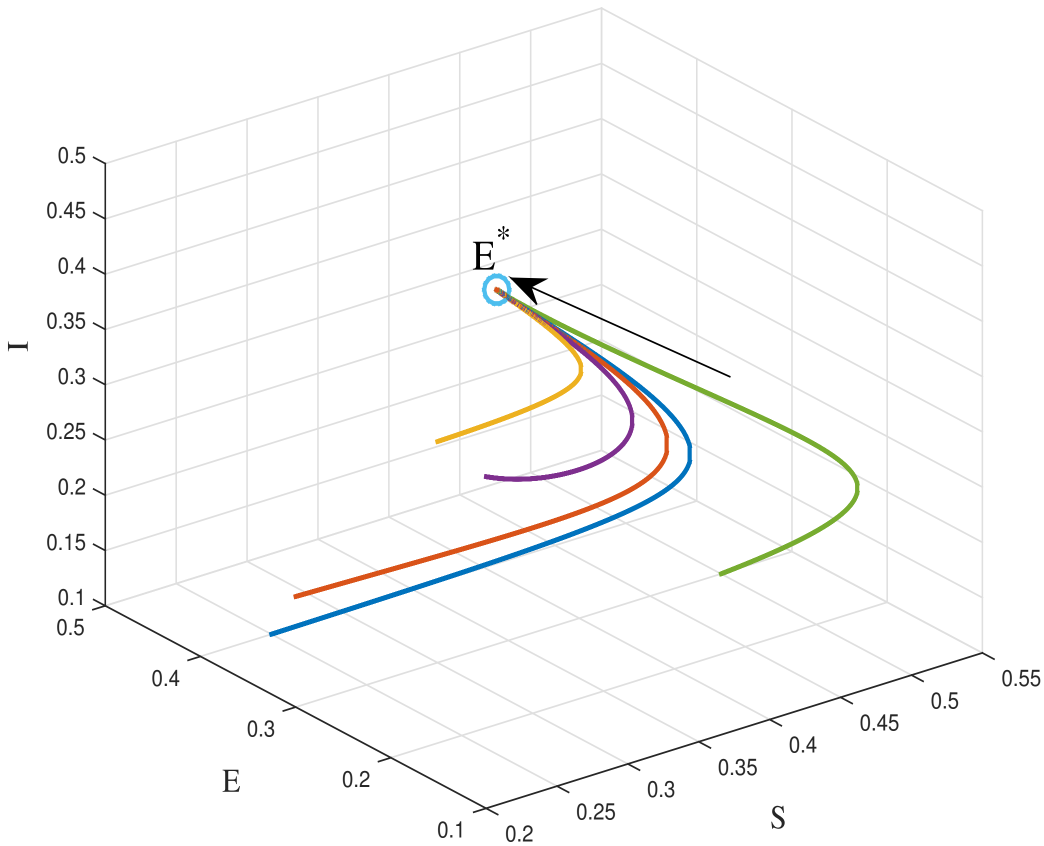

FIGURE 1. Approximate solutions generated by the NSFD scheme (3.2) in Example 3.4.

Example 3.5 (When the condition (H_1) is satisfied)**.** Consider the model (1.3) with the data

$$h(S) = \frac{S}{1+\alpha S}, \quad \alpha = 0.8, \quad \beta = 0.9, \quad \varepsilon = 0.6, \quad p = 0.3, \quad \gamma = 0.1, \quad \mu = 0.2.$$

In this case, $\mathscr{R}_0 = 1.600 > 1$ and $E^* = (0.3757, 0.2081, 0.4162)$. It is easy to check that

$$\beta h'(S^*)I^* > \frac{\mu(\gamma+\varepsilon+\mu)}{\gamma+\varepsilon},$$

which implies that condition (H_1) is satisfied. Clearly, numerical solutions in Figure 2 show that E^* is globally asymptotically stable. From Examples 3.4 and 3.5, we see that the validity of Theorems 2.4 and 2.5 is confirmed and the condition (H_1) is only technical one

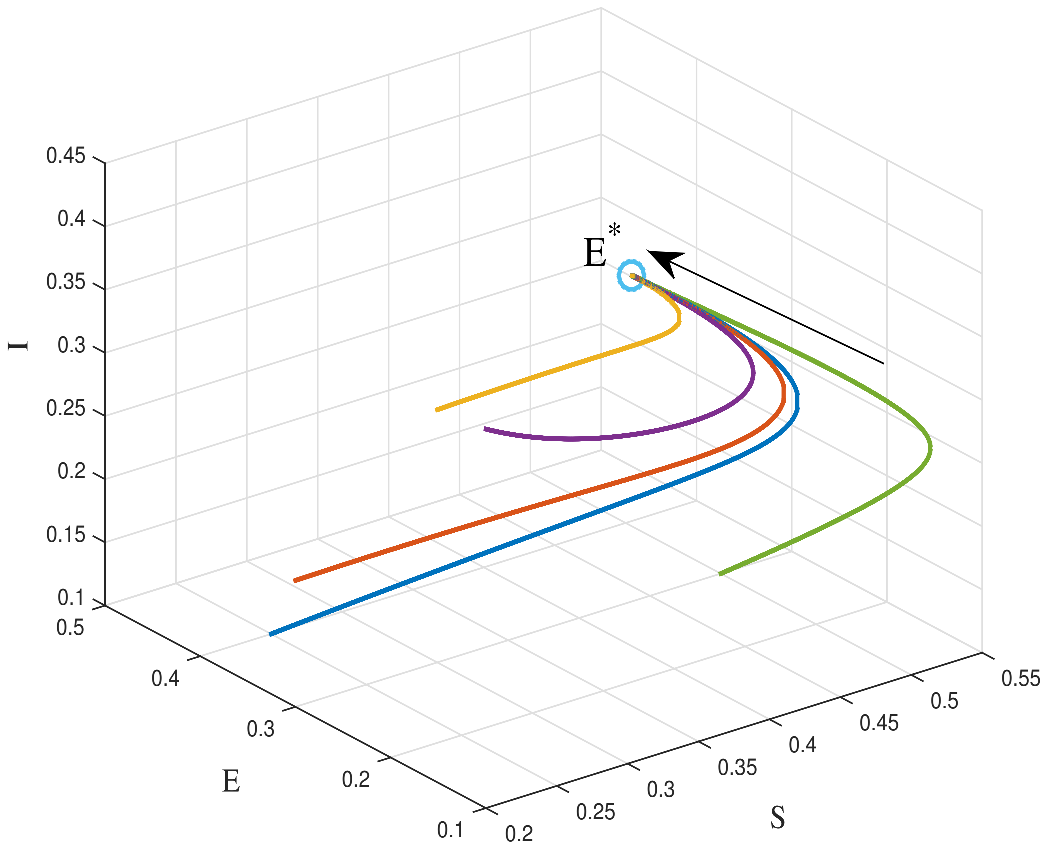

FIGURE 2. Approximate solutions generated by the NSFD scheme (3.2) in Example 3.5.

Example 3.6 (Comparison of NSFD scheme and standard finite difference schemes)**.** In this example, we compare dynamics of the NSFD scheme (3.2) and the standard Euler scheme and the classical fourth order Runge-Kutta (RK4) scheme . For this purpose, we consider the model (1.3) with the parameters

$$h(S) = \frac{S}{1+\alpha S}, \quad \alpha = 0.8, \quad \beta = 0.9, \quad \varepsilon = 0.6, \quad p = 0.3, \quad \gamma = 0.1, \quad \mu = 0.75,$$

and the initial data

$$\big(S(0), E(0), I(0)\big) = (0.5, 0.25, 0.2).$$

In this case, $\mathscr{R}_0 = 0.5359 < 1$, and therefore, the equilibrium point $E_0 = (1,0,0)$ is globally asymptotically stable.

Numerical solutions obtained by the Euler scheme, the RK4 scheme and the NSFD scheme are sketched in Figures 3-5. We see that the Euler scheme and the RK4 scheme destroy not only the positivity and the boundedness but also the stability of the model (1.3). This is a common drawback of the standard schemes. In general, standard schemes can not preserve essential properties of the continuous models for all finite step sizes. Conversely, NSFD schemes reflect dynamics of the continuous model exactly. Actually, NSFD schemes can operate well for any finite step size, and hence, they provide reliable numerical approximations.

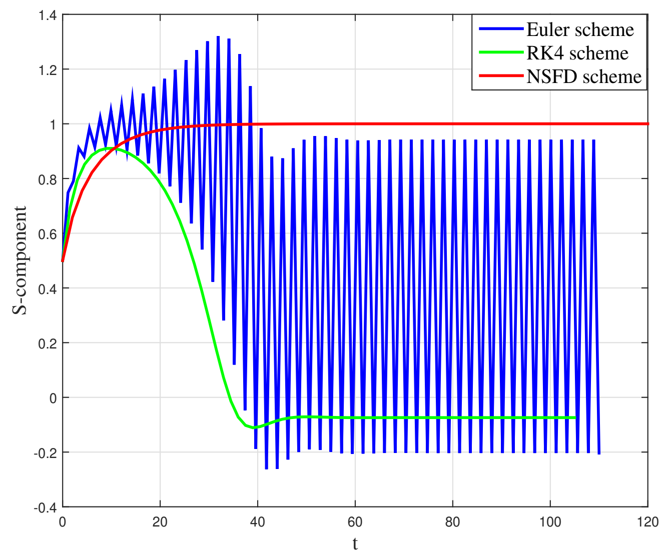

FIGURE 3. S-components generated by the NSFD scheme with $h = 2$ and $t \in [0,120]$, the Euler scheme with $h = 1.1$ and $t \in [0,110]$ and the RK4 scheme with $h = 1.5$ and $t \in [0,105]$ in Example 3.6.

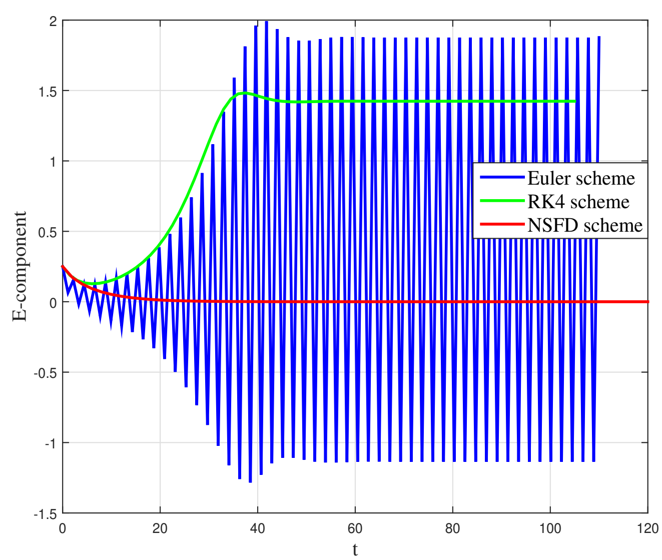

FIGURE 4. E-components generated by the NSFD scheme with $h = 2$ and $t \in [0,120]$, the Euler scheme with $h = 1.1$ and $t \in [0,110]$ and the RK4 scheme with $h = 1.5$ and $t \in [0,105]$ in Example 3.6.

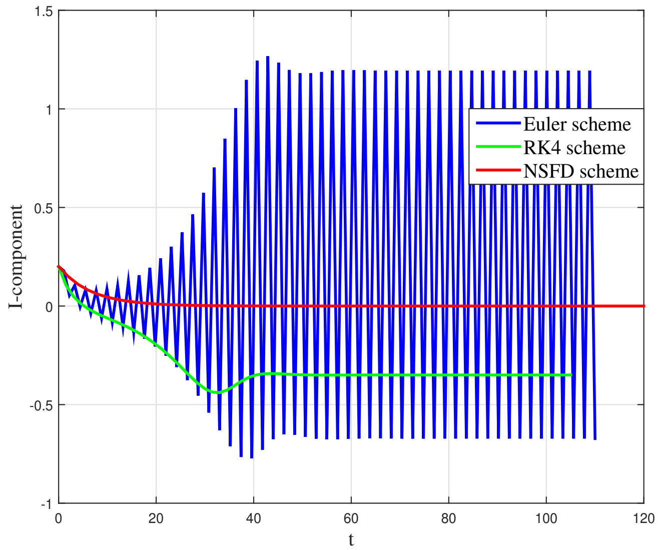

FIGURE 5. I-components generated by the NSFD scheme with $h = 2$ and $t \in [0, 120]$, the Euler scheme with $h = 1.1$ and $t \in [0, 110]$ and the RK4 scheme with $h = 1.5$ and $t \in [0, 105]$ in Example 3.6.

4. THE CONCLUSION

In this paper, we provided a rigorously mathematical analysis for the stability of an SEIRS model with nonlinear incidence and vertical transmission. It is worth noting that the assumption proposed in the previous work [28] for the local asymptotic stability of the endemic equilibrium point was freed completely. Especially, we proved that if the endemic equilibrium point exists, then it is not only locally asymptotically stable but also globally asymptotically stable. Furthermore, NSFD schemes which preserve the positivity and boundedness of the model have been proposed and used. The numerical simulations show that the NSFD schemes captured the essential dynamical features of the model. Meanwhile, the standard schemes fail to preserve the dynamics of the continuous model. Importantly, as pointed out through test cases in this work, the proposed NSFD schemes are able to generate numerical schemes that validate theoretical results proved in Theorems 2.4 and 2.5. Our main objective in the future is to apply the approach in this paper for other epidemic models described by ordinary or partial differential equations. In addition, the complete global stability of the delay model (1.1) will be also considered.

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