J. Nonlinear Funct. Anal. 2020 (2020), Article ID 36 https://doi.org/10.23952/jnfa.2020.36

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

SOME FIXED POINT RESULTS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we obtain some fixed point results for nonexpansive mappings defined on an intersection of a finite number of closed bounded and convex nonempty subsets in Banach spaces. Our results refine those established by Lin [Unconditional bases and fixed points of nonexpansive mappings, Pacific J. Math. 116 (1985), 69-76].

Keywords. Nonexpansive mapping; Fixed point; Minimal weakly compact convex subset; Goebel-Karlovitz lemma; Suzuki mapping.

1. Introduction

In the setting of Banach spaces, the study of fixed points for nonexpansive self-mappings defined on weakly compact convex subsets is an interesting subject in nonlinear functional analysis. After the contributions of Browder, Göhde and Kirk (see [1, 2, 3]), this field was extensively explored. The notion of the normal structure was the first powerful tool to study the existence of fixed points for such mappings. In addition, it is well known that compact convex subsets have normal structure (see [4], p. 39), which is not the case of weakly compact convex subsets. The first example linked to this question was established by Alspach who constructed a fixed point free nonexpansive self-mapping on the set

$$\widetilde{K}=\left\{f\in L^1([0,1]),\int_0^1 f(t)dt=1,\ 0\leq f\leq 2\ \ a.e.\right\}.$$

For more details, we refer to [5].

A Banach space X is said to have the weak fixed point property (in abbreviation, wfpp) if, for each weakly compact convex subset K of X, every nonexpansive self-mapping $T:K\longrightarrow K$ has a fixed point. As an example, uniformly convex Banach spaces have the wfpp since it was proved that closed bounded convex subsets of such reflexive spaces have the normal structure, covering

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Received December 6, 2019; Accepted August 19, 2020.

in particular Lebesgue spaces $L_p([0,1])(1<p<\infty)$. Unfortunately, the problem whether every reflexive Banach space has the wfpp is still open. If X is a Banach space with the Schur property, it is obvious to assert that it has the wfpp since in this case, the existence of fixed points is an immediate consequence of Schauder's fixed point theorem.

One of the most important problems in the fixed point theory is related to Banach spaces with unconditional basis and it was solved partially by Lin [6] who proved by using the techniques of the ultraproduct that every Banach space X with an unconditional basis constant $\lambda_0 < \dfrac{\sqrt{33}-3}{2}$ has the wfpp. On this subject, we refer to, for example, [6, 7, 8] and the references therein.

In this paper, inspired by the ideas given in [6], we show the existence of fixed points for nonexpansive self-mappings defined on an intersection of finite number of closed bounded convex subsets of a Banach space when the sum of their diameters satisfy appropriate estimations together with some convenable additional assumptions. We also establish a general setting, extending in particular the case of Banach spaces with 1-unconditional basis. Our results are illustrated by some examples.

2. Preliminaries

In this section, we give some notations and lemmas which will be used in the sequel.

Definition 2.1. Let K be a nonempty subset of a Banach space X and let $T : K \longrightarrow K$ be a self-mapping. A sequence (x_n) in K is called an approximate fixed point sequence for T if

$$\lim_{n\longrightarrow+\infty} \|x_n - Tx_n\| = 0.$$

Definition 2.2. Let K be a nonempty subset of a Banach space X. A self-mapping $T : K \longrightarrow K$ is said to be nonexpansive if, for all $x, y \in K$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Remark 2.3. The family of nonexpansive self-mappings on a nonempty subset can be seen as a semigroup. This idea allows several authors to study common fixed points for semi-topological semigroups acting on convex subsets of Banach spaces. For a good reading on this area, we refer to the contributions of Lau and his coauthors [9, 10, 11, 12, 13, 14, 15].

Let us now give the following useful lemma.

Lemma 2.4. *(see [6, 16]) Let K be a nonempty closed convex bounded subset of a Banach space X and let $T : K \longrightarrow K$ be a nonexpansive self-mapping. Then T posses an approximate fixed point sequence in K.*

The following lemma was established independently at the same time by Karlovitz [17] and Goebel ([4, 18]) and plays an important role in the investigation of fixed points of nonexpansive self-mappings.

Lemma 2.5. *Let K be a subset of a Banach space X which is minimal with respect to being nonempty weakly compact convex and T-invariant for a nonexpansive self-mapping T. Then, for every approximate fixed point sequence (x_n) in K, we have*

$$\text{for each } x \in K,\ \lim_{n\longrightarrow+\infty} \|x - x_n\| = diam(K).$$

The following corollary is an immediate consequence of Lemma 2.5.

Corollary 2.6. *Let K be a minimal weakly compact convex subset for a nonexpansive self-mapping T on K. If $0 \in K$ and $diam(K) = 1$, then*

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that } \|y\| > 1 - \varepsilon \text{ whenever } \|Ty - y\| < \delta.$$

3. Main Results

Our first main result in this section is given by the following theorem.

Theorem 3.1. *Let X be a Banach space and let $A_1, A_2, \ldots, A_{n+1} (n \geq 2)$ be a closed bounded and convex subsets of X such that $\bigcap_{i=1}^{n+1} A_i \neq \emptyset$. Assume that there exists $j_0 \in \{1, 2, \ldots, n+1\}$ such that A_{j_0} is weakly compact and there exist bounded linear operators $S_1, S_2, \ldots., S_n$ on X satisfying the following assumptions:*

($\imath$): *For all $w_0 \in \bigcap_{i=1}^{n+1} A_i$, there exists $z_i \in Ker(S_i) \bigcap \left(A_{i+1} \setminus \bigcup_{k=1, k \neq i+1}^{n+1} A_k \right)$ such that*

$$\|w_0 - z_i\| \leq \frac{diam(A_{i+1})}{2}, \quad i = 1, \ldots, n; \tag{3.1}$$

($\imath\imath$): *For all $w_0 \in \bigcap_{i=1}^{n+1} A_i$, there exists $z_{n+1} \in Ker\left(nI - \sum_{i=1}^{n} S_i\right) \bigcap \left(A_1 \setminus \bigcup_{i=2}^{n+1} A_i\right)$ such that*

$$\|w_0 - z_{n+1}\| \leq \frac{diam(A_1)}{2}. \tag{3.2}$$

Let $\beta = \max\left(\max(\|S_i\|)_{i=1}^n, \|nI - \sum_{i=1}^{n} S_i\|\right)$. If each $A_i, i = 1, \ldots., n+1$ is invariant under a nonexpansive self-mapping T and $\beta \sum_{i=1}^{n+1} diam(A_i) < 2n$, then T has a fixed point in $\bigcap_{i=1}^{n+1} A_i$

Proof. Assume that, for all $i \in \{1, \ldots, n+1\}, A_i$ is invariant under T. Since A_{j_0} is weakly compact, then it is easy to observe that $\bigcap_{i=1}^{n+1} A_i$ is a nonempty (by assumption) weakly compact and convex subset of X, which is invariant under T. If T has no fixed points in $\bigcap_{i=1}^{n+1} A_i$, then $\bigcap_{i=1}^{n+1} A_i$ must contain an approximate fixed point sequence. Let $w_0 \in \bigcap_{i=1}^{n+1} A_i$. Then there exist

$$z_i \in Ker(S_i) \bigcap \left(A_{i+1} \setminus \bigcup_{k=1, k \neq i+1}^{n+1} A_k \right) \quad (i = 1, \ldots, n)$$

and

$$z_{n+1} \in Ker\left(nI - \sum_{i=1}^{n} S_i\right) \bigcap \left(A_1 \setminus \bigcup_{i=2}^{n+1} A_i\right).$$

such that (3.1) and (3.2) are satisfied. It follows that

$$\begin{aligned}
\|w_0\| =& \frac{1}{n}\left[\left\|\left(nI - \sum_{i=1}^{n} S_i\right)(w_0) + \sum_{i=1}^{n} S_i(w_0)\right\|\right] \\
\leq& \frac{1}{n}\left[\left\|\left(nI - \sum_{i=1}^{n} S_i\right)(w_0)\right\| + \sum_{i=1}^{n} \|S_i(w_0)\|\right] \\
\leq& \frac{1}{n}\left[\left\|\left(nI - \sum_{i=1}^{n} S_i\right)(w_0 - z_{n+1})\right\| + \sum_{i=1}^{n} \|S_i(w_0 - z_i)\|\right] \\
\leq& \frac{1}{n}\left[\left\|nI - \sum_{i=1}^{n} S_i\right\| \|w_0 - z_{n+1}\| + \sum_{i=1}^{n} \|S_i\| \|w_0 - z_i\|\right] \\
\leq& \frac{\beta}{2n}\left[diam(A_1) + \sum_{i=1}^{n} diam(A_{i+1})\right] \\
=& \frac{\beta}{2n} \sum_{i=1}^{n+1} diam(A_i) < 1,
\end{aligned}$$

which is a contradiction by Corollary 2.6. □

Remark 3.2. In the proof of Theorem 3.1, it is easy to observe that if each $A_i, i = 2, \ldots\ldots, n+1$ is a ball centered at z_i together with A_1 a ball centered at z_{n+1} having respectively the radius r_i and r_{n+1}, then the real number $\dfrac{diam(A_i)}{2}$ can be replaced by r_i for $i = 2, \ldots., n+1$ and by r_{n+1} for $i = 1$.

Example 3.3. Let X be a Banach space having 1-unconditional basis $(e_n)_{n=1}^{\infty}$. Denote by $\widetilde{X}$ the ultraproduct space of X, which is the quotient space of

$$l^{\infty}(X) = \{(x_n) : x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \|(x_n)_n\| = \sup \|x_n\| < +\infty\}$$

by the closed subspace $\widetilde{M}$ of $l^{\infty}(X)$ given by

$$\widetilde{M} = \{(x_n) \in l^{\infty}(X) : \lim_{n \longrightarrow \mathfrak{I}} \|x_n\| = 0\},$$

where $\lim_{n \longrightarrow \mathfrak{I}} \|x_n\|$ is the limit of $\|x_n\|$ over the free ultrafilter $\mathfrak{I}$ on $\mathbb{N}$. Let C be a closed bounded and convex subset of X and let $T : C \longrightarrow C$ be a nonexpansive self-mapping on C. To prove that T has at least a fixed point, we will show that Lin's reasoning given in [6] becomes a particular case of Theorem 3.1. Assume that T is a fixed point free mapping. Then C must contain a minimal weakly compact and convex subset K. Without loss of generality, we can assume that $0 \in K$ and $diam(K) = 1$. Denote by $\tilde{K} \subseteq \tilde{X}$, the set

$$\tilde{K} = \{(x_n) \in \tilde{X} : x_n \in K \text{ for all } n \in \mathbb{N}\}$$

and let $\tilde{T} : \tilde{K} \longrightarrow \tilde{K}$ be the self-mapping on $\tilde{K}$ given by $\tilde{T}((x_n)) = (T(x_n))$. It is obvious to see that if K is a closed bounded and convex subset of X, then $\tilde{K}$ inherits these properties.

Furthermore, since X has a 1-unconditional basis and T is nonexpansive, then $\tilde{T}$ is nonexpansive with at least two fixed points $\widetilde{y_1}, \widetilde{y_2} \in \tilde{K}, \widetilde{y_1} \neq \widetilde{y_2}$ with $\|\widetilde{y_1} + \widetilde{y_2}\| = \|\widetilde{y_1} - \widetilde{y_2}\| = 1$ (it suffices to take $\widetilde{y_1} = (x_n)_n$ and $\widetilde{y_2} = (x_{n+1})_n$). Let A_1, A_2 and A_3 be the subsets of $\widetilde{X}$ defined by

$$A_1 = \left\{ \widetilde{w} \in \tilde{K} : \text{there exists } x \in K : \|\widetilde{w} - \tilde{x}\| \leq \frac{1}{2} \right\},$$

where $\tilde{x} = (x, x, x, \ldots\ldots..) \in \widetilde{X}$,

$$A_2 = \left\{ \widetilde{w} \in \tilde{K} : \|\widetilde{w} - \tilde{y_1}\| \leq \frac{1}{2} \right\},$$

$$A_3 = \left\{ \widetilde{w} \in \tilde{K} : \|\widetilde{w} - \tilde{y_2}\| \leq \frac{1}{2} \right\}.$$

We have the following claims.

Claim 1. $\bigcap_{i=1}^{3} A_i$ is a nonempty bounded closed and convex $\widetilde{T}$-invariant subset of $\widetilde{X}$.

Indeed, for all $x \in K, \widetilde{x} = (x, x, x, \ldots\ldots) \in A_1$, we have $\widetilde{y_1} \in A_2$ and $\widetilde{y_2} \in A_3$. On the other hand, it is easy to see that each $A_i, i = 1, 2, 3$ is a closed bounded and convex $\widetilde{T}$-invariant subset of $\widetilde{K}$. Furthermore, since

$$\left\| \frac{\widetilde{y_1} + \widetilde{y_2}}{2} - \widetilde{y_1} \right\| = \left\| \frac{\widetilde{y_2} - \widetilde{y_1}}{2} \right\| = \frac{1}{2},$$

$$\left\| \frac{\widetilde{y_1} + \widetilde{y_2}}{2} - \widetilde{y_2} \right\| = \left\| \frac{\widetilde{y_1} - \widetilde{y_2}}{2} \right\| = \frac{1}{2},$$

and

$$\left\| \frac{\widetilde{y_1} + \widetilde{y_2}}{2} - 0 \right\| = \frac{1}{2},$$

we get

$$\frac{\widetilde{y_1} + \widetilde{y_2}}{2} \in \bigcap_{i=1}^{3} A_i.$$

Claim 2. $\widetilde{x} \in (A_1 \backslash (A_2 \bigcup A_3)), \widetilde{y_1} \in (A_2 \backslash (A_1 \bigcup A_3))$ and $\widetilde{y_2} \in (A_3 \backslash (A_1 \bigcup A_2))$.

This follows immediately from the following

$$\|\widetilde{y_1} - \widetilde{y_2}\| = \|\widetilde{y_1} - \widetilde{x}\| = \|\widetilde{y_2} - \widetilde{x}\| = \lim_{n \longrightarrow +\infty} \|x_n - x\| = 1 > \frac{1}{2}.$$

Let $\widetilde{P_1}$ and $\widetilde{P_2}$ a natural bounded projections on $\widetilde{X}$ associated to the basis $(e_n)_n$ for which

$$\widetilde{P_1}(\widetilde{y_1}) = \widetilde{y_1}, \widetilde{P_2}(\widetilde{y_2}) = \widetilde{y_2}, \widetilde{P_1}(\widetilde{x}) = \widetilde{P_2}(\widetilde{x}) = 0.$$

Denote $S_1 = I_{\widetilde{X}} - \widetilde{P_1}$ and $S_2 = I_{\widetilde{X}} - \widetilde{P_2}$.

Claim 3. Mappings S_1 and S_2 satisfy the assumptions $(\imath)$ and $(\imath\imath)$ of Theorem 3.1.

This claim is an immediate consequence of the following assertions:

(a): $\|S_1\| \leq 1, \|S_2\| \leq 1, \|2I_{\widetilde{X}} - (S_1 + S_2)\| = \|\widetilde{P_1} + \widetilde{P_2}\| \leq 1.$

(b): $\widetilde{y_1} \in Ker(S_1) \bigcap (A_2 \backslash (A_1 \bigcup A_3))$ and $\|w_0 - \widetilde{y_1}\| \leq \frac{1}{2}$ for all $w_0 \in \bigcap_{i=1}^{3} A_i$.

(c)**:** $\widetilde{y_2} \in Ker(S_2)\bigcap(A_3\backslash(A_1\bigcup A_2))$ and $\|w_0-\widetilde{y_2}\| \leq \dfrac{1}{2}$ for all $w_0 \in \bigcap_{i=1}^{3} A_i$.

(d)**:** $\widetilde{x} \in Ker(2I_{\widetilde{X}}-(S_1+S_2))\bigcap(A_1\backslash(A_2\bigcup A_3))$ for all $x \in K$ and if $w_0 \in \bigcap_{i=1}^{3} A_i$, then necessarily we have

$$\|w_0-\widetilde{z}\| \leq \frac{1}{2}$$

for some $\widetilde{z} \in Ker(2I_{\widetilde{X}}-(S_1+S_2))\bigcap(A_1\backslash(A_2\bigcup A_3))$.

By taking $\beta = 1$, all the assumptions of Theorem 3.1 are satisfied. Next, following Remark 3.6, we can replace the real number $\dfrac{diam(A_i)}{2}, i = 1,2,3$ by $\dfrac{1}{2}$. Hence, by Theorem 3.1 $\widetilde{T}$ has a fixed point $\widetilde{y} = (x_1, x_2, \ldots\ldots\ldots)$ in $\bigcap_{i=1}^{3} A_i \subseteq \widetilde{K}$, but $\|\widetilde{y}\| = \|\widetilde{y}-0\| = \lim_{n\longrightarrow+\infty} \|x_n-0\| = 1$, which contradicts the proof of Theorem 3.1. From this, we conclude that T must have a fixed point in C.

Corollary 3.4. *Let C be a closed bounded and convex subset of a reflexive Banach space X. Assume that there exist $x_1, x_2, \ldots, x_{n+1} \in C$ such that $x_i \neq x_j$ for all $i, j = 1, 2, \ldots, n+1, i \neq j$. Under assumptions of Theorem 3.1, assume that $0 < \beta r < \dfrac{n}{n+1}$. If each $\overline{B}(x_i, r)(i = 1, \ldots, n+1)$ is invariant by a nonexpansive self-mapping T and $\bigcap_{i=1}^{n+1} \overline{B}(x_i, r) \neq \emptyset$, then T has a fixed point in $\bigcap_{i=1}^{n+1} \overline{B}(x_i, r)$.*

Proof. Here, each $\overline{B}(x_i, r)(i = 1, \ldots.., n+1)$ is a convex and weakly compact subset since X is reflexive. Furthermore, we have $diam(\overline{B}(x_i, r)) = 2r$ and the result follows immediately. □

Now, we are in a position to prove the following theorem.

Theorem 3.5. *Let C be a weakly compact convex subset of a Banach space X and let $T : C \longrightarrow C$ be a nonexpansive self-mapping. Then, for all integer $n \geq 2$, we can not construct a finite family $(A_i)_{i=1}^{n+1}$ of closed bounded and convex subsets of C, which are invariant under T and having the following properties:*

(1) $\bigcap_{i=1}^{n+1} A_i$ *contains an element w_0 of norm 1.*

(2) There exist $(T_i)_{i=1}^{n}$ a bounded linear operators on X such that

$(\imath)$**:** $\max_{1\leq i\leq n} \|I-nT_i\| \leq \alpha_1 \ (\alpha_1 \geq 1)$ *and* $\left\|I-n\sum_{i=1}^{n} T_i\right\| \leq \alpha_2 \ (\alpha_2 > 0)$;

$(\imath\imath)$**:** *For all $y_0 \in \bigcap_{i=1}^{n+1} A_i$, there exists $x_i \in Ker(I-T_i)\bigcap\left(A_{i+1}\backslash \bigcup_{k=1,k\neq i+1}^{n+1} A_k\right)$ satisfying that*

$$\|y_0-x_i\| \leq \frac{diam(A_{i+1})}{2}, \quad i = 1, \ldots, n; \tag{3.3}$$

$(\imath\imath)$**:** *For all* $y_0 \in \bigcap_{i=1}^{n+1} A_i$*, there exists* $x_{n+1} \in Ker\left(\sum_{i=1}^{n} T_i\right) \bigcap \left(A_1 \setminus \bigcup_{i=2}^{n+1} A_i\right)$ *satisfying that*

$$\|y_0 - x_{n+1}\| \leq \frac{diam(A_1)}{2}. \tag{3.4}$$

(3) $\max_{1\leq i\leq n+1} diam(A_i) < \dfrac{4}{\alpha_1+1}$ *and* $diam(A_1) < \dfrac{n\left[4-(\alpha_1+1)\max_{1\leq i\leq n+1} diam(A_i)\right]}{\alpha_2+1}$.

Proof. Suppose that it was not true. From our assumptions, there exist $x_i \in A_{i+1}(i = 1,2,...,n)$ and $x_{n+1} \in A_1$ such that $(\imath)$ and $(\imath\imath)$ of (2) are satisfied if we replace y_0 by w_0. From the Hahn-Banach theorem, let $f_0 \in X^\star$ be such that $f_0(w_0) = 1 = \|f_0\|$. Hence

$$1 - f_0(x_i) = f_0(w_0 - x_i) \leq \|f_0\|\|w_0 - x_i\| \leq \frac{diam(A_{i+1})}{2}$$

and

$$1 - f_0(x_{n+1}) = f_0(w_0 - x_{n+1}) \leq \|f_0\|\|w_0 - x_{n+1}\| \leq \frac{diam(A_1)}{2}. \tag{3.3}$$

So

$$1 - \frac{diam(A_{i+1})}{2} \leq f_0(x_i), \quad i = 1,2,...,n$$

and

$$1 - \frac{diam(A_1)}{2} \leq f_0(x_{n+1}).$$

Putting

$$\alpha_0 = f_0\left[\left(I - \sum_{i=1}^{n} T_i\right)(w_0)\right],$$

we arrive at

$$\begin{aligned} 1-\alpha_0 &= f_0(w_0) - f_0\left[\left(I - \sum_{i=1}^{n} T_i\right)(w_0)\right] \\ &= f_0\left(\left(\sum_{i=1}^{n} T_i\right)(w_0)\right) \\ &= f_0(T_1(w_0)) + f_0(T_2(w_0)) + ... + f_0(T_n(w_0)). \end{aligned} \tag{3.4}$$

Hence, there exists necessarily $l_0 \in \{1,2,...,n\}$ such that

$$f_0(T_{l_0}(w_0)) \leq \frac{1-\alpha_0}{n}.$$

Combining (3.3) and (3.4), we get

$$n(1-\alpha_0) - \frac{diam(A_1)}{2} \leq n[f_0(T_1(w_0)) + ... + f_0(T_n(w_0))] - f_0(w_0 - x_{n+1}).$$

The linearity of f_0 and assumptions $(\imath\imath)$ and $(\imath)$ of (2) give

$$\begin{aligned}
n(1-\alpha_0)-\frac{diam(A_1)}{2} \leq & f_0[(nT_1+....+nT_n)(w_0)]-f_0(w_0-x_{n+1}) \\
= & f_0\left[\left(n\sum_{i=1}^{n}T_i-I\right)(w_0-x_{n+1})\right] \\
\leq & \|f_0\|\left\|I-n\sum_{i=1}^{n}T_i\right\|\|w_0-x_{n+1}\| \\
\leq & \alpha_2\ \frac{diam(A_1)}{2}.
\end{aligned}$$

Thus

$$n(1-\alpha_0)\leq(\alpha_2+1)\frac{diam(A_1)}{2}. \tag{3.5}$$

Next, we have

$$\alpha_0+(1-\frac{diam(A_{l_0+1})}{2})=(1-\frac{diam(A_{l_0+1})}{2})+1-(1-\alpha_0).$$

Now, for

$$x_{l_0}\in\left(A_{l_0+1}\backslash\bigcup_{k=1,k\neq l_0+1}^{n+1}A_k\right)\bigcap Ker(I-T_{l_0}),$$

we obtain

$$\begin{aligned}
\alpha_0+(1-\frac{diam(A_{l_0+1})}{2}) = & (1-\frac{diam(A_{l_0+1})}{2})+1-(1-\alpha_0) \\
\leq & f_0(w_0)+f_0(x_{l_0})-nf_0(T_{l_0}(w_0)) \\
\leq & f_0(w_0-x_{l_0})+nf_0(x_{l_0})-nf_0(T_{l_0}(w_0)) \\
= & f_0(w_0-x_{l_0})+nf_0(T_{l_0}(x_{l_0}))-nf_0(T_{l_0}(w_0)) \\
= & f_0(w_0-x_{l_0})+nf_0\left[T_{l_0}(x_{l_0}-w_0)\right] \\
= & f_0[(I-nT_{l_0})(w_0-x_{l_0})] \\
\leq & \|f_0\|\|I-nT_{l_0}\|\|w_0-x_{l_0}\| \\
\leq & \alpha_1\frac{diam(A_{l_0+1})}{2}.
\end{aligned}$$

Consequently

$$\alpha_0+1\leq(\alpha_1+1)\frac{diam(A_{l_0+1})}{2}. \tag{3.6}$$

Multiplying (3.6) by n, we infer that

$$n(\alpha_0+1)\leq n(\alpha_1+1)\frac{diam(A_{l_0+1})}{2}. \tag{3.7}$$

From inequalities (3.5) and (3.7), we deduce

$$n\left[1-(\alpha_1+1)\frac{diam(A_{l_0+1})}{2}+1\right]\leq(\alpha_2+1)\frac{diam(A_1)}{2}.$$

This leads to

$$n\left[4-(\alpha_1+1)\max_{1\leq i\leq n+1} diam(A_i)\right]\leq(\alpha_2+1)diam(A_1).$$

Finally, we have

$$\frac{n\left[4-(\alpha_1+1)\max_{1\leq i\leq n+1} diam(A_i)\right]}{\alpha_2+1}\leq diam(A_1),$$

which is a contradiction. This completes the proof. □

Remark 3.6. In the proof of Theorem 3.5, it is easy to see that if each $A_i, i=2,......,n+1$ is a ball centered at x_i together with A_1 a ball centered at x_{n+1} having respectively the radius r_i and r_{n+1}, then the real number $\frac{diam(A_i)}{2}$ can be replaced by r_i for $i=2,....,n+1$ and by r_{n+1} for $i=1$.

In the next result, we show that the framework of Theorem 3.5 enable us to give a second proof to the fact that Banach spaces with 1-unconditional basis have the wfpp. More precisely, we have the following.

Corollary 3.7. *(see also Theorems 1 and 2 in [6]) Let C be a weakly compact convex subset of a Banach space X with a 1-unconditional basis and let $T: C\longrightarrow C$ be a nonexpansive mapping. Then T has a fixed point in C.*

Proof. If T has no fixed points in C, then there exists a minimal weakly compact convex subset K of C. Denote by $\widetilde{K}$ its corresponding subset in the ultraproduct space $\widetilde{X}$ of X. Let A_1, A_2 and A_3 be the closed bounded convex subsets of $\widetilde{X}$ defined in Example 3.3. It is easy to see that the assumption (1) of Theorem 3.5 is satisfied. The assumption (2) holds by taking $\alpha_1=\alpha_2=1, T_1=I_{\tilde{X}}-S_1=\tilde{P}_1, T_2=I_{\tilde{X}}-S_2=\tilde{P}_2, x_1=\tilde{y}_1, x_2=\tilde{y}_2$ and $x_3=\tilde{x}$ (depending on w_0) for some $x\in K$ while the third assumption is an immediate consequence of the fact that $diam(A_i)=1<2$ for all $i=1,2,3$. By Theorem 3.5, this is a contradiction. Hence, necessarily, T has a fixed point in C. □

From Theorem 3.5, Corollary 3.7 can be deduced from the following general result.

Corollary 3.8. *Let C be a weakly compact convex subset of a Banach space X and let $T: C\longrightarrow C$ be a nonexpansive mapping. If for every nontrivial minimal weakly compact convex invariant subset K for T, the corresponding subset $\tilde{K}$ in the ultraproduct space $\tilde{X}$, contains a finite family $(A_i)_{i=1}^{n+1} (n\geq 2)$ of closed bounded and convex subsets of $\tilde{X}$ having properties (1), (2) and (3) indicated in Theorem3.5, then T has a fixed point in C.*

In 2008, Suzuki [19] introduced the C_λ-mappings as an extension of nonexpansive mappings.

Definition 3.9. Let K be a nonempty subset of a Banach space X. A self-mapping $T: K\longrightarrow K$ is said to be a C_λ-mapping if for some $\lambda\in(0,1)$ and all $x,y\in K$,

$$\lambda\|x-Tx\|\leq\|x-y\|\Longrightarrow\|Tx-Ty\|\leq\|x-y\|.$$

If $\lambda=\dfrac{1}{2}, thenT$ is said to be a C-mapping.

Remark 3.10. Define the set $\widetilde{S_x}$ by

$$\widetilde{S_x} = \{y \in K : \lambda \|x - Tx\| \leq \|x - y\|\}.$$

Then $Tx \in \widetilde{S_x}$ and $\widetilde{S_x}$ is nonempty.

Remark 3.11. It can be seen that every nonexpansive self-mapping is a C_λ-mapping for every $\lambda \in (0,1)$ but the converse is not true as the following example shows.

Example 3.12. Let $T : [0,3] \longrightarrow [0,3]$ be defined by

$$Tx = \begin{cases} 0, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

Then T is a C-mapping. However, T fails to be nonexpansive since it is not continuous at $x_0 = 3$ (for more details, see [19, 20]).

Remark 3.13. Let K be a nonempty bounded and convex subset of a Banach space X and $\lambda \in (0,1)$. Assume that T is a C_λ-self-mapping on K. Then T has an approximate fixed point sequence $(x_n)_n$ in K (see Lemma 2.2 in [21] and Lemma 3.1 in [22]).

Lemma 3.14. *(see [21], Lemma 2.4) Let T be a continuous C_λ-mapping ($\lambda \in (0,1)$) defined on a minimal weakly compact and convex set K and let $(x_n)_n$ be an approximate fixed point sequence for T in K. Then there exists a real number ρ such that, for all $x \in K$, $\lim_{n\to+\infty} \|x_n - x\| = \rho$. If $\lambda = \frac{1}{2}$, then the continuity assumption can be dropped.*

Remark 3.15. We do not know if Theorems 3.1 and 3.5 also hold for the case of continuous C_λ-mappings ($\lambda \in (0,1)$).

4. Conclusion

In this paper, our fixed point results were established for nonexpansive self-mappings by means of some properties of the intersection of finite number of bounded closed and convex subsets of a Banach space X. They were obtained independently on the geometrical properties of theses subsets but are related directly to their diameters and a specific structure of the Banach algebra $\mathscr{L}(X)$ of bounded linear operators on X.

Acknowledgements
This paper was supported by the research team RPC (Controllability and Perturbation Results) in the laboratory of Informatics and Mathematics (LIM) at the university of Souk-Ahras, Algeria. The authors are very grateful to the anonymous referees for their valuable comments and suggestions which helped us to improve the quality of the manuscript.

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