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INERTIAL MODIFIED TSENG'S EXTRAGRADIENT ALGORITHMS FOR SOLVING MONOTONE VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. For solving monotone variational inequalities and fixed point problems of a quasi-nonexpansive mapping in real Hilbert spaces, we introduce two new algorithms which combine the inertial Tseng's extragradient method and the hybrid-projection method, respectively. Weak and strong convergence theorems are established under some appropriate conditions. Finally, we provide some numerical experiments to show the effectiveness and advantages of the proposed algorithms.

Keywords. Variational inequality; Tseng's extragradient method; Monotone operator; Inertial method; Self-adaptive method.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively.

The variational inequality problem (VIP) for a mapping A on set C is to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The VIP plays an important role in a lot of real world problems, such as, single processing, transportation, machine learning and medical imaging; see, e.g., [1, 2, 3, 4]. From now on, the set of solutions of the VIP is denoted by $VI(C,A)$. It is known that the VIP is equivalent to the following fixed point problem

$$x^* = P_C(I - \lambda A)x^*, \tag{1.2}$$

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where $P_C : H \to C$ is the metric projection and λ is a positive real number. Thus, we have the following simple projection iterative algorithm

$$x_{n+1} = P_C(I - \lambda A)x_n, \tag{1.3}$$

where $A : H \to H$ is an η-strongly monotone and L-Lipschitz continuous mapping. The iteration is strongly convergent under appropriate conditions of parameters. If A is inverse-strongly monotone (see below), it is weakly convergent under some certain conditions. In order to avoid the strong assumption on the monotonicity, Korpelevich [5] proposed the following extragradient method in 1976

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n). \end{cases} \tag{1.4}$$

The convergence of this method only requires that the operator A is monotone and L-Lipschitz continuous in a Hilbert space. The conditions of the extragradient method are weakened, but this method still needs to calculate two projections from H onto its closed convex set C, and the projection onto the nonempty closed convex subset C might be difficult to calculate.

To overcome these difficulties, various modification of the extragradient method were proposed. In 2000, Tseng [6] studied the following iterative method

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda(A y_n - A x_n), \end{cases} \tag{1.5}$$

where A is monotone, L-Lipschitz continuous and $\lambda \in (0, 1/L)$. There is only projection on set C in Tseng's extragradient method. In 2011, the subgradient extragradient method (SEGM) was proposed by Censor et al. [7]:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H \mid \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \le 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{1.6}$$

where A is L-Lipschitz continuous, monotone and $\lambda \in (0, 1/L)$. In particular, the second projection onto the set C of the extragradient method is replaced by a projection onto a special half-space, which improves the efficiency of the algorithm.

Inertial algorithms are efficient in variational inequality problems, split fixed point problems, and equilibrium problems; see, for instance, [8, 9, 10, 11, 12] and the references therein. Next, let us mention the inertial method, which is based upon a discrete version of a second order dissipative dynamical system in time. In particular, Alvarez and Attouch [13] solved the problem of finding zero of a maximal monotone operator with the inertial proximal method: find $x_{n+1} \in H$ such that $0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1})$, where $x_{n-1}, x_n \in H$, $\theta_n \in [0, 1)$ and $\lambda_n > 0$. It also can be written in the following form:

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})),$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$. The advantage of the inertial method is that it can speed up the convergence of original algorithms.

In 2017, Dong, Cho and Zhang [14] studied the following inertial projection-contraction method

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ d(w_n, y_n) = (w_n - y_n) - \lambda(Aw_n - Ay_n), \\ x_{n+1} = w_n - \gamma\beta_n d(w_n, y_n), \end{cases} \tag{1.7}$$

for each $k \geq 1$, where $\gamma \in (0,2), \lambda > 0$,

$$\beta_n := \begin{cases} \varphi(w_n, y_n)/\|d(w_n, y_n)\|^2, & d(w_n, y_n) \neq 0, \\ 0, & d(w_n, y_n) = 0, \end{cases} \tag{1.8}$$

$$\varphi(w_n, y_n) = \langle w_n - y_n, d(w_n, y_n)\rangle.$$

In 2017, Thong and Hieu [15] proposed the following algorithm

Algorithm 1.1. Step 1: Choose $x_0 \in H$, $\gamma > 0$, $l \in (0,1)$, and $\mu \in (0,1)$.
Step 2: Given the current iterate x_n, compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \dots\}$ satisfying

$$\lambda\|Ax_n - Ay_n\| \leq \mu\|x_n - y_n\|.$$

If $y_n = x_n$, then stop and x_n is the solution of the variational inequality problem. Otherwise, go to Step 3.
Step 3: Compute the new iterate x_{n+1} via the following iterate formula:

$$x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n).$$

Set $n := n+1$ and return to Step 2.

This iterative algorithm does not need to know the knowledge of the Lipschitz constant of the operator A. It is a new self-adaptive method. Under appropriate conditions, the sequence $\{x_n\}$ generated by (1.5), (1.6), (1.7) and Algorithm 1.1 all converge weakly to an element of $VI(C,A)$. Since the weak convergence is not desirable, efforts have been made to various modifications so that the strong convergence is guaranteed. In 2017, Dong et al. [16] employed the hybrid-projection method to modify an inertial forward-backward algorithm for solving zero point problems in Hilbert spaces:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = (I + r_n B)^{-1}(y_n - r_n A y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2\|x_{n-1} - x_n\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.9}$$

They proved that $\{x_n\}$ converges strongly to $P_{(A+B)^{-1}(0)}x_0$ under some suitable conditions.

In this paper, motivated by the above results, we present two new algorithms, which are based on the Tseng's extragradient method, for solving a monotone variational inequality problem and the fixed-point problem of a quasi-nonexpansive mapping. The operator A involved in the variational inequality problem is monotone and Lipschitz continuous. We combine the inertial

Tseng's extragradient method with self-adaptive technique and the hybrid-projection method, respectively. Weak and strong convergence theorems are established under some appropriate conditions. Finally, we give numerical examples to illustrate the efficiency and advantages of the proposed algorithms and compare with existing methods. This paper is organized as follows. In Section 2, we recall some definitions and lemmas for sequel use. In Section 3, two convergence theorems are proved. In Section 4, we perform numerical examples and comparisons. This paper ends with a conclusion remark in Section 5

2. Preliminaries

Assume that H is a real Hilbert space and C is a nonempty closed convex subset of H. In this paper, we use the following notations:

- $\to$ denotes strong convergence.
- $\rightharpoonup$ denotes weak convergence.
- $\omega_w(x_n) := \{x| \, there\ exists\ \{x_{n_j}\}_{j=0}^{\infty} \subset \{x_n\}_{n=0}^{\infty}\ such\ that\ x_{n_j} \rightharpoonup x\}$ denotes the weak cluster point set of $\{x_n\}_{n=0}^{\infty}$.

Let H be a real Hilbert space, for all $x, y \in H$ and $\lambda \in \mathbb{R}$, we have
$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;
$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$;
$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Then
$\|P_C x - P_C y\|^2 \le \langle x-y, P_C x - P_C y\rangle, \quad \forall x, y \in H$;
$\|x - P_C x\|^2 + \|y - P_C x\|^2 \le \|x-y\|^2, \quad \forall x \in H, y \in C$.

Given $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if there holds the inequality $\langle x-z, y-z\rangle \le 0$, $\forall y \in C$.

Let $T : H \to H$ be the nonlinear operators. Recall the following definitions. T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x-y\|, \quad \forall x, y \in H.$$

T is said to be firmly nonexpansive if

$$\langle Tx - Ty, x-y\rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

It is easy to see that a firmly nonexpansive mapping is always nonexpansive by using the Cauchy-Schwarz inequality. T is said to be α-averaged with $0 < \alpha < 1$ if

$$T = (1-\alpha)I + \alpha S,$$

where $S : H \to H$ is nonexpansive. It is obvious that $Fix(S) = Fix(T)$. It is easy to see that a firmly nonexpansive mapping is $\frac{1}{2}$-averaged. T is said to be L-Lipschitz continuous with $L \ge 0$ if

$$\|Tx - Ty\| \le L\|x-y\|, \quad \forall x, y \in H.$$

We call T a contractive mapping if $0 \le L < 1$. T is said to be quasi-nonexpansive if $Fix(T) \ne \emptyset$ and

$$\|Tx - p\| \le \|x - p\|, \quad \forall x \in H, p \in Fix(T).$$

Let $A : H \to H$ be an operator. Recall the following definitions. A is said to be monotone if

$$\langle x-y, Ax - Ay\rangle \ge 0, \quad \forall x, y \in H.$$

A is said to be η-strongly monotone with $\eta > 0$ if

$$\langle x-y, Ax-Ay\rangle \geq \eta \|x-y\|^2, \quad \forall x,y \in H.$$

A is said to be v-inverse-strongly monotone (v-ism) with $v > 0$ if

$$\langle x-y, Ax-Ay\rangle \geq v\|Ax-Ay\|^2, \quad \forall x,y \in H.$$

We can easily show that a v-ism mapping is $\frac{1}{v}$-Lipschitz continuous by using the Cauchy-Schwarz inequality.

Definition 2.1. [17] Assume that $T: H \to H$ is a nonlinear operator with $Fix(T) \neq \emptyset$. Then $I-T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H the following implication holds

$$x_n \rightharpoonup x \quad and \quad (I-T)x_n \to 0 \Rightarrow x \in Fix(T).$$

If T is nonexpansive, we know that $I-T$ is demiclosed at zero. However, there exists a quasi-nonexpansive mapping T with the fact that $I-T$ is not demiclosed at zero. Next, we give an example.

Example 2.2. Let $H = \mathbb{R}$ and $C = [0, \frac{3}{2}]$. Define the operator T on C by

$$Tx = \begin{cases} \frac{x}{2}, & if\ x \in [0,1], \\ x\cos 2\pi x, & if\ x \in (1, \frac{3}{2}]. \end{cases}$$

Obviously, $Fix(T) = \{0\}$.

For any $x \in [0,1]$, we have

$$|Tx - 0| = |\frac{x}{2} - 0| \leq |x-0|.$$

On the other hand, for any $x \in (1, \frac{3}{2}]$, we have

$$|Tx-0| = |x\cos 2\pi x - 0| = |x\cos 2\pi x| \leq |x| = |x-0|.$$

Thus, operator T is quasi-nonexpansive.

By taking $\{x_n\} \subset (1, \frac{3}{2}]$ and $x_n \to 1$ as $n \to \infty$, we have

$$|(I-T)x_n| = |x_n - Tx_n| = |x_n - x_n\cos 2\pi x_n| = |x_n| \cdot |1-\cos 2\pi x_n| \to 0 \quad (n \to \infty).$$

But $1 \notin Fix(T)$. We conclude that $I-T$ is not demiclosed at zero.

Lemma 2.3. [13] *Let $\{\varphi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0,+\infty)$ such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n, \forall n \geq 1, \sum_{n=1}^{\infty} \delta_n < +\infty,$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:
(i) $\sum_{n=1}^{\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$, where $[t]_+ := max\{t, 0\}$;
(ii) there exists $\varphi^* \in [0,+\infty)$ such that $\lim_{n\to\infty} \varphi_n = \varphi^*$.

Lemma 2.4. [18] *(Minty). If $A: C \to H$ is a continuous and monotone mapping, then x^* is a solution of the VIP if and only if x^* is a solution of the following problem*

$$find\ x \in C\ such\ that\ \langle Ay, y-x\rangle \geq 0, \forall y \in C.$$

Lemma 2.5. [19] *Let C be a nonempty closed and convex subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H. If*
(i) for all $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;
(ii) every sequential weak cluster point of the sequence $\{x_n\}$ is in C,
then the sequence $\{x_n\}$ converges weakly to a point in C.

3. Main results

In this section, we propose two new iterative algorithms. We combine a modified inertial Tseng's extragradient method and the hybrid-projection method, respectively. The first algorithm is weakly convergent while the second one is strongly convergent. They are mainly used to solve the monotone and Lipschitz continuous variational inequality problem and the fixed-point problem of a quasi-nonexpansive mapping in real Hilbert spaces. We assume that the operator $A : H \to H$ is monotone and Lipschitz continuous, and $U : H \to H$ is quasi-nonexpansive.

3.1. **Weak convergence.** In this subsection, we propose the weakly convergent algorithm: Mann-type Tseng's extragradient algorithm, which is described as follows.

Algorithm 3.1. Initialization: Give $\gamma > 0$, $l \in (0,1)$, and $\mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrarily fixed.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1: Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$, and compute

$$y_n = P_C(w_n - \tau_n A w_n),$$

where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, \dots\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \le \mu\|w_n - y_n\|. \tag{3.1}$$

Step 2: Compute

$$z_n = y_n - \tau_n(Ay_n - Aw_n).$$

Step 3: Compute

$$x_{n+1} = (1-\beta_n)w_n + \beta_n U z_n.$$

If $w_n = y_n = x_{n+1}$ then $w_n \in Fix(U) \cap VI(C,A)$. Set $n := n+1$ and go to step 1.

The following lemmas are important to prove the convergence of the above algorithm.

Lemma 3.2. [15] *The Armijo-like search rule (3.1) is well defined and*

$$min\{\gamma, \frac{\mu l}{L}\} \le \tau_n \le \gamma.$$

Lemma 3.3. *If $w_n = y_n = x_{n+1}$, then $w_n \in Fix(U) \cap VI(C,A)$.*

Proof. Since $y_n = P_C(w_n - \tau_n A w_n)$, and $w_n = y_n$, we have $w_n \in VI(C,A)$. From $z_n = y_n - \tau_n(Ay_n - Aw_n)$, we have $y_n = z_n$. On the other hand, if $w_n = y_n = x_{n+1}$, we conclude from $x_{n+1} = (1-\beta_n)w_n + \beta_n U z_n$ that $w_n = (1-\beta_n)w_n + \beta_n U w_n$. Therefore, $Uw_n = w_n$, which means $w_n \in Fix(U)$. Thus, $w_n \in Fix(U) \cap VI(C,A)$. □

Lemma 3.4. *Let $\{z_n\}$ be a sequence generated by Algorithm 3.1, then, for all $p \in VI(C,A)$, we have*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1-\mu^2)\|w_n - y_n\|^2.$$

Proof. Letting $p \in VI(C,A)$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - p\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|(y_n - w_n) + (w_n - p)\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|y_n - w_n\|^2 + \|w_n - p\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|y_n - w_n\|^2 + \|w_n - p\|^2 + 2\langle y_n - w_n, y_n - p\rangle \\
&\quad -2\langle y_n - w_n, y_n - w_n\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - y_n\|^2 + \|w_n - p\|^2 + 2\langle y_n - w_n, y_n - p\rangle - 2\|w_n - y_n\|^2 \\
&\quad +\tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle. \qquad (3.2)
\end{aligned}$$

Since $y_n = P_C(w_n - \tau_n Aw_n)$, we find that

$$\langle y_n - w_n + \tau_n Aw_n, y_n - p\rangle \leq 0,$$

which implies that

$$\langle y_n - w_n, y_n - p\rangle \leq -\tau_n\langle Aw_n, y_n - p\rangle. \qquad (3.3)$$

From (3.2) and (3.3), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\tau_n\langle Aw_n, y_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\tau_n\langle Ay_n - Aw_n, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle Ay_n, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\quad -2\tau_n\langle Ay_n - Ap, y_n - p\rangle - 2\tau_n\langle Ap, y_n - p\rangle \\
&\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 \\
&\leq \|w_n - p\|^2 - (1-\mu^2)\|y_n - w_n\|^2.
\end{aligned}$$

Therefore,

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1-\mu^2)\|y_n - w_n\|^2, \quad \forall p \in VI(C,A).$$

This completes the proof. □

Theorem 3.5. *Let $A : H \to H$ be a monotone and L-Lipschitz continuous mapping, and let $U : H \to H$ be a quasi-nonexpansive mapping. Let $\{\alpha_n\}$ be a non-decreasing real sequence such that $0 \leq \alpha_n \leq \alpha \leq \frac{1}{4}$ and $\{\beta_n\}$ is a real sequence such that $0 < \beta \leq \beta_n \leq \frac{1}{2}$. Assume that $I - U$ is demiclosed at zero. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to an element of $Fix(U) \cap VI(C,A)$.*

Proof. Let $p \in Fix(U) \cap VI(C,A)$. From Lemma 3.4, we have

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1-\mu^2)\|w_n - y_n\|^2.$$

This implies that

$$\|z_n - p\|^2 \le \|w_n - p\|^2.$$

Thus,

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\beta_n)(w_n - p) + \beta_n(Uz_n - p)\|^2 \\ &= (1-\beta_n)\|w_n - p\|^2 + \beta_n\|Uz_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2 \\ &\le (1-\beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2 \\ &\le (1-\beta_n)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2 \\ &= \|w_n - p\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2. \end{aligned} \tag{3.4}$$

Since $x_{n+1} = (1-\beta_n)w_n + \beta_n Uz_n$, we have

$$Uz_n - w_n = \frac{1}{\beta_n}(x_{n+1} - w_n). \tag{3.5}$$

Combining (3.4) with (3.5), and borrowing $\beta_n \le \frac{1}{2}$, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|w_n - p\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - w_n\|^2 \\ &\le \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2. \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} \|w_n - p\|^2 &= \|(1+\alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\|^2 \\ &= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 \\ &\quad + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\ge \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 \\ &\quad - 2\alpha_n\|x_{n+1} - x_n\| \cdot \|x_n - x_{n-1}\| \\ &\ge \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - \alpha_n\|x_{n+1} - x_n\|^2 \\ &\quad - \alpha_n\|x_n - x_{n-1}\|^2 \\ &= (1-\alpha_n)\|x_{n+1} - x_n\|^2 + (\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.8}$$

Since the sequence $\{\alpha_n\}$ is non-decreasing, we conclude from (3.6), (3.7) and (3.8) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\quad - (1-\alpha_n)\|x_{n+1} - x_n\|^2 - (\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2 \\ &= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - (1-\alpha_n)\|x_{n+1} - x_n\|^2 \\ &\quad + [\alpha_n(1+\alpha_n) - (\alpha_n^2 - \alpha_n)]\|x_n - x_{n-1}\|^2 \\ &\le (1+\alpha_{n+1})\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 - (1-\alpha_n)\|x_{n+1} - x_n\|^2 \\ &\quad + 2\alpha_n\|x_n - x_{n-1}\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} & \|x_{n+1}-p\|^2-\alpha_{n+1}\|x_n-p\|^2+2\alpha_{n+1}\|x_{n+1}-x_n\|^2 \\ \leq\ & \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2 \\ & +2\alpha_{n+1}\|x_{n+1}-x_n\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2 \\ =\ & \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2 \\ & +(2\alpha_{n+1}-1+\alpha_n)\|x_{n+1}-x_n\|^2. \end{aligned}$$

Putting $\Gamma_n := \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2$, we have

$$\Gamma_{n+1}-\Gamma_n \leq (2\alpha_{n+1}-1+\alpha_n)\|x_{n+1}-x_n\|^2.$$

Since $0\leq\alpha_n\leq\alpha\leq\frac{1}{4}$, we have $-(2\alpha_{n+1}-1+\alpha_n)\geq\delta$, where $\delta=\frac{1}{4}$. Thus, we have $\Gamma_{n+1}-\Gamma_n\leq-\delta\|x_{n+1}-x_n\|^2$, where $\delta=\frac{1}{4}$. It implies that the sequence $\{\Gamma_n\}$ is non-increasing. Note that

$$\begin{aligned} \Gamma_n &= \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2 \\ &\geq \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2 \end{aligned}$$

and

$$\begin{aligned} \Gamma_{n+1} &= \|x_{n+1}-p\|^2-\alpha_{n+1}\|x_n-p\|^2+2\alpha_{n+1}\|x_{n+1}-x_n\|^2 \\ &\geq -\alpha_{n+1}\|x_n-p\|^2. \end{aligned}$$

From $0\leq\alpha_n\leq\alpha$, we have

$$\begin{aligned} \|x_n-p\|^2 &\leq \alpha_n\|x_{n-1}-p\|^2+\Gamma_n \\ &\leq \alpha\|x_{n-1}-p\|^2+\Gamma_1 \\ &\leq \dots \\ &\leq \alpha^n\|x_0-p\|^2+(1+\cdots+\alpha^{n-1})\Gamma_1 \\ &\leq \alpha^n\|x_0-p\|^2+\frac{\Gamma_1}{1-\alpha}, \end{aligned}$$

which implies that sequence $\{x_n\}$ is bounded. Hence,

$$\begin{aligned} -\Gamma_{n+1} &\leq \alpha_{n+1}\|x_n-p\|^2 \\ &\leq \alpha\|x_n-p\|^2 \\ &\leq \alpha^{n+1}\|x_0-p\|^2+\frac{\alpha\Gamma_1}{1-\alpha} \end{aligned}$$

and

$$\begin{aligned} \delta\sum_{n=1}^{k}\|x_{n+1}-x_n\|^2 &\leq \Gamma_1-\Gamma_{k+1} \\ &\leq \Gamma_1+\alpha^{k+1}\|x_0-p\|^2+\frac{\alpha\Gamma_1}{1-\alpha} \\ &= \alpha^{k+1}\|x_0-p\|^2+\frac{\Gamma_1}{1-\alpha} \\ &\leq \|x_0-p\|^2+\frac{\Gamma_1}{1-\alpha}, \end{aligned}$$

which means $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2<+\infty$, and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Since $\alpha_n\le\alpha$, we have

$$\|x_{n+1}-w_n\| \le \|x_{n+1}-x_n\|+\alpha\|x_n-x_{n-1}\|.$$

It follows that $\lim_{n\to\infty}\|x_{n+1}-w_n\|=0$. In view of

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +2\alpha_n\|x_n-x_{n-1}\|^2\\ &\le (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2,\end{aligned}$$

we have $\lim_{n\to\infty}\|x_n-p\|^2=l$. It follows that $\lim_{n\to\infty}\|w_n-p\|^2=l$ and $\lim_{n\to\infty}\|x_n-w_n\|^2=0$. Observe that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\beta_n)\|w_n-p\|^2+\beta_n\|z_n-p\|^2-\beta_n(1-\beta_n)\|Uz_n-w_n\|^2\\ &\le (1-\beta_n)\|w_n-p\|^2+\beta_n\|z_n-p\|^2,\end{aligned}$$

that is,

$$\|z_n-p\|^2\ge\frac{\|x_{n+1}-p\|^2-\|w_n-p\|^2}{\beta_n}+\|w_n-p\|^2.$$

From the fact that the sequence $\{\beta_n\}$ is bounded, we have

$$\liminf_{n\to\infty}\|z_n-p\|^2\ge\lim_{n\to\infty}\|w_n-p\|^2=l$$

and

$$\limsup_{n\to\infty}\|z_n-p\|^2\le\lim_{n\to\infty}\|w_n-p\|^2=l.$$

It follows that $\lim_{n\to\infty}\|z_n-p\|^2=l$ and $\lim_{n\to\infty}\|y_n-w_n\|=0$. From

$$\begin{aligned}\|z_n-y_n\| &= \|y_n-\tau_n(Ay_n-Aw_n)-y_n\|\\ &= \tau_n\|Aw_n-Ay_n\|\\ &\le \mu\|w_n-y_n\|,\end{aligned}$$

we have $\|z_n-y_n\|\le\mu\|w_n-y_n\|$ and $\lim_{n\to\infty}\|z_n-y_n\|=0$. It follows that

$$\lim_{n\to\infty}\|z_n-w_n\|\le\lim_{n\to\infty}(\|z_n-y_n\|+\|y_n-w_n\|)=0,$$

which implies $\lim_{n\to\infty}\|z_n-w_n\|=0$ and $\lim_{n\to\infty}\|Uz_n-w_n\|=0$. Since

$$\lim_{n\to\infty}\|Uz_n-z_n\|\le\lim_{n\to\infty}(\|Uz_n-w_n\|+\|w_n-z_n\|)=0,$$

we have $\lim_{n\to\infty}\|Uz_n-z_n\|=0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q\in H$ such that $x_{n_k}\rightharpoonup q$. So, we have $w_{n_k}\rightharpoonup q$ and $z_{n_k}\rightharpoonup q$. Since $z_{n_k}\rightharpoonup q$ and $I-U$ is demiclosed at zero, we have $q\in Fix(U)$. By $y_{n_k}=P_C(w_{n_k}-\tau_{n_k}Aw_{n_k})$ and the monotonicity of A, we get

$$\begin{aligned}0 &\le \langle y_{n_k}-w_{n_k}+\tau_{n_k}Aw_{n_k},x-y_{n_k}\rangle, \quad \forall x\in C\\ &= \langle y_{n_k}-w_{n_k},x-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},x-y_{n_k}\rangle, \quad \forall x\in C\\ &= \langle y_{n_k}-w_{n_k},x-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},w_{n_k}-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},x-w_{n_k}\rangle, \quad \forall x\in C\\ &\le \langle y_{n_k}-w_{n_k},x-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},w_{n_k}-y_{n_k}\rangle+\tau_{n_k}\langle Ax,x-w_{n_k}\rangle, \quad \forall x\in C.\end{aligned}$$

From Lemma 3.2, we assume that the limit of $\{\tau_{n_k}\}$ exists. By taking the limit, we get $\langle Ax,x-q\rangle\ge 0, \quad \forall x\in C$. It follows that $q\in Fix(U)\cap VI(C,A)$. By Lemma 2.5, we get the conclusion that the sequence $\{x_n\}$ converges weakly to an element of $Fix(U)\cap VI(C,A)$. This completes the proof. $\square$

3.2. **Strong convergence.** In this section, we introduce a strong convergence algorithm which was based on the inertial hybrid method and the Mann-type Tseng's extragradient algorithm.

Let H be a Hilbert space. Let C be a nonempty closed convex subset of H. Let $A: H \to H$ be a mapping, and let $U: H \to H$ be a quasi-nonexpansive mapping.

Algorithm 3.6. Let $x_0, x_1 \in H$ be arbitrarily fixed. Calculate x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}),\\ y_n = P_C(w_n - \tau_n A w_n),\\ z_n = y_n - \tau_n(Ay_n - Aw_n),\\ v_n = (1-\beta_n)w_n + \beta_n U z_n,\\ C_n = \{u \in H : \|v_n - u\|^2 \le \|x_n - u\|^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2\|x_{n-1} - x_n\|^2\},\\ Q_n = \{u \in H : \langle u - x_n, x_1 - x_n\rangle \le 0\},\\ x_{n+1} = P_{C_n\cap Q_n} x_1, \end{cases} \tag{3.9}$$

for each $n \ge 1$, where $\tau_n > 0$. If $y_n = w_n$, then calculate x_{n+1} and the next iterative process steps; otherwise, set $n := n+1$ and go to (3.9) to calculate the next iterate x_{n+2}.

Theorem 3.7. *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $A: H \to H$ be a monotone and L-Lipschitz continuous mapping, and let U be a quasi-nonexpansive mapping. Assume that $Fix(U)\cap VI(C,A)$ is nonempty, and $\{\alpha_n\}$ is non-decreasing real sequence such that $0 \le \alpha_n \le \alpha < 1$, $\{\beta_n\} \subset [a,b] \subset (0,1)$, and $0 < \tau \le \tau_n \le \frac{1}{L}$. Assume that $I-U$ is demiclosed at zero. Then the sequence $\{x_n\}$ generated by Algorithm 3.6 converges strongly to $x^* = P_{Fix(U)\cap VI(C,A)} x_1$.*

Proof. The proof is split into four steps.

Step 1. Prove that $Fix(U)\cap VI(C,A) \subset C_n \cap Q_n$ for each $n \in \mathbb{N}$.

Obviously, C_n and Q_n are self-spaces for each $n \in \mathbb{N}$. Let $u \in Fix(U)\cap VI(C,A)$, we obtain

$$\begin{aligned} \|v_n - u\|^2 &= (1-\beta_n)\|w_n - u\|^2 + \beta_n\|Uz_n - u\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2\\ &\le (1-\beta_n)\|w_n - u\|^2 + \beta_n\|z_n - u\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2\\ &\le (1-\beta_n)\|w_n - u\|^2 + \beta_n\|w_n - u\|^2 - \beta_n(1-\beta_n)\|Uz_n - w_n\|^2\\ &\le \|w_n - u\|^2. \end{aligned}$$

By use of the expression of w_n, we have

$$\|w_n - u\|^2 = \|x_n - u\|^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2\|x_{n-1} - x_n\|^2.$$

It follows that

$$\|v_n - u\|^2 \le \|x_n - u\|^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2\|x_{n-1} - x_n\|^2.$$

Therefore, $u \in C_n$ for each $n \in \mathbb{N}$. So, $Fix(U)\cap VI(C,A) \subset C_n$ for each $n \in \mathbb{N}$. For $n = 1$, we have $Q_1 = H$ and hence $Fix(U)\cap VI(C,A) \subset C_1 \cap Q_1$. Assume that x_k is given and $Fix(U)\cap VI(C,A) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. It follows that

$$\langle y - x_{k+1}, x_1 - x_{k+1}\rangle \le 0, \quad \forall y \in Fix(U)\cap VI(C,A).$$

Thus, it implies that $Fix(U)\cap VI(C,A)\subset Q_{k+1}$. So, $Fix(U)\cap VI(C,A)\subset C_{k+1}\cap Q_{k+1}$. By introduction, we obtain $Fix(U)\cap VI(C,A)\subset C_n\cap Q_n$ for each $n\in\mathbb{N}$.

Step 2. Prove that $\{x_n\}$ is bounded. In view of $\langle y-x_n,x_1-x_n\rangle\leq 0, \forall y\in Q_n$, we have $x_n=P_{Q_n}x_1$ and hence $\|x_n-x_1\|\leq\|x_1-y\|, \forall y\in Q_n$. Since $Fix(U)\cap VI(C,A)\subset Q_n$, we obtain $\|x_n-x_1\|\leq \|x_1-y\|, \forall y\in Fix(U)\cap VI(C,A)$. Since $x_{n+1}\in Q_n$, we have $\|x_n-x_1\|\leq\|x_{n+1}-x_1\|$. Thus, $\lim_{n\to\infty}\|x_n-x_1\|$ exists. It implies that $\{x_n\}$ is bounded.

Step 3. Prove that $\omega_w(x_n)\subset Fix(U)\cap VI(C,A)$.

By use of $x_n=P_{Q_n}x_1$, and $x_{n+1}\in Q_n$, we have

$$\|x_{n+1}-x_n\|^2\leq\|x_{n+1}-x_1\|^2-\|x_n-x_1\|^2.$$

Therefore, $x_{n+1}-x_n\to 0$, as $n\to\infty$. Note that $\|w_n-x_n\|=\alpha_n\|x_n-x_{n-1}\|$ and $\{x_n\}$ is bounded, we obtain $w_n-x_n\to 0$, as $n\to\infty$. Since $x_{n+1}\in C_n$, we have

$$\|v_n-x_{n+1}\|^2 \leq \|x_n-x_{n+1}\|^2+2\alpha_n\|x_n-x_{n+1}\|\|x_{n-1}-x_n\|+\alpha_n^2\|x_{n-1}-x_n\|^2.$$

Thus $v_n-x_{n+1}\to 0$, as $n\to\infty$ and $v_n-w_n\to 0$, as $n\to\infty$. Since $v_n=(1-\beta_n)w_n+\beta_nUz_n$, we have $Uz_n-w_n=\frac{1}{\beta_n}(v_n-w_n)$ and $Uz_n-w_n\to 0$, as $n\to\infty$. Since

$$\|v_n-u\|^2-\|w_n-u\|^2=\|w_n-v_n\|^2-2\langle w_n-u,w_n-v_n\rangle.$$

we have $\|v_n-u\|^2-\|w_n-u\|^2\to 0$, as $n\to\infty$. Observe that

$$\begin{aligned}\|v_n-u\|^2 &= \|(1-\beta_n)(w_n-u)+\beta_n(Uz_n-u)\|^2\\ &= (1-\beta_n)\|w_n-u\|^2+\beta_n\|Uz_n-u\|^2-\beta_n(1-\beta_n)\|Uz_n-w_n\|^2\\ &\leq (1-\beta_n)\|w_n-u\|^2+\beta_n\|z_n-u\|^2.\end{aligned}$$

Thus, $\|z_n-u\|^2-\|w_n-u\|^2\geq\frac{1}{\beta_n}(\|v_n-u\|^2-\|w_n-u\|^2)$. It follows that

$$\|z_n-u\|^2-\|w_n-u\|^2\leq 0.$$

Therefore, $\|z_n-u\|^2-\|w_n-u\|^2\to 0$, as $n\to\infty$ and $\|w_n-y_n\|\to 0$, as $n\to\infty$. Thus, $\|z_n-w_n\|\to 0$, as $n\to\infty$ and $Uz_n-z_n\to 0$, as $n\to\infty$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q\in H$ such that $x_{n_k}\rightharpoonup q$. So, we have $w_{n_k}\rightharpoonup q$ and $z_{n_k}\rightharpoonup q$. Since $z_{n_k}\rightharpoonup q$ and $I-U$ is demiclosed at zero, we have $q\in Fix(U)$. From the facts that $w_{n_k}\rightharpoonup q$, $y_{n_k}=P_C(w_{n_k}-\tau_{n_k}Aw_{n_k})$ and A is monotone, we get

$$\begin{aligned}0 &\leq \langle y_{n_k}-w_{n_k}+\tau_{n_k}Aw_{n_k},x-y_{n_k}\rangle\\ &= \langle y_{n_k}-w_{n_k},x-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},x-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},x-w_{n_k}\rangle\\ &\leq \langle y_{n_k}-w_{n_k},x-y_{n_k}\rangle+\tau_{n_k}\langle Aw_{n_k},w_{n_k}-y_{n_k}\rangle+\tau_{n_k}\langle Ax,x-w_{n_k}\rangle, \quad \forall x\in C.\end{aligned}$$

We assume that the limit of $\{\tau_{n_k}\}$ exists. By taking the limit, we get $\langle Ax,x-q\rangle\geq 0, \forall x\in C$. By Lemma 2.5, we have $q\in VI(C,A)$. Thus, $q\in Fix(U)\cap VI(C,A)$. Therefore, we obtain $\omega_w(x_n)\subset Fix(U)\cap VI(C,A)$.

Step 4. Prove that $x_n\to x^*$, as $n\to\infty$.

Since the norm is convex and lower semicontinuity and $z\in Fix(U)\cap VI(C,A)$, it follows that

$$\|x_1-x^*\|\leq\|x_1-z\|\leq\liminf_{i\to\infty}\|x_{n_i}-x_1\|\leq\limsup_{i\to\infty}\|x_{n_i}-x_1\|\leq\|x_1-x^*\|.$$

Thus,

$$\lim_{i\to\infty}\|x_{n_i}-x_1\| = \|x_1-z\| = \|x_1-x^*\|.$$

Since $x^* = P_{Fix(U)\cap VI(C,A)}x_1$, we have $z = x^*$. So, $\lim_{n\to\infty}\|x_n-x_1\| = \|x_1-x^*\|$. Since $x_n \rightharpoonup x^*, \quad n\to\infty$, we obtain $x_n - x_1 \rightharpoonup x^*-x_1$. Since the space is Hilbert, we can obtain $x_n-x_1 \to x^*-x_1$. Thus, $x_n \to x^*, \quad n\to\infty$. The proof is completed. □

4. NUMERICAL EXPERIMENTS

In this section, we consider some numerical examples to illustrate the efficiency and advantages of our algorithms in comparisons with Algorithm 3.1 [20], the extragradient method and the gradient method. The projections over C are computed effectively by the function *quadprog* in Matlab 7.0 Optimization Toolbox.

In the following, we give the following examples.

Example 4.1. Let $C = [-2,5]$, and $H = \mathbb{R}$. Let $A:\mathbb{R}\to\mathbb{R}$ be defined by $Ax := x+\sin x$ and $U:\mathbb{R}\to\mathbb{R}$ be defined by $Ux := \frac{x}{2}\sin x$. For all $x,y\in H$, we have

$$\|Ax-Ay\| = \|x+\sin x-y-\sin y\| \le \|x-y\| + \|\sin x-\sin y\| \le 2\|x-y\|,$$

and

$$\langle Ax-Ay, x-y\rangle = (x+\sin x-y-\sin y)(x-y) = (x-y)^2 + (\sin x-\sin y)(x-y) \ge 0.$$

Thus, $\|Ax-Ay\| \le L\|x-y\|$, where $L=2$ and $\langle Ax-Ay,x-y\rangle \ge 0$. So, A is L-Lipschitz continuous and monotone. It is easy to find that $VI(C,A) = \{0\}$. If $x\neq 0$ and $Ux = x$, then $x = \frac{x}{2}\sin x$, and $\sin x = 2$, which is impossible. So, we obtain $x = 0$, which implies $Fix(U) = \{0\}$. For all $x\in\mathbb{R}$, we have

$$\|Ux-0\| = \|\frac{x}{2}\sin x\| \le \|\frac{x}{2}\| < \|x\| = \|x-0\|,$$

which implies that U is quasi-nonexpansive. Letting $x = 2\pi$ and $y = \frac{3\pi}{2}$, we have

$$\|Ux-Uy\| = \|\frac{2\pi}{2}\sin 2\pi - \frac{3\pi}{4}\sin\frac{3\pi}{2}\| = \frac{3\pi}{4} > \|2\pi-\frac{3\pi}{2}\| = \frac{\pi}{2},$$

which implies that U is not a nonexpansive mapping. Since U is continuous, C is finite-dimensional, it satisfies the demiclosed principle. We denote $x^* = 0$. The numerical results for this example are shown in Figure 1 and Table 1. The starting point is $x_0 = x_1 = 1\in\mathbb{R}$ for Algorithm 3.1. From Figure 1, we see that Algorithm 3.1 converges faster than the Algorithm 3.1 studied in [20]. For Algorithm 3.6, we choose $x_0 = x_1 = 1,2,3\in\mathbb{R}$, respectively. Moreover, we use $\|x_n-x^*\| \le 10^{-5}$ as the stopping criterion. From Table 1, it is easy to see that Algorithm 3.6 converges in a shorter iterate number than the extragradient method and the gradient method.

Example 4.2. Consider the operator $U:\mathfrak{R}^m\to\mathfrak{R}^m$ with $Ux = -\frac{1}{2}x$ and a linear operator $A:\mathfrak{R}^m\to\mathfrak{R}^m$ $(m = 10,20)$ in the form $A(x) = Mx+q$ [21, 22], where

$$M = NN^T + S + D,$$

N is a $m\times m$ matrix, S is a $m\times m$ skew-symmetric matrix, D is a $m\times m$ diagonal matrix which its diagonal entries are nonnegative, and $q\in\mathfrak{R}^m$ is a vector. Therefore M is positive definite. The feasible set is

$$C = \{x = (x_1,\dots,x_m)\in\mathfrak{R}^m : -2\le x_i\le 5,\, i = 1,2,\dots,m\}.$$

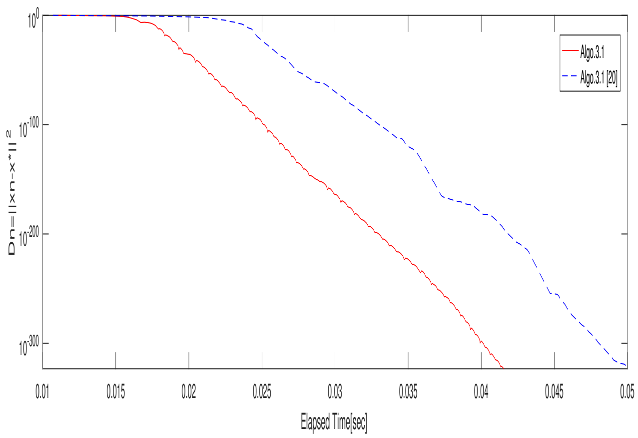

FIGURE 1. Experiment for Example 4.1

TABLE 1. Numerical results as regards Example 4.1

x_1	λ	Algo. 3.6.		Extragradient method		Gradient method	
		Iter.	Time [s]	Iter.	Time [s]	Iter.	Time [s]
1	0.05	14	0.88	124	3.67	111	1.88
	0.02	17	0.92	297	8.04	285	3.77
	0.01	18	0.94	586	14.81	575	7.13
2	0.05	16	0.89	137	3.63	119	1.79
	0.02	19	0.92	318	8.04	305	3.96
	0.01	20	1.01	628	15.73	616	7.45
3	0.05	16	0.92	139	3.82	126	1.84
	0.02	19	0.93	334	8.46	322	4.21
	0.01	20	0.97	661	16.21	648	8.07

It is obvious that A is monotone and Lipschitz continuous. For the experiments, q is equal to zero vector, all the entries of N, S are generated randomly and uniformly in $[-2,2]$, and the diagonal entries of D are in $(0,2)$. We choose $x_0 = x_1 = (1,1,\dots,1) \in \mathfrak{R}^m$. Moreover, it is easy to see that $Fix(U) \cap VI(C,A) = \{(0,0,\cdots,0)^T\}$. Denote $x^* = (0,0,\cdots,0)^T$. The results are described in Figures 2 and 3. According to Figures 2 and 3, we see that the proposed algorithm has the competitive advantages over the Algorithm 3.1 in [20].

Example 4.3. Let $H = \mathbb{R}^m$. Define the feasible set by $C = \mathbb{R}^m$ and $A : \mathbb{R}^m \to \mathbb{R}^m$ is a linear operator in the form $Ax := Mx$ and $U : \mathbb{R}^m \to \mathbb{R}^m$ is given by $Ux := -\frac{1}{2}x$. for each $x \in \mathbb{R}^m$, where $M = (a_{i,j})_{1 \le i,j \le m}$ is a matrix in $\mathbb{R}^{m \times m}$ whose terms are given by

$$a_{i,j} = \begin{cases} -1, & \text{if} \quad j = m+1-i \quad \text{and} \quad j > i, \\ 1, & \text{if} \quad j = m+1-i \quad \text{and} \quad j < i, \\ 0, & \text{otherwise.} \end{cases}$$

Then A is monotone and $\|M\|$-Lipschitz continuous. This is the classical example of the problem that ordinary gradient methodes does not converge. It is easy to see that $VI(C,A) = A^{-1}(0)$, the zero vector is the unique element in $VI(C,A)$, and $Fix(U) \cap VI(C,A) = \{(0,0,\cdots,0)^T\}$.

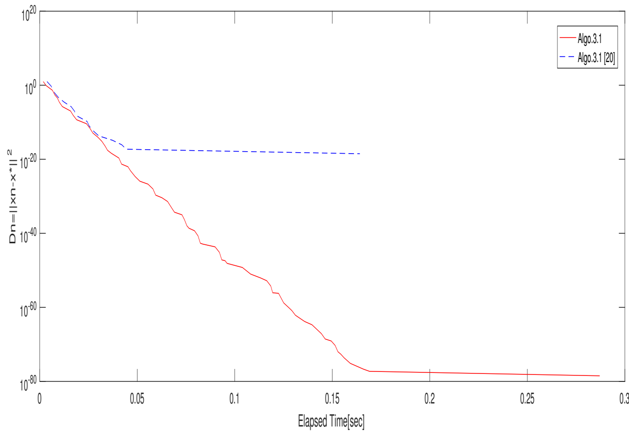

FIGURE 2. Experiment with m=10 for Example 4.2.

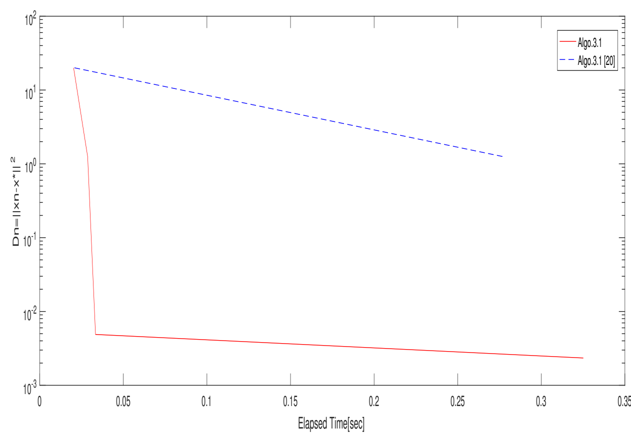

FIGURE 3. Experiment with m=20 for Example 4.2.

TABLE 2. Numerical results as regards Example 4.3

m	λ	Algo 3.6		Extragradient method	
		Iter.	Time [s]	Iter.	Time [s]
20	$0.1/\|M\|$	78	0.87	765	0.10
	$0.05/\|M\|$	83	0.97	3046	0.35
40	$0.1/\|M\|$	65	0.92	835	0.11
	$0.05/\|M\|$	61	0.85	3323	0.43
80	$0.1/\|M\|$	93	1.34	905	0.17
	$0.05/\|M\|$	99	1.41	3601	0.65

Denote $x^* = (0,0,\cdots,0)^T$, and take $\|x_n - x^*\| \leq 10^{-1}$ as the stopping criterion. Choose $x_1 = (1,1,\cdots,1)^T$ for each iterative scheme and take $\tau_n = \tau$ in the iterative scheme. Taking $x_0 = (2,2,\cdots,2)^T$ and $\alpha_n = 1/2$ in our iterative scheme, We show the numerical results for the case $m = 20, 40, 80$, respectively, in Table 2.

Example 4.4. Consider the Algorithm 3.6 in an infinite-dimensional Hilbert space $H = L^2([0,1])$ with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \quad \forall x, y \in H,$$

and the induced norm

$$\|x\| := (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}, \quad \forall x \in H.$$

Let the operator $A : H \to H$ be defined by $(Ax)(t) = \max\{0, x(t)\}$, $\forall x \in H$. It is easy to show that the operator $A : H \to H$ is monotone and 1-Lipschitz continuous:

$$(Ax)(t) = \max\{0, x(t)\} = \frac{x(t) + |x(t)|}{2}, \quad \forall x \in H.$$

$$\begin{aligned} \langle Ax - Ay, x - y\rangle &= \int_0^1 (Ax(t) - Ay(t))(x(t) - y(t))dt \\ &= \int_0^1 \frac{x(t) - y(t) + |x(t)| - |y(t)|}{2}(x(t) - y(t))dt \\ &= \int_0^1 \frac{1}{2}[(x(t) - y(t))^2 + (|x(t)| - |y(t)|)(x(t) - y(t))]dt \\ &\geq 0. \end{aligned}$$

Thus, the operator A is monotone.

$$\begin{aligned} \|Ax - Ay\|^2 &= \int_0^1 |Ax(t) - Ay(t)|^2 dt \\ &= \int_0^1 |\frac{x(t) - y(t) + |x(t)| - |y(t)|}{2}|^2 dt \\ &= \frac{1}{4}\int_0^1 |x(t) - y(t) + |x(t)| - |y(t)||^2 dt \\ &\leq \int_0^1 |x(t) - y(t)|^2 dt \\ &= \|x - y\|^2. \end{aligned}$$

Therefore, the operator A is 1-Lipschitz continuous. Let $C := \{x \in H : \|x\| \leq 1\}$. The set of solutions to the variational inequality VIP is $VI(C, A) = \{0\} \neq \emptyset$. Let $U : L^2([0,1]) \to L^2([0,1])$ be of the form $(Ux)(t) = \int_0^1 tx(s)ds, t \in [0,1]$. Indeed, $0 \in Fix(U)$. So $Fix(U) \neq \emptyset$ and

$$\begin{aligned} |Ux(t) - Uy(t)|^2 &= |\int_0^1 t(x(s) - y(s))ds|^2 \\ &\leq (\int_0^1 t|x(s) - y(s)|ds)^2 \\ &\leq \int_0^1 |x(s) - y(s)|^2 ds. \end{aligned}$$

Hence, $\|Ux - Uy\|^2 \leq \|x - y\|^2$. We have that U is nonexpansive. Since $Fix(U) \cap VI(C, A) = \{0\} \neq \emptyset$, we denote $x^* = 0$, and choose $x_0(t) = x_1(t) = t^2$ and $x_0(t) = x_1(t) = \frac{t}{3}$, respectively.

TABLE 3. Numerical results as regards Example 4.4

x_0	x_1	ε	Algo. 3.6	
			Iter.	*Time*[*s*]
t^2	t^2	10^{-2}	21	9.59
		7×10^{-3}	23	11.04
$\frac{t}{3}$	$\frac{t}{3}$	10^{-2}	15	2.31
		7×10^{-3}	16	2.46

TABLE 4. Numerical results as regards Example 4.5

x_0	x_1	ε	Algo. 3.6	
			Iter.	*Time*[*s*]
$\frac{1}{5}t^2e^{-3t}$	$\frac{1}{4}t^2e^{-3t}$	10^{-3}	3	4.68
		7×10^{-4}	4	5.97
$\cos t$	$3\cos t$	10^{-3}	1	4.67
		7×10^{-4}	1	4.84

The numerical results are shown in Table 3. We use the condition $\|x_n\|\leq\varepsilon$ to terminate Algorithm 3.6. We mainly consider the iteration step and iteration time of Algorithm 3.6 to verify its effectiveness.

Example 4.5. Suppose that $H=L^2([0,2\pi])$ with norm

$$\|x\|=(\int_0^{2\pi}|x(t)|^2dt)^{\frac{1}{2}},\quad \forall x\in H,$$

and inner product

$$\langle x,y\rangle=\int_0^{2\pi}x(t)y(t)dt,\quad \forall x,y\in H.$$

Consider the operator $A:H\to H$ defined by

$$Ax(t)=\frac{1}{2}\max(0,x(t)),\quad t\in[0,2\pi],\ \forall\, x\in H.$$

From Example 4.4, we see that A is Lipschitz continuous and monotone on H. The feasible set is $C=\{x\in H:\int_0^{2\pi}(t^2+1)x(t)dt\leq1\}$. It is known [23] that a projection formula on a half-space is

$$P_C(x)=\begin{cases}\frac{b-\langle a,x\rangle}{\|a\|^2}a+x, & \langle a,x\rangle>b,\\ x, & \langle a,x\rangle\leq b,\end{cases}$$

where $C:=\{x\in L^2([0,2\pi]):\langle a,x\rangle\leq b\}$, $0\neq a\in L^2([0,2\pi])$ and $b\in\mathbb{R}$. Let $U:L^2([0,2\pi])\to L^2([0,2\pi])$, $(Ux)(t)=x(t)$. It is easy to see that $Fix(U)\cap VI(C,A)=\{0\}\neq\emptyset$. We choose $x_0(t)=\frac{1}{5}t^2e^{-3t}$, $x_1(t)=\frac{1}{4}t^2e^{-3t}$ and $x_0(t)=\cos t$, $x_1(t)=3\cos t$, respectively. The numerical results are shown in Table 4. We use the condition $\|x_{n+1}-x_n\|\leq\varepsilon$ to terminate Algorithm 3.6.

5. CONCLUSION

In this paper, we presented two new algorithms, which are based on the Tseng's extragradient method for solving the monotone variational inequality problem and the fixed-point problem of a quasi-nonexpansive mapping in a real Hilbert space. Under suitable conditions, we proved the convergence of the Algorithms. It is worth mentioning that Algorithm 3.1 does not need to require the information of the Lipschitz constant of the operator A and only has one projection in each iteration. Algorithm 3.6 is a strong convergence iterative method with the inertial acceleration. Some numerical experiments are performed to illustrate the advantages of our algorithms compared with existing ones.

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REFERENCES

[1] N.T. An, Solving k-center problems involving sets based on optimization techniques, J. Global Optim. 76 (2020), 189–209.
[2] D.R. Sahu, J.C. Yao, M. Verma, K.K. Shukla, Convergence rate analysis of proximal gradient methods with applications to composite minimization problems, Optimization, (2020), 10.1080/02331934.2019.1702040.
[3] T.H. Cuong, J.C. Yao, N.D. Yen, Qualitative properties of the minimum sum-of-squares clustering problem, Optimization, (2020), 10.1080/02331934.2020.1778685.
[4] X. Qin, N.T. An, Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets, Comput. Optim. Appl. 74 (2019), 821-850.
[5] G.M. Koepelevich, The extragradient method for finding saddle points and other problem, Ekon. Mat. Metody 12 (1976), 747-756.
[6] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J.Control Optim. 38 (2000), 431-446.
[7] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, J.Optim. Theory Appl. 148 (2011), 318-335.
[8] L. Liu, A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings, J. Nonlinear Convex Anal. 20 (2019), 471-488.
[9] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019), 1497-1506.
[10] Y. Shehu, O.S. Iyiola, Convergence analysis for the proximal split feasibility problem using an inertial extrapolation term method, J.Fixed Point Theory Appl. 19 (2017), 2483-2510.
[11] B. Tan, S. Xu, S. Li, Inertial shrinking projection algorithms for solving hierarchical variational inequality problems, J. Nonlinear Convex Anal. 21 (2020), 871–884.
[12] R.I. Bot, E.R. Csetnek, An inertial forward–backward–forward primal-dual splitting algorithm for solving monotone inclusion problems, Numer. Algo. 71 (2016), 519–540.
[13] F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Anal. 9 (2001), 3-11.
[14] Q.L. Dong, Y.J. Cho, L.L. Zhong, T.M. Rassias, Inertial projection and contraction algorithms for variational inequalites, J. Glob. Optim. 70 (2018), 687-704.
[15] D.V. Thong, D.V. Hieu, Weak and strong convergence theorems for variational inequality problems, Numer. Algo. 78 (2017), 1045-1060.
[16] Q.L. Dong, D. Jiang, P. Cholamjiak, Y. Shehu, A strong convergence result involving an inertial forward-backward algorithm for monotone inclusions, J. Fixed Point Theory Appl. 19 (2017), 3097-3118.
[17] K. Goebel, S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Marcel Dekker, Nwe York, 1984.
[18] R.W. Cottle, J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl. 75 (1992), 281-295.

[19] H.K. Xu, Averaged mappings and the gradient-projection algorithm, J. Optim. Theory Appl. 150 (2011), 360-378.
[20] D.V. Thong, D.V. Hieu, New extragradient methods for solving variational inequality problems and fixed point problems, J. Fixed Point Theory Appl. 20 (2018), 129.
[21] P.T. Harker, J.S. Pang, A damped-Newton method for the linear complementarity problem. In: Allgower, G., Georg K. (eds.) Computational Solution of Nonlinear Systems of Equations, Lectures in Applied Mathematics, vol. 26, pp. 265–284. American Mathematical Society, Providence, RI, 1990.
[22] D.V. Hieu, P.K. Anh, L.D. Muu, Modified hybrid projection methods for finding common solutions to variational inequality problems, Comput. Optim. Appl. 66 (2017), 75-96.
[23] A. Cegielski, Iterative Method for Fixed Point Problem in Hilbert Spaces, Springer, Berlin., 2012.