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A SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR SOLVING SPLIT EQUILIBRIUM AND FIXED POINT PROBLEMS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, a new iterative algorithm with a self-adaptive step size is proposed for split feasibility problem involving bifunctions and Bregman quasi-nonexpansive mappings. A strong convergence theorem is obtained in the framework of real reflexive Banach spaces. An application and an example are presented to illustrate the performance of our algorithm.

Keywords. Equilibrium problem; Strongly pseudomonotone; Strong convergence; Quasi-ϕ-nonexpansive mapping; Fixed point.

1. Introduction

Let C be a nonempty, closed and convex subset of a Banach space E. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a fixed point of T iff $x = Tx$. We denote by $Fix(T)$ the set of fixed points set T, that is, $Fix(T) = \{x \in C : x = Tx\}$.

Recall that a bifunction $g : C \times C \to \mathbb{R}$ is said to be:

(i) monotone on C iff $g(x,y) + g(y,x) \leq 0$, $\forall x,y \in C$;
(ii) pseudomonotone on C iff $g(x,y) \geq 0 \implies g(y,x) \leq 0$, $\forall x,y \in C$;
(iii) strongly pseudomonotone on C iff there exists a constant $\gamma > 0$ such that $g(x,y) \geq 0 \implies g(y,x) \leq -\gamma||x-y||^2$, $\forall x,y \in C$;
(iv) Lipschitz-type condition on C iff there exist two positive constants c_1, c_2 such that

$$g(x,y) + g(y,z) \geq g(x,z) - c_1||x-y||^2 - c_2||y-z||^2, \ \forall x,y,z \in C.$$

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It is easy to see that (i) $\implies$ (ii) and (iii) $\implies$ (iv). For $x,y \in C$, we denote by $\partial g(x,y)$ the subdifferential of the convex function $g(x,\cdot)$ at the second variable, that is,

$$\partial g(x,y) := \{v \in E : g(x,z) \geq g(x,y) + \langle v, z-y\rangle,\ \forall z \in C\}.$$

Let C and Q be nonempty, closed and convex subsets of real reflexive Banach spaces E_1 and E_2 with duals E_1^* and E_2^* respectively. The Split Feasibility Problem (SFP), which introduced by Censor and Elfving [1], consists of finding a point

$$\hat{x} \in C \text{ such that } L\hat{x} \in Q, \tag{1.1}$$

where $L : E_1 \to E_2$ is a bounded linear operator. The SFP has been found useful in the study of many real-world problems, such as, signal processing, medical image reconstruction, intensity modulated therapy and so on (see [2, 3]). The SFP with the sets C and Q being fixed-point or common fixed-point sets of mappings and solutions of other optimization problems have recently been considered in the literature; see, e.g., [4, 5, 6, 7, 8, 9] and the references therein.

Let C be a nonempty, closed and convex subset of a real Banach space E with dual E^*. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction. Let $B : C \to E^*$ be a nonlinear mapping and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a real valued function. The Generalized Mixed Equilibrium Problem (GMEP) is to find a point $\hat{x} \in C$ such that

$$\Theta(\hat{x}, y) + \langle B\hat{x}, y - \hat{x}\rangle + \varphi(y) - \varphi(\hat{x}) \geq 0,\ \forall y \in C. \tag{1.2}$$

We denote by GMEP(Θ,φ) the solution set of GMEP (1.2).

The GMEP is general in the sense that it includes several optimization problems such as the Mixed Equilibrium Problem (MEP) with $B = 0$ in (1.2), the Generalized Equilibrium Problem (GEP) if in (1.2) we set $\varphi = 0$. In particular, the GMEP is reduced to the classical Equilibrium Problem (EP) if $B = \varphi = 0$ in (1.2). That is, the EP is defined as follows: Find a point $\hat{x} \in C$ such that

$$\Theta(\hat{x}, x) \geq 0, \quad \forall x \in C. \tag{1.3}$$

The EP was initially introduced as the Ky Fan inequality (see [10]). This class of problems has been extensively studied due to its vast applications. The EP includes numerous problems arising in physics, economics, optimization theory e.t.c. as special cases. For solving the EP (1.3), where the underlining bifunction is monotone, one of the most employed method is the proximal point algorithm. However, this method cannot be adapted for solving EP when the bifunction is pseudomonotone. For this reasons, a number of authors have introduced several extragradient-like algorithms and their modifications for solving the EP involving pseudomonotone bifunctions; see, e.g., [11, 12, 13, 14] and the references therein. Also, some known methods involve the use of step sizes which depends on Lipschitz constants which are often unknown or difficult to estimate.

Iterative algorithms for finding common solutions of EP and fixed point problem and related optimization problems have been investigated in the Hilbert spaces and other spaces in recent research papers (see [6, 7, 15, 16, 17]). In 2016, Dihn et al. [18] studied the split equilibrium problem involving pseudomonotone and monotone bifunctions and nonexpansive mappings in real Hilbert spaces. Jouymandi and Moradlou [19] extended this study to the framework of Banach space. They considered a single EP involving a pseudomonotone bifunction. They proposed an extragradient and linesearch algorithm for finding a common element of the set of solutions of an EP and a fixed point of a relatively nonexpansive mapping. Very recently, Yang and Liu [20] proposed a new algorithm for obtaining common solutions of a pseudomonotone equilibrium problem and a fixed-point problem of a quasi-nonexpansive mapping. Their algorithm used a new stepsize, which does not depend on the Lipschitz constants of the bifunction.

Motivated by the research interest going on in this direction, we propose a new algorithm for finding a common solution in the solution set of a split equilibrium problem with a fixed-point constraint. We prove a strong convergence theorem for finding a common solution of SFP for the EP involving monotone and pseudomonotone bifunctions and the fixed-point problem of Bregman quasi-nonexpansive mapping in the framework of real reflexive Banach spaces. Our result extends the result of [20] and other related results in the literature.

2. Preliminaries

Let E be a real Banach space with the norm $||\cdot||$ and dual space E^*. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous function. Define by $domf := \{x \in E : f(x) < +\infty\}$ the domain of f. Let $x \in intdomf$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) := \{x^* \in E^* : f(x) - f(y) \leq \langle y - x, x^* \rangle, \quad y \in E\}.$$

The Fénchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ given by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}.$$

The function f is known to satisfy the Young-Fénchel inequality

$$\langle x^*, x \rangle \leq f^*(x^*) + f(x), \quad x \in E,\ x^* \in E^*.$$

If $x^* \in \partial f(x)$, then

$$\langle x^*, x \rangle = f^*(x^*) + f(x), \quad x \in E,\ x^* \in E^*.$$

Let $x \in intdomf$ and $y \in E$ be given. The right hand derivative of f at x in the direction of y is evaluated as:

$$f^\circ(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(y)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if the limit in (2.1) exist for any y. In this case, the gradient of f at x is the linear function $\bigtriangledown f(x)$ defined by $\langle y, \bigtriangledown f(x) \rangle := f^\circ(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in intdomf$. If the limit (2.1) is attained uniformly for any $y \in E$ with $||y|| = 1$ as t tends to zero, then f is Fréchet differentiable at x. The function $f : E \to (-\infty, +\infty]$ is called Legendre if it satisfies the following two restrictions:

(i) f is Gâteaux differentiable, $intdomf = dom \bigtriangledown f \neq \emptyset$.
(ii) f^* is Gâteaux differentiable, $intdomf^* = dom \bigtriangledown f^* \neq \emptyset$.

For more on Lengendre function, we refer to [21, 22] and the references therein.

The Bregman distance function with respect to f is defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \bigtriangledown f(y) \rangle.$$

It is worth mentioning that the bifunction $D_f : domf \times intdomf \to (-\infty, +\infty]$ defined above is not a distance in the usual sense. We refer the readers to [22, 23] for more properties of the Bregman distance.

Definition 2.1. [23] Let C be a nonempty, closed and convex subset of a reflexive real Banach space E. A Bregman projection of $x \in intdomf$ onto $C \subset intdomf$ is the unique vector $P_C^f \in C$ which satisfies $D_f(P_C^f x, x) = \inf\{D_f(y,x) : y \in C\}$.

Lemma 2.2. [24] *Let C be a nonempty, closed and convex subset of E and $x \in E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Then*

(i) $q = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(q), y - q \rangle \leq 0$, *for all* $y \in C$;

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y,x)$, *for all* $y \in C$.

We also need the function $V_f : E \times E^* \to [0, +\infty)$ associated with f ([24]) by

$$V_f(x, x^*) = f(x) + f^*(x^*) - \langle x, x^* \rangle, \quad \forall x^* \in E^*,\ x \in E.$$

It is easy to see that $V_f(x,x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Also, from the subdifferential inequality, we have

$$V_f(x,x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*), \tag{2.2}$$

for all $x \in E$ and $x^*, y^* \in E^*$ (see [25]).

Definition 2.3. Let E be a Banach space and let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous function. Let C be a nonempty subset of $intdomf$. A mapping $T : C \to intdomf$ is said to be Bregman Quasi-Nonexpansive (BQNE) (see [26]) if $Fix(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p,x), \quad \forall x \in E,\ p \in Fix(T).$$

Let B and S be the close unit ball and unit sphere of a Banach space E, respectively. Let $rB = \{x \in E : ||x|| \leq r\}$ for all $r > 0$. Then a function f is said to be uniformly convex on bounded subsets if $\rho_r(t) > 0$ for all $r,t > 0$, where the gauge of uniform convexity ρ_r of f is given by

$$\rho_r(t) = \inf_{x,y \in rB} \left\{ \frac{(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)}{\lambda(1-\lambda)} : ||x||, ||y|| \leq r,\ \lambda \in (0,1), ||x-y|| = t \right\}.$$

It is known that ρ_r is a non decreasing function.

The gauge of uniform smoothness of f is defined by

$$\sigma_r(t) = \sup_{x \in rB,\ y \in S} \left\{ \frac{(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)}{\lambda(1-\lambda)} : ||x||, ||y|| \leq r,\ \lambda \in (0,1), ||x-y|| = t \right\}.$$

Then the function f is said to be uniformly smooth on bounded subsets if $\lim_{t \to 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$.

Definition 2.4. A function $f : E \to \mathbb{R}$ is said to be super coercive if

$$\lim_{x \to \infty} \frac{f(x)}{||x||} = +\infty.$$

We recall from [27] that the normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{v \in E^* : \langle v, y - x \rangle \leq 0, \forall y \in C\}.$$

Lemma 2.5. [28] *Let C be a nonempty closed and convex subset of a real Banach space E. Let $g : E \to (-\infty, \infty]$ be a Gáteaux differentiable and lower semicontinuous function on C. Then, $\hat{x}$ is a solution to the following convex problem* $\min\{g(x) : x \in C\}$ *if and only if* $0 \in \partial g(\hat{x}) + N_C(\hat{x})$, *where $\partial g(\cdot)$ is the subdifferential of g and $N_C(\cdot)$ is the normal cone of C at $\hat{x}$.*

Lemma 2.6. [29] *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following relation $a_{k+1} \le (1-b_k)a_k + b_k c_k$, $k \ge 0$, where $\{b_k\} \subset (0,1)$ and $\{c_k\} \subset \mathbb{R}$ satisfy the conditions $\sum\limits_{k=0} b_k = \infty$ and $\limsup\limits_{k\to\infty} c_k \le 0$. Then, $\lim\limits_{k\to\infty} a_k = 0$.*

Lemma 2.7. [22, 30] *Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{k_j\}$ of $\{k\}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} < a_{m_k+1}$ and $a_k < a_{m_k+1}$. In fact, $m_k = \max\{i \le k : a_i < a_{i+1}\}$.*

For solving the EP, the following conditions will be needed.

Assumption A: The bifunction $g : C \times C \to \mathbb{R}$ satisfies the following restrictions:

(A1) g is pseudomonotone on C and $g(x,x) = 0$, for all $x \in C$;

(A2) g satisfies the Bregman-Lipschitz type condition on C, that is, there exist two positive constants c_1 and c_2 such that

$$g(x,y) + g(y,z) \ge g(x,z) - c_1 D_f(y,x) - c_2 D_f(y,z),\ \forall x,y,z \in C,$$

where $f : C \to (-\infty, +\infty]$ is a Lengendre function. The constants c_1, c_2 are called Bregman-Lipschitz coefficients with respect to f;

(A3) $g(x,\cdot)$ is convex, lower semicontinuous and subdifferentiable on C for all $x \in C$;

(A4) g is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_k\}$ and $\{y_k\}$ converges weakly to x and y respectively, then $g(x_k, y_k) \to g(x,y)$ as $k \to \infty$.

Assumption B: The bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following restrictions:

(B1) $\Theta(x,x) = 0$, for all $x \in Q$;

(B2) Θ is monotone on Q;

(B3) $\limsup\limits_{t\downarrow 0} \Theta(x + t(z-x), y) \le \Theta(x,y)$, for all $x,y,z \in Q$;

(B4) the function $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

Lemma 2.8. [31] *Let C be a nonempty, closed and convex subset of a reflexive Banach space E with dual E^*. Let $f : E \to (-\infty, +\infty]$ be a convex and Gáteaux differentiable function. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying restrictions $B1$-$B4$, $B : C \to E^*$ be a continuous mapping and $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous function. Let $r > 0$ be any given number and $x \in E$ be any given point. Then, the following hold:*

(1) there exists $z \in C$, such that

$$\Theta(z,y) + \langle Bz, y-z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y-z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall y \in C;$$

(2) the mapping

$$K_r^{\Theta}(x) = \{z \in C : \Theta(z,y) + \langle Bz, y-z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y-z, \nabla f(z) - \nabla f(x)\rangle \ge 0,\ \forall y \in C\},\ x \in E,$$

has the following properties:

(a) $K_r^{\Theta}(x) \neq \emptyset$, $\forall x \in E$;
(b) K_r^{Θ} *is single valued;*
(c) K_r^{Θ} *is firmly nonexpansive, that is,*

$$\langle K_r^{\Theta}(z) - K_r^{\Theta}(y), \nabla f(K_r^{\Theta}(z)) - \nabla f(K_r^{\Theta}(y))\rangle \leq \langle K_r^{\Theta}(z) - K_r^{\Theta}(y), \nabla f(z) - \nabla f(y)\rangle, \ \forall z, y \in E;$$

(d) $Fix(K_r^{\Theta}) = GMEP(\Theta, B, \varphi)$;
(e) $Fix(K_r^{\Theta})$ *is closed and convex;*
(f) $D_f(p, K_r^{\Theta}(x)) + D_f(K_r^{\Theta}(x), x) \leq D_f(p, x), \ p \in Fix(K_r^{\Theta}), \ x \in E.$

3. MAIN RESULTS

In this section, we give a concise and precise statement of our algorithm, discuss some of its elementary properties and its convergence analysis.

Let C and Q be nonempty, closed and convex subset of real reflexive Banach spaces E_1 and E_2 with duals E_1^* and E_2^* respectively, $L : E_1 \to E_2$ be a bounded linear operator. Let $f_1 : E_1 \to \mathbb{R}$ be super coercive Legendre function which is bounded, uniformly Fr*é*chet differentiable and totally convex on bounded subsets of E_1 and let $f_2 : E_2 \to \mathbb{R}$ be convex, continuous and injective on E_2 such that f_2^{-1} is continuous. Let $g : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction satisfying assumptions A and let $\Theta : Q \times Q \to \mathbb{R}$ be a monotone bifunction satisfying assumptions B. Let $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function and let $B : Q \to E_2^*$ be a continuous and monotone mapping. Assume that $T : C \to C$ and $S : Q \to Q$ are Bregman quasi-nonexpansive mapping such that $I - T$ and $I - SK_r^{\Theta}$ are demiclosed at zero, where K_r^{Θ} is defined as in Lemma 2.8. We consider the problem of finding a point $p \in C$ such that

$$p \in EP(g) \cap F(T) \text{ such that } Lp \in GMEP(\Theta, \varphi) \cap F(S). \tag{3.1}$$

Assume $\Gamma = \{p \in EP(g) \cap F(T) : Lp \in GMEP(\Theta, \varphi) \cap F(S)\} \neq \emptyset$. It follows from [32, Lemma 15.5], [27, Lemma 2.14] and Lemma 2.8 (e) that Γ is well defined.

First, we state and prove the following important Lemma.

Lemma 3.1. *Let $S : Q \to Q$ be a Bregman quasi-nonexpansive mapping and let K_r^{Θ} be defined as in Lemma 2.8. Then, $Fix(SK_r^{\Theta}) = Fix(S) \cap Fix(K_r^{\Theta})$.*

Proof. Clearly, $Fix(S) \cap Fix(K_r^{\Theta}) \subseteq Fix(SK_r^{\Theta})$. We only need to show that $Fix(SK_r^{\Theta}) \subseteq Fix(S) \cap Fix(K_r^{\Theta})$. Indeed, let $\hat{x} \in Fix(SK_r^{\Theta})$ and $\hat{y} \in Fix(S) \cap Fix(K_r^{\Theta})$, one has

$$\begin{aligned} D_f(\hat{y}, \hat{x}) &= D_f(\hat{x}, SK_r^{\Theta}\hat{x}) \\ &\leq D_f(\hat{y}, K_r^{\Theta}\hat{x}). \end{aligned} \tag{3.2}$$

From Lemma 2.8 (f) and (3.2), we get

$$\begin{aligned} D_f(K_r^{\Theta}\hat{x}, \hat{x}) &\leq D_f(\hat{x}, \hat{x}) - D_f(\hat{y}, K_r^{\Theta}\hat{x}) \\ &\leq D_f(\hat{y}, \hat{x}) - D_f(\hat{y}, \hat{x}) \\ &= 0. \end{aligned} \tag{3.3}$$

Hence, $\hat{x} \in Fix(S)$. This implies $\hat{x} \in Fix(S) \cap Fix(K_r^{\Theta})$. Therefore, $Fix(SK_r^{\Theta}) = Fix(S) \cap Fix(K_r^{\Theta})$. □

For solutions of problem (3.1), we consider the following iterative algorithm.

We pick two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ in $(0, 1)$ satisfying:

(i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(ii) $0 \le \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 1$.

We now present our subgradient extragradient algorithm.

Algorithm 3.2. Subgradient extragradient algorithm

Step 0: Choose the sequences $\{\alpha_k\}$ and $\{\beta_k\} \subset (0,1)$ satisfying the conditions (i) and (ii) above, and let $\rho \in (0,1)$ and $\lambda_0 > 0$. For $u \in C$, select initial point $x_0 \in C$. For each $k \ge 0$, given the k-th iterate $\{x_k\}$, execute the following steps:

Step 1: Solve the following strong convex programming

$$y_k = CP(x_k) = \arg\min\left\{g(x_k,y) + \frac{1}{2\lambda_k} D_{f_1}(x_k,y) : y \in C\right\}.$$

Step 2: Choose $t_k \in \partial g(x_k,y_k)$ such that $\nabla f_1(x_k) - \lambda_k t_k - \nabla f_1(y_k) \in N_C(y_k)$, define $T_k := \{w \in E_1 : \langle \nabla f_1(x_k) - \lambda_k t_k - \nabla f_1(y_k), w - y_k\rangle \le 0\}$ and compute

$$z_k = \arg\min\left\{g(y_k,y) + \frac{1}{2\lambda_k} D_{f_1}(x_k,y) : y \in T_k\right\}.$$

Step 3: Compute

$$\begin{cases} v_k = \nabla f_1^*(\beta_k \nabla f_1(z_k) + (1-\beta_k)\nabla f_1(Tz_k)), \\ w_k = \nabla f_1^*((1-\gamma_k)\nabla f_1(v_k) + \gamma_k L^* \nabla f_2((SK_r^{\Theta} - I)Lv_k)), \\ x_{k+1} = \nabla f_1^*(\alpha_k \nabla f_1(u) + (1-\alpha_k)\nabla f_1(w_k)), \end{cases}$$

and

$$\lambda_{k+1} = \begin{cases} \min\left\{\dfrac{\rho(D_{f_1}(y_k,x_k) + D_{f_1}(y_k,z_k))}{2(g(x_k,z_k) - g(x_k,y_k) - g(y_k,z_k))}, \lambda_k\right\}, & g(x_k,z_k) - g(x_k,y_k) - g(y_k,z_k) > 0, \\ \lambda_k & \text{otherwise.} \end{cases}$$

Step 4: Set $k := k+1$ and go to step 1.

Remark 3.3. If $x_k = y_k$, then we e reach the optimal solution of $CP(x_k)$.

Remark 3.4. We remark that t_k exists and $C \subset T_k$. Indeed, since

$$y_k = \arg\min\left\{g(x_k,y) + \frac{1}{2\lambda_k} D_{f_1}(x_k,y) : y \in C\right\},$$

we obtain from Lemma 2.5 that $0 \in \partial(g(x_k,y) + \frac{1}{2\lambda_k} D_{f_1}(x_k,y))(y_k) + N_C(y_k)$. That is, there exists $t_k \in \partial g(x_k,y_k)$ such that $\nabla f_1(x_k) - \lambda_k t_k - \nabla f_1(y_k) \in N_C(y_k)$. Thus, there exists $w \in N_C(y_k)$ such that $\lambda_k t_k + \nabla f_1(y_k) - \nabla f_1(x_k) + w = 0$. Therefore,

$$\langle \nabla f_1(x_k) - \nabla f_1(y_k), y - y_k\rangle = \langle \lambda_k t_k, y - y_k\rangle + \langle w, y - y_k\rangle \le \lambda_k \langle t_k, y - y_k\rangle, \forall y \in C,$$

that is,

$$\langle \nabla f_1(x_k) - \lambda_k t_k - \nabla f_1(y_k), y - y_k\rangle \le 0, \forall y \in C.$$

Hence, $C \subset T_k$.

The lemma below gives a property of the sequence $\{\lambda_k\}$ given in Algorithm 3.2.

Lemma 3.5. *The sequence $\{\lambda_k\}$ generated by Algorithm 3.2 is monotonically nonincreasing with a lower bound equal to*

$$\min\left\{\frac{\rho}{2\max\{c_1,c_2\}},\lambda_0\right\}.$$

Proof. It is easy to see that $\{\lambda_k\}$ is a monotonically nonincreasing sequence. Since g satisfies the Lipschitz-type condition with constants c_1 and c_2, in the case $g(x_k,z_k)-g(x_k,y_k)-g(y_k,z_k)>0$, we have

$$\frac{\rho(D_{f_1}(y_k,x_k)+D_{f_1}(y_k,z_k))}{2(g(x_k,z_k)-g(x_k,y_k)-g(y_k,z_k))}\geq\frac{\rho(D_{f_1}(y_k,x_k)+D_{f_1}(y_k,z_k))}{2(c_1D_{f_1}(y_k,x_k)+c_2D_{f_1}(z_k,y_k))}\geq\frac{\rho}{2\max\{c_1,c_2\}}.$$

Therefore, the sequence $\{\lambda_k\}$ is bounded below by

$$\min\left\{\frac{\rho}{2\max\{c_1,c_2\}},\lambda_0\right\}.$$

Remark 3.6. It is obvious that the limit of $\{\lambda_k\}$ exists and we say $\lambda_k\to\lambda$ as $k\to\infty$. Clearly λ is positive. If $\lambda_0\leq\frac{\rho}{2\max\{c_1,c_2\}}$, then $\{\lambda_k\}$ is a constant sequence.

Next we prove the boundedness of the sequences generated by Algorithm 3.2.

Lemma 3.7. *Let T and S be Bregman quasi nonexpansive mappings defined on C and Q respectively. Let $\{x_k\}$ be the sequence given by Algorithm 3.2 and $\Gamma\neq\emptyset$. Then the sequences $\{x_k\},\{y_k\},\{z_k\},\{v_k\}$ and $\{w_k\}$ are bounded.*

Proof. From Lemma 2.5, we have that

$$z_k=\arg\min\left\{g(y_k,y)+\frac{1}{2\lambda_k}D_{f_1}(x_k,y):y\in T_k\right\},$$

if and only if $0\in\lambda_k\partial g(y_k,z_k)+\frac{1}{2\lambda_k}D_{f_1}(z_k,x_k)+N_C(z_k)$. This implies that there exists $v\in\partial g(y_k,x_k)$ and $\bar{v}\in N_C(z_k)$ such that

$$0=\lambda_kv+\nabla f_1(z_k)-\nabla f_1(x_k)+\bar{v},\tag{3.4}$$

so from the definition of $\partial(y_k,x_k)$, we obtain

$$\langle v,y-z_k\rangle\leq g(y_k,y)-g(y_k,z_k)\tag{3.5}$$

for all $y\in T_k$. From the definition of the $N_C(z_k)$ and equality (3.4), we get

$$\lambda_k\langle v,z_k-y\rangle\leq\langle\nabla f_1(z_k)-\nabla f_1(x_k),y-z_k\rangle\tag{3.6}$$

for all $y\in T_k$. Combining (3.5) and (3.6), we obtain

$$\lambda_k(g(y_k,y)-g(y_k,z_k))\geq\langle\nabla f_1(x_k)-\nabla f_1(z_k),y-z_k\rangle,\ \forall y\in T_k.\tag{3.7}$$

Observe that $\Gamma\subset C\subset T_k$. Letting $p\in\Gamma$, substituting p for y in (3.7), we get

$$\lambda_k(g(y_k,p)-g(y_k,z_k))\geq\langle\nabla f_1(x_k)-\nabla f_1(z_k),p-z_k\rangle.\tag{3.8}$$

Since $p\in Ep(g)\subset\Gamma$, one has $g(p,y_k)\geq0$ and then we get $g(y_k,p)\leq0$. Hence, it follows from (3.8) that

$$-\lambda_kg(y_k,z_k)\geq\lambda_k\langle\nabla f_1(x_k)-\nabla f_1(z_k),p-z_k\rangle.\tag{3.9}$$

From $t_k \in \partial g(x_k, y_k)$, we get

$$g(x_k, y) - g(x_k, y_k) \geq \langle t_k, y - y_k \rangle, \ \forall y \in E_1. \tag{3.10}$$

If $y = z_k$ in (3.10), then

$$\lambda_k(g(x_k, z_k) - g(x_k, y_k)) \geq \lambda_k(\langle t_k, z_k - y_k \rangle), \ \forall y \in E_1. \tag{3.11}$$

From the definition of T_k, we have

$$\lambda_k \langle t_k, z_k - y_k \rangle \geq \langle \nabla f_1(x_k) - \nabla f_1(y_k), z_k - y_k \rangle. \tag{3.12}$$

Combining (3.9), (3.11) and (3.12), we get

$$\begin{aligned}
&\lambda_k(g(x_k, z_k) - g(z_k, y_k) - g(y_k, z_k)) \\
&\geq \langle \nabla f_1(x_k) - \nabla f_1(y_k), z_k - y_k \rangle + \langle \nabla f_1(x_k) - \nabla f_1(z_k), p - z_k \rangle \\
&= D_{f_1}(z_k, y_k) + D_{f_1}(y_k, x_k) - D_{f_1}(z_k, x_k) - D_{f_1}(p, x_k) + D_{f_1}(p, z_k) + D_{f_1}(z_k, x_k).
\end{aligned}$$

Thus,

$$D_{f_1}(p, z_k) \leq D_{f_1}(p, x_k) - D_{f_1}(y_k, x_k) - D_{f_1}(z_k, y_k) + \lambda_k(g(x_k, z_k) - g(x_k, y_k) - g(y_k, z_k)). \tag{3.13}$$

From the definition of λ_k and (3.13), we obtain

$$\begin{aligned}
&D_{f_1}(p, z_k) \\
&\leq D_{f_1}(p, x_k) - D_{f_1}(y_k, x_k) - D_{f_1}(z_k, y_k) + \lambda_k(g(x_k, z_k) - g(x_k, y_k) - g(y_k, z_k)) \\
&= D_{f_1}(p, x_k) - D_{f_1}(y_k, x_k) - D_{f_1}(z_k, y_k) + \frac{\lambda_k}{\lambda_{k+1}} \lambda_{k+1}(g(x_k, z_k) - g(x_k, y_k) - g(y_k, z_k)) \\
&\leq D_{f_1}(p, x_k) - D_{f_1}(y_k, x_k) - D_{f_1}(z_k, y_k) + \frac{\lambda_k}{\lambda_{k+1}} \rho (D_{f_1}(y_k, x_k) + D_{f_1}(y_k, z_k)).
\end{aligned} \tag{3.14}$$

We note that $\lim_{k \to \infty} \frac{\lambda_k}{\lambda_{k+1}} \rho = \rho$, $\rho \in (0, 1)$. That is, there exists $N \geq 0$ such that, for all $k \geq N$, $0 < \frac{\lambda_k}{\lambda_{k+1}} \rho < 1$. We obtain, for all $k \geq N$,

$$D_{f_1}(p, z_k) \leq D_{f_1}(p, x_k). \tag{3.15}$$

From Algorithm 3.2 and [33, Lemma 2.2], we have

$$\begin{aligned}
D_{f_1}(p, v_k) &= D_{f_1}(p, \nabla f_1^*(\beta_k \nabla f_1(z_k) + (1 - \beta_k) \nabla f_1(T z_k))) \\
&= \beta_k D_{f_1}(p, z_k) + (1 - \beta_k) D_{f_1}(T z_k) - \beta_k (1 - \beta_k) \rho_r^*(||\nabla f_1(z_k) - \nabla f_1(T z_k)||) \\
&\leq \beta_k D_{f_1}(p, z_k) + (1 - \beta_k) D_{f_1}(z_k) - \beta_k (1 - \beta_k) \rho_r^*(||\nabla f_1(z_k) - \nabla f_1(T z_k)||) \\
&= D_{f_1}(p, z_k) - \beta_k (1 - \beta_k) \rho_r^*(||\nabla f_1(z_k) - \nabla f_1(T z_k)||) \\
&\leq D_{f_1}(p, z_k).
\end{aligned} \tag{3.16}$$

Again, from Algorithm 3.2 and [31, Remark 2.2], we have

$$
\begin{aligned}
&D_{f_1}(p,w_k)\\
&= D_{f_1}(p,\nabla f_1^*((1-\gamma_k)\nabla f_1(v_k)-\gamma_k L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)))\\
&= f_1(p)+f_1^*((1-\gamma_k)\nabla f_1(v_k)-\gamma_k L^*\nabla f_2((I-SK_r^{\Theta}Lv_k)))\\
&-\langle p,(1-\gamma_k)\nabla f_1(v_k)-\gamma_k L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)\rangle\\
&\leq f_1(p)+(1-\gamma_k)f_1^*(\nabla f_1(v_k))+\gamma_k f_1^*(-L^*\nabla f_2((I-SK_r^{\Theta})Lv_k))-\langle p,\nabla f_1(v_k)\rangle\\
&+\gamma_k\langle p,\nabla f_1(v_k)\rangle+\gamma_k\langle p,\nabla f_1(v_k)\rangle+\gamma_k\langle p,L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)\rangle\\
&\leq f_1(p)+f_1^*(\nabla(v_k))-\langle p,\nabla f_1(v_k)\rangle+\gamma_k[f_1^*(-L^*\nabla f_2((I-SK_r^{\Theta})Lv_k))\\
&+\langle p,\nabla f_1(v_k)+L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)\rangle]\\
&\leq D_{f_1}(p,v_k)+\gamma_k[\sup_{x\in E_1}\{\langle -x,L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)-f_1(x)\}+\langle p,\nabla f_1(v_k)\rangle\\
&+\langle p,L^*\triangledown f_2((I-SK_r^{\Theta})Lv_k)\rangle]\\
&\leq D_{f_1}(p,v_k)+\gamma_k[\sup_{x\in E_1}\{\langle -x,L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)-f_1(x)\}+f_1(p+v_k)-f_1(v_k)\\
&+\langle p,L^*\nabla f_2((I-SK_r^{\Theta})Lv_k)\rangle]\\
&\leq D_{f_1}(p,v_k)+\gamma_k[\langle -Lv_k,\nabla f_2((I-SK_r^{\Theta})Lv_k)\rangle-f_1(v_k)]\\
&\leq D_{f_1}(p,v_k)+\gamma_k[f_2(-SK_r^{\Theta}Lv_k)-f_2((I-SK_r^{\Theta})Lv_k))-f_1(v_k)]\\
&= D_{f_1}(p,v_k)-\gamma_k[f_1(v_k)+f_2((I-SK_r^{\Theta})Lv_k)+f_2(SK_r^{\Theta}Lv_k)].
\end{aligned}
\tag{3.17}
$$

Suppose that there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$, $|f_1(v_k)|\geq|f_2(SK_r^{\Theta})Lv_k||$,

$$
\gamma_k=\sigma_k\frac{|f_2((I-SK_r^{\Theta})Lv_k)|}{|f_2((I-SK_r^{\Theta})Lv_k)|+|f_1(v_k)|}
$$

and

$$
\begin{aligned}
&\gamma_k[f_1(v_k)+f_2((I-SK_r^{\Theta})Lv_k)+f_2(SK_r^{\Theta}Lv_k)]\\
&\geq -\gamma_k|f_2((I-SK_r^{\Theta})Lv_k)+f_1(v_k)+f_2(SK_r^{\Theta}Lv_k)|\\
&\geq -\gamma_k[|f_2(SK_r^{\Theta}Lv_k)|-|f_2((I-SK_r^{\Theta})Lv_k)|-|f_1(v_k)|]\\
&= \gamma_k[|f_2((I-SK_r^{\Theta})Lv_k)|+|f_1(v_k)|-|f_2(SK_r^{\Theta}Lv_k)|]\\
&= |f_2((I-SK_r^{\Theta})Lv_k)|\sigma_k\left(1-\frac{|f_2(SK_r^{\Theta}Lv_k)|}{|f_2((I-SK_r^{\Theta})Lv_k)|+|f_1(v_k)|}\right)\\
&\geq 0.
\end{aligned}
\tag{3.18}
$$

Conversely, if there exists n_1 such that

$$
|f_1(v_k)|\leq|f_2((I-SK_r^{\Theta})Lv_k)|
$$

for all $n\geq n_1$, we have the same conclusion as in (3.18). Hence, we obtain from (3.17), that

$$
D_{f_1}(p,w_k)\leq D_{f_1}(p,x_k).
$$

Furthermore, we have also from 3.2, that

$$\begin{aligned}
D_{f_1}(p,x_{k+1}) &= V_{f_1}(\nabla f_1(u)+(1-\alpha_k)\nabla f_1(w_k)) \\
&= f_1(p)-\langle p,\nabla f_1(u)+(1-\alpha_k)\nabla f_1(w_k)\rangle + f_1^*(\nabla f_1(u)+(1-\alpha_k)\nabla f_1(w_k)) \\
&\quad -\alpha_k(1-\alpha_k)\rho_r^*(||\nabla f_1(u)-\nabla f_1(w_k)||) \\
&\leq f_1(p)-\alpha_k\langle p,\nabla f_1(u)\rangle-(1-\alpha_k)\langle p,\nabla f_1(w_k)\rangle+\alpha_k f_1^*(\nabla f_1(u)) \\
&\quad +(1-\alpha_k)f_1^*(\nabla f_1(w_k))-\alpha_k(1-\alpha_k)\rho_r^*(||\nabla f_1(u)-\nabla f_1(w_k)||) \\
&\leq \alpha_k D_{f_1}(p,u)+(1-\alpha_k)D_{f_1}(p,v_k)-\alpha_k(1-\alpha_k)\rho_r^*(||\nabla f_1(u)-\nabla f_1(w_k)||) \\
&\leq \alpha_k D_{f_1}(p,u)+(1-\alpha_k)D_{f_1}(p,x_k)-\alpha_k(1-\alpha_k)\rho_r^*(||\nabla f_1(u)-\nabla f_1(w_k)||) \\
&\leq \max\{D_{f_1}(p,u),D_{f_1}(p,x_k)\} \\
&\ \vdots \\
&\leq \max\{D_{f_1}(p,u),D_{f_1}(p,x_0)\}, \quad \forall\, k\geq 0.
\end{aligned} \tag{3.19}$$

This implies that sequence $D_{f_1}(p,x_k)$ is bounded. [21, Lemma 1] asserts that $\{x_k\}$ is bounded. Conversely, $\{y_k\}$, $\{v_k\}$, $\{w_k\}$ and $\{z_k\}$ are also bounded. □

In the following lemma, we show that the element p is a solution of $EP(g)$.

Lemma 3.8. *Let $\{x_k\}$ be the sequence given by Algorithm 3.2. Given a subsequence of $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j}\rightharpoonup p$ and $||x_{k_j}-z_{k_j}||\to 0$, as $j\to\infty$, one has $p\in EP(g)$.*

Proof. Following the same procedure that leads to (3.7). It follows that

$$\lambda_k(g(x_{k_j},y)-g(x_{k_j},y_{k_j}))\geq\langle\nabla f_1(x_{k_j})-\nabla f_1(y_{k_j}),y-y_{k_j}\rangle,\ \forall y\in C. \tag{3.20}$$

Using assumptions $(A1)$, $(A4)$ and $\lim_{j\to\infty}\lambda_{k_i}=\lambda=0$, we get $g(p,y)\geq 0$, for all $y\in C$. Hence $p\in EP(g)$. This completes the proof. □

We now state and prove our main theorem.

Theorem 3.9. *Let C and Q be nonempty, closed and convex subset of real reflexive Banach spaces E_1 and E_2 with duals E_1^* and E_2^* respectively. Let $L:E_1\to E_2$ be a bounded linear operator. Let $f_1:E_1\to\mathbb{R}$ be a super coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E_1 and let $f_2:E_2\to\mathbb{R}$ be convex, continuous and injective on E_2 such that f_2^{-1} is continuous. Let $g:C\times C\to\mathbb{R}$ be a pseudomonotone bifunction satisfying assumption A and let $\Theta:Q\times Q\to\mathbb{R}$ be a monotone bifunction satisfying assumption B. Let $\varphi:Q\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semi-continuous convex function and let $B:Q\to E_2^*$ be a continuous and monotone mapping. Let $T:C\to C$ and $S:Q\to Q$ be Bregman quasi-nonexpansive mappings such that $I-T$ and $I-SK_r^{\Theta}$ are demiclosed at zero. Let $\Gamma=\{p:EP(C,f)\cap F(T):Lp\in GMEP(\Theta,\varphi)\cap F(S)\}\neq\emptyset$. Then, the sequence $\{x_k\}$ generated by Algorithm 3.2 converges strongly to $p=P_\Gamma^f u$, where P_Γ^f is the Bregman projection of $\operatorname{int}\operatorname{dom} f$ onto Γ.*

Proof. We start the proof by considering the following two possible cases.
Case 1. Suppose there exists $k_0\in\mathbb{N}$ such that $\{D_{f_1}(p,x_k)\}$ is monotonically nonincreasing for all $k\geq k_0$. Then, $\{D_{f_1}(p,x_k)\}$ converges and hence $D_{f_1}(p,x_k)-D_{f_1}(p,x_{k+1})\to 0$ as $k\to\infty$.

From (3.16) and (3.19), we have

$$\begin{aligned}&\lim_{k\to\infty}\beta_k(1-\beta_k)\rho_r^*||\nabla f_1(z_k)-\nabla f_1(Tz_k)||\\&\le\lim_{k\to\infty}\alpha_k(D_{f_1}(p,u)-D_{f_1}(p,v_k))+D_{f_1}(p,x_k)-D_{f_1}(p,x_{k+1}).\end{aligned}$$

Using conditions (i) and (ii), we obtain

$$\lim_{k\to\infty}\rho_r^*||\nabla f_1(z_k)-\nabla f_1(Tz_k)||=0.$$

Next, we show that $||\nabla f_1(z_k)-\nabla f_1(Tz_k)||\to 0$ as $k\to\infty$. If not, there exists $\varepsilon>0$ and a subsequence $\{k_m\}$ of $\{k\}$ such that $||\nabla f_1(z_{k_m})-\nabla f_1(Tz_{k_m})||\ge\varepsilon$. Since ρ_r^* is an increasing function, we have $\rho_r^*(\varepsilon)\le\rho_r^*(||\nabla f_1(z_k)-\nabla f_1(Tz_k)||)$ for all $m\in\mathbb{N}$. Letting $m\to\infty$, we have $\rho_r^*\le 0$ and this contradicts the uniform convexity of f_1^* on bounded sets. Hence, by [34, Propositions 3.6.3, 3.6.4] and the fact that ∇f_1^* is uniformly continuous on bounded subsets of E_1^*, we obtain

$$\lim_{k\to\infty}||\nabla f_1(z_k)-\nabla f_1(Tz_k)||=0.$$

By [34, Proposition 3.63], we have that $\triangledown f_1^*$ is uniformly continuous on bounded subsets of E_1^*, hence, we obtain

$$\lim_{k\to\infty}||z_k-Tz_k||=0. \tag{3.21}$$

Observe from Algorithm 3.2, that

$$\begin{aligned}&D_{f_1}(z_k,v_k) &(3.22)\\
=\;&f_1(z_k)+f_1^*(\beta_k\nabla f_1(z_k)+(1-\beta_k)\nabla f_1(Tz_k))-\langle z_k,\beta_k\nabla f_1(z_k)+(1-\beta_k)\nabla f_1(Tz_k)\rangle\\
\le\;&\beta_kf_1(z_k)+(1-\beta_k)f_1(z_k)+\beta_kf_1^*(\nabla f_1(z_k))+(1-\beta_k)f_1^*(\nabla f_1(Tz_k))-\beta_k\langle z-k,\nabla f_1(z_k)\rangle\\
&-(1-\beta_k)\langle z_k,\nabla f_1(Tz_k)\rangle\beta_k(1-\beta_k)\rho_r^*||\nabla f_1(z_k)-\nabla f_1(Tz_k)||\\
\le\;&(1-\beta_k)D_{f_1}(z_k,Tz_k)-\beta_k(1-\beta_k)\rho_r^*||\nabla f_1(z_k)-\nabla f_1(Tz_k)||.\end{aligned}$$

It follows from (3.21) that $D_{f_1}(z_k,v_k)\to 0$ as $k\to\infty$. By using [35, Proposition 2.2, pp 3], we obtain that

$$\lim_{k\to\infty}||z_k-v_k||=0. \tag{3.23}$$

From (3.14), we have

$$\begin{aligned}D_{f_1}(p,z_k)&\le D_{f_1}(p,x_k)-D_{f_1}(y_k,x_k)-D_{f_1}(z_k,y_k)+\frac{\lambda_k}{\lambda_{k+1}}\rho\,(D_{f_1}(y_k,x_k)+D_{f_1}(z_k,y_k))\\&\le D_{f_1}(p,x_k)-\left(1-\frac{\lambda_k}{\lambda_{k+1}}\rho\right)(D_{f_1}(y_k,x_k)+D_{f_1}(z_k,y_k)).\end{aligned}$$

Using this and (3.19), we have

$$\begin{aligned}&\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)(D_{f_1}(y_k,x_k)+D_{f_1}(z_k,y_k))\\&\le\alpha_k(D_{f_1}(p,u)-D_{f_1}(p,z_k))+D_{f_1}(p,x_k)-D_{f_1}(p,x_{k+1})\to 0,\end{aligned}$$

as $k \to \infty$. Since $1 - \frac{\lambda_k}{\lambda_{k+1}}\rho = 1 - \rho > 0$, we obtain $D_{f_1}(y_k, x_k) \to 0$ and $D_{f_1}(z_k, y_k) \to 0$ as $k \to \infty$. Using [35, Proposition 2.2, pp 3], we obtain

$$\lim_{k\to\infty} ||y_k - x_k|| = 0 = \lim_{k\to\infty} ||z_k - x_k||. \tag{3.24}$$

Consequently,

$$\lim_{k\to\infty} ||z_k - x_k|| = 0. \tag{3.25}$$

Next, we show that $||(I - SK_r^\Theta)Lv_k|| \to 0$ as $k \to \infty$. From (3.17), we have

$$D_{f_1}(p, w_k) \leq D_{f_1}(p, x_k) - \gamma_k[f_1(v_k) + f_2((I - SK_r^\Theta)Lv_k) + f_2(SK_r^\Theta Lv_k)].$$

Substituting (3.26) into (3.19), we have

$$D_{f_1}(p, x_{k+1}) \leq \alpha_k D_{f_1}(p, u) + (1 - \alpha_k) D_{f_1}(p, w_k),$$

which implies

$$\begin{aligned} D_{f_1}(p, x_{k+1}) \leq [D_{f_1}(p, u) - D_{f_1}(p, w_k)] + D_{f_1}(p, x_k) \\ - \gamma_k[f_1(v_k) + f_2((I - SK_r^\Theta)Lv_k) + f_2(SK_r^\Theta Lv_k)]. \end{aligned}$$

Thus

$$\begin{aligned} &[f_1(v_k) + f_2((I - SK_r^\Theta)Lv_k) + f_2(SK_r^\Theta Lv_k)] \\ &\leq \alpha_k(D_{f_1}(p, u) - D_{f_1}(p, w_k)) + D_{f_1}(p, x_k) - D_{f_1}(p, x_{k+1}). \end{aligned}$$

Using condition (i), we get

$$[f_1(v_k) + f_2((I - SK_r^\Theta)Lv_k) + f_2(SK_r^\Theta Lv_k)] \to 0, \text{ as } k \to \infty.$$

Now, we suppose there exists k_0 such that $|f_1(v_k)| \geq |f_2(SK_r^\Theta Lv_k)||$ for all $k \geq k_0$. Then

$$\gamma_k = \frac{\sigma_k |f_2((I - SK_r^\Theta)Lv_k)|}{|f_2((I - SK_r^\Theta)Lv_k)| + |f_1(v_k)|}.$$

and

$$\lim_{k\to\infty} \frac{\sigma_k |f_2((I - SK_r^\Theta)Lv_k)|}{|f_2((I - SK_r^\Theta)Lv_k)| + |f_1(v_k)|} |f_2((I - SK_r^\Theta)Lv_k) + f_2(SK_r^\Theta Lv_k) + f_1(v_k)| = 0.$$

Clearly,

$$|f_2(SK_r^\Theta Lv_k)| - |f_2((I - SK_r^\Theta)Lv_k)| - |f_1(v_k)| \leq |f_2((I - SK_r^\Theta)Lv_k) + f_2(SK_r^\Theta Lv_k) + f_1(v_k)|.$$

So,

$$\lim_{k\to\infty} \sigma_k |f_2((I - SK_r^\Theta)Lv_k)| \big(\frac{|f_2(SK_r^\Theta Lv_k)|}{|f_2((I - SK_r^\Theta)Lv_k)| + |f_1(v_k)|} - 1 \big) = 0.$$

which together with the condition on σ_k and the fact that $\frac{|f_2(SK_r^\Theta Lv_k)|}{|f_2((I - SK_r^\Theta)Lv_k)|} - 1 > 0$ obtains $\lim_{k\to\infty} f_2((I - SK_r^\Theta)Lv_k) = 0$. Since f_2^{-1} is continuous, we have

$$\lim_{k\to\infty} (I - SK_r^\Theta)Lv_k = 0. \tag{3.26}$$

From (3.2), we have

$$\begin{aligned} D_{f_1}(w_k, x_{k+1}) &= D_{f_1}(w_k, \nabla f_1^*(\alpha_k \nabla f_1(u) + (1 - \alpha_k)\nabla f_1(w_k))) \\ &\leq \alpha_k D_{f_1}(w_k, u) + (1 - \alpha_k) D_{f_1}(w_k, w_k) \to 0, \text{ as } k \to \infty. \end{aligned}$$

Using [35, Proposition 2.2, pp 3] and the boundedness of $\{w_k\}$, we get

$$\lim_{k\to\infty}||w_k - x_{k+1}|| = 0. \tag{3.27}$$

On the other hand, by the boundedness of ∇f_1 on bounded subsets of E, we have

$$D_{f_1}(x_{k+1}, x_k) = D_{f_1}(x_k, p) - D_{f_1}(x_{k+1}, p) + \langle x_k - p, \nabla f_1 p - \nabla f_1 x_{k+1}\rangle \to 0, \text{ as } k \to \infty.$$

Again, by [35, Proposition 2.2, pp 3], we get

$$||x_{k+1} - x_k|| \to 0 \text{ as } k \to \infty. \tag{3.28}$$

From (3.27) and (3.28), we have

$$\lim_{k\to\infty}||w_k - x_k|| = 0. \tag{3.29}$$

By use of (3.23) and (3.25), we also have

$$\lim_{k\to\infty}||v_k - x_k|| \le \lim_{k\to\infty}||z_k - x_k|| + \lim_{k\to\infty}||v_k - z_k|| = 0. \tag{3.30}$$

Since $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup q$. We have from (3.25), (3.29) and (3.30) that $\{z_{k_i}\}$, $\{w_{k_i}\}$ and $\{v_{k_i}\}$ converge to q. Hence, by use of (3.21) and the demiclosedness assumption on $I - T$, we have $q \in F(T)$. From this fact and Lemma 3.8, we have $q \in EP(g) \cap F(T)$. On the other hand, from the linearity of L, we obtain $Lv_{k_i} \rightharpoonup Lq$. By use of (3.26), (3.30) and the demiclosedness assumption on $I - SK_r^{\Theta}$, we obtain $Lq \in F(SK_r^{\Theta}) = GMEP(\Theta, \varphi) \cap F(S)$.

Next, we show that $\{x_k\}$ converges strongly to p. From Algorithm 3.2 and (2.2), we have

$$\begin{aligned}
D_{f_1}(p, x_{k+1}) &= V_{f_1}(p, \alpha_k \nabla f_1(u) + (1-\alpha_k)\nabla f_1(w_k)) \\
&\le V_{f_1}(p, \alpha_k \nabla f_1(u) + (1-\alpha_k)\nabla f_1(w_k)) - \alpha_k(\nabla f_1(u) - \nabla f_1(p)) + \\
&\quad \langle \nabla f_1^*(\alpha_k \nabla f_1(u) + (1-\alpha_k)\nabla f_1(w_k)) - p, \alpha_k(\nabla f_1(u) - \nabla f_1(p))\rangle \\
&= V_{f_1}(p, \alpha_k \nabla f_1(p) + (1-\alpha_k)\nabla f_1(w_k)) + \alpha_k\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle \\
&\le \alpha_k D_{f_1}(p,p) + (1-\alpha_k)D_{f_1}(p, w_k) + \alpha_k\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle \\
&\le (1-\alpha_k)D_{f_1}(p, x_k) + \alpha_k\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle.
\end{aligned} \tag{3.31}$$

We only have to show that $\limsup_{k\to\infty}\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle \le 0$ and then apply Lemma 2.6. Since $\{x_k\}$ is bounded, we find that there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup q$ and $\limsup_{k\to\infty}\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle = \lim_{j\to\infty}\langle x_{k_j+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle$. Since $||x_{k+1} - x_k|| \to 0$ as $k \to \infty$, we obtain $x_{k_j+1} \rightharpoonup q$. Using Lemma 2.2 (i), we get

$$\begin{aligned}
\limsup_{k\to\infty}\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle &= \lim_{j\to\infty}\langle x_{k_j+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle \\
&= \langle q - p, \nabla f(u) - \nabla f(p)\rangle \\
&\le 0.
\end{aligned}$$

By using this, Lemma 2.6 and (3.31), we have that $\{x_k\}$ converges strongly to $p = P_\Gamma^f u$.

Case 2: Suppose that $\{\Phi_k = ||x_k - p||\}$ is not a monotone sequence. Then there exists a subsequence $\{k_i\}$ of $\{k\}$ such that $||x_{k_i} - p|| < ||x_{k_i+1} - p||$, $\forall i \in \mathbb{N}$. For some k_0 large enough, define a mapping $\tau(k) := \max\{j \in \mathbb{N} : j \le k, \Phi_j \le \Phi_{j+1}\}$. Clearly, τ is a nondecreasing sequence, $\tau(k) \to 0$ as $k \to \infty$ and $0 \le \Phi_{\tau(k)} \le \Phi_{\tau(k)+1}$, $\forall k \ge k_0$. By the same argument as in Case 1, we have $||z_{\tau(k)} - Tz_{\tau(k)}|| \to 0$, $||(SK_r^{\Theta} - I)Lv_{\tau(k)}|| \to 0$ and $||x_{\tau(k)+1} - x_{\tau(k)}|| \to 0$

as $k \to \infty$ and $\limsup_{k\to\infty}\langle x_{k+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle \le 0$. Since $x_{\tau(k)}$ is bounded, there exists a subsequence $\{x_{\tau(k_j)}\}$ such that $\{x_{\tau(k_j)}\} \rightharpoonup \bar{q} \in C$. Also by the linearity of L, we have $Lv_{\tau(k_j)} \rightharpoonup L\bar{q} \in Q$. Following similar arguments as in the first case, we can conclude $\bar{q} \in \Gamma$. By applying Lemma 2.6, we have $\Phi_{\tau(k)+1} \le (1 - b_{\tau(k)})\Phi_{\tau(k)} + b_{\tau(k)} c_{\tau(k)}$, where $b_{\tau(k)} = \alpha_{\tau(k)}$, $c_{\tau(k)} = \langle x_{\tau(k)+1} - p, \nabla f_1(u) - \nabla f_1(p)\rangle$. Note that $b_{\tau(k)} \to 0$ as $k \to \infty$ and $\limsup_{k\to\infty} c_{\tau(k)} \le 0$. Since $\Phi_{\tau(k)} \le \Phi_{\tau(k+1)}$ and $b_{\tau(k)} > 0$, we have $||x_{\tau(k)} - p|| \le c_{\tau(k)}$. This implies $\limsup\limits_{k\to\infty} ||x_{\tau(k)} - p||^2 \le 0$, and hence

$$\lim_{k\to\infty} ||x_{\tau(k)} - p|| = 0. \tag{3.32}$$

By using $\lim\limits_{k\to\infty} ||x_{\tau(k)+1} - x_{\tau(k)}|| = 0$ and (3.32), we have that

$$\lim_{k\to\infty} ||x_{\tau(k)+1} - p|| \le \lim_{k\to\infty} (||x_{\tau(k)+1} - x_{\tau(k)}|| + ||x_{\tau(k)} - p||) = 0. \tag{3.33}$$

Furthermore, for $k \ge k_0$, it is easy to see that $\Phi_{\tau(k)} \le \Phi_{\tau(k)+1}$ if $k \ne \tau(k)$ (that is $\tau(k) < k$), because $\Phi_j \ge \Phi_{j+1}$ for $\tau(k)+1 \le j \le k$. As a consequence, we obtain, for all $k \ge k_0$, $0 \le \Phi_k \le \max\{\Phi_{\tau(k)}, \Phi_{\tau(k)+1}\} = \Phi_{\tau(k)+1}$. By using (3.33), we can conclude that $\lim\limits_{k\to\infty} \Phi_k = 0$, that is, $\{x_k\}$ converges strongly to $p = P^f_\Gamma u$. This completes the proof. □

4. THE APPLICATION

We define Θ in (1.3) by:

$$\Theta(x,y) := \begin{cases} \langle Ax, y - x\rangle, & \text{if } x, y \in C, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $A : C \to E_1^*$ is a pseudomonotone mapping. Then EP (1.3) is reduced to the following variational inequality problem (VIP) (see [?, ?]: Find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \ge 0, \ \forall\, y \in C. \tag{4.2}$$

The VIP is a very useful tool in studying optimization problems, differential equations, minmax problems and has applications in economics and mechanics theory; see, e.g., [10, 26, ?].

We recall that the operator A is pseudomonotone if, for any $x, y \in C$,

$$\langle Ax, y - x\rangle \ge 0 \implies \langle Ay, y - x\rangle \ge 0.$$

We note that if Θ is pseudomonotone (monotone), then the operator A is pseudomotone (monotone). We denote the set of solutions of (4.2) by $VIP(C,A)$.

Remark 4.1. Let $g(x,y) = \langle Ax, y - x\rangle$, $\forall x, y \in E$. If A is Lipschitz continuous, i.e, there exists $L^1 > 0$ such that $||Ax - Ay|| \le L^1 ||x - y||$, $\forall x, y \in E$. Then the condition $(A2)$ holds for g with $c_1 = c_2 = \frac{L^1}{2}$. It is clear that g satisfies $(A1)$ and $(A3)$.

The following useful result is from [27].

Lemma 4.2. *Let C be a nonempty, closed and convex subset of a reflexive Banach space E, $A : C \to E^*$ be a mapping and let $f : E \to \mathbb{R}$ be Legendre function. Then*

$$P^f_C(\nabla f^*[\nabla f(x) - \lambda A(y)]) = \arg\min_{w \in C}\{\langle w - y, A(y)\rangle + \frac{1}{2\lambda} D_f(w,x)\},$$

for all $x \in X$, $y \in C$ and $\lambda \in (0, +\infty)$.

In this situation, Algorithm 3.2 provides a new method for solving the VIP and the fixed-point problem for a Bregman quasi-nonexpansive mapping. We give the new method as follows.

Let $\{\alpha_k\}$ and $\{\beta_k\}$ in $(0,1)$ be two sequences satisfying: (i) $\lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=1}^{\infty}\alpha_k = \infty$; and(ii) $0 \le \liminf_{k\to\infty}\beta_k \le \limsup_{k\to\infty}\beta_k < 1$.

Theorem 4.3. *Let C and Q be nonempty, closed and convex subset of real reflexive Banach spaces E_1 and E_2 with duals E_1^* and E_2^* respectively, let $L : E_1 \to E_2$ be a bounded linear operator. Let $f_1 : E_1 \to \mathbb{R}$ be super coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E_1 and $f_2 : E_2 \to \mathbb{R}$ be convex, continuous and injective on E_2 such that f_2^{-1} is continuous. Let $A : C \to E_1^*$ be a Lipschitz continuous, pseudomonotone operator that is bounded on bounded sets. Assume that $T : C \to C$ and $S : Q \to Q$ are Bregman quasi nonexpansive mapping such that $I-T$ and $I-S$ are demiclosed at zero. Let $Sol = \{p : VIP(C,A)\cap F(T) : Lp \in F(S)\} \neq \emptyset$. Let $\{x_k\}$ be the sequence generated by*

$$\begin{cases} \lambda_0 > 0, x_0 \in C, \mu \in (0,1) \\ y_k = P_C^f(\nabla f^*[\nabla f(x_k) - \lambda A(x_k)]) \\ T_k := \{w \in E_1 : \langle \nabla f_1(x_k) - \lambda_k t_k - \nabla f_1(y_k), w - y_k\rangle \le 0\} \\ z_k = P_{T_k}^f(\nabla f^*[\nabla f(x_k) - \lambda A(y_k)]) \\ v_k = \nabla f_1^*(\beta_k \nabla f_1(z_k) + (1-\beta_k)\nabla f_1(Tz_k)) \\ w_k = \nabla f_1^*((1-\gamma_k)\nabla f_1(v_k) + \gamma_k L^* \nabla f_2((S-I)Lv_k)) \\ x_{k+1} = \nabla f_1^*(\alpha_k \nabla f_1(u) + (1-\alpha_k)\nabla f_1(w_k)). \end{cases}$$

Then $\{x_k\}$ converges strongly to $p = P_{Sol}^f u$, where P_{Sol}^f is the Bregman projection of $\operatorname{int} \operatorname{dom} f$ onto Sol.

5. The Numerical Example

In this section, we give a numerical example to illustrate the performance of our algorithm and compare it with a related method in the literature.

Example 5.1. Let $E_1 = E_2 = \mathbb{R}^m$ be the linear spaces of square-summable sequences $\{x_i\}_{i=1}^m$ of scalars in $\mathbb{R}^m$, that is, $\mathbb{R}^m := \left\{x = (x_1, x_2 \cdots, x_m \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty}|x_i|^2 < \infty\right\}$, with the inner product $\langle\cdot,\cdot\rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by $\langle x, y\rangle := \sum_{i=1}^m x_i y_i$, the norm $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ defined by $\|x\| := \left(\sum_{i=1}^m |x_i|^2\right)^{\frac{1}{2}}$, where $x = \{x_i\}_{i=1}^m$, $y = \{y_i\}_{i=1}^m$ and $f(\cdot) = \frac{1}{2}\|\cdot\|^2$. Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be given by $Lx = \left(\frac{x_1}{5}, \frac{x_2}{5}, \cdots, \frac{x_m}{5}\right)$ for all $x = \{x_i\}_{i=1}^m \in \mathbb{R}^m$. Then $L^*y = \left(\frac{y_1}{5}, \frac{y_2}{5}, \cdots, \frac{y_m}{5}\right)$ for each $y = \{y_i\}^m \in \mathbb{R}^m$. Define the sets $C := \{x \in \mathbb{R}^m : \|x\| \le 1\}$ and $Q := \{y \in \mathbb{R}^m : \|y\| \le 1\}$. Let the bifunction $\Theta : Q \times Q \to \mathbb{R}$ be defined by $\Theta(w,u) = uw - w^2 + 5u - 5w$, $B : Q \to E_2^*$ be defined by $B(w) := w$, and $\varphi : Q \to \mathbb{R}\cup\{+\infty\}$ be given by $\varphi(u) = 0$ for all $u = \{u_i\}_{i=1}^m \in \mathbb{R}^m$ and $w = \{w_i\}_{i=1}^m \in \mathbb{R}^m$. It is easy to check that Θ, B and φ all satisfy the assumption of the Theorem 3.9 and $K_r^{\Theta}L(x) = \dfrac{x - 30r}{5(r+1)}$. Define the mapping $g : C \times C \to \mathbb{R}$ by $g(x,y) = 2xy(y-x) + xy|y-x|$, $\forall x, y \in C$. Then, g is pseudomonotone and satisfies the assumption A. Let the mapping $T : C \to C$ and

TABLE 1. The comparison between Algorithm 3.2 and Algorithm 3 [18].

		Algorithm 3.2	Algorithm 3 [18]
Case (i)	CPU time (sec)	0.5652	3.7105
	No. of Iter.	22	234
Case (ii)	CPU time (sec)	1.0684	2.2114
	No. of Iter.	42	120
Case (iii)	CPU time (sec)	1.3392	1.9052
	No. of Iter.	48	122
Case (iv)	CPU time (sec)	2.2049	2.2096
	No. of Iter.	60	90

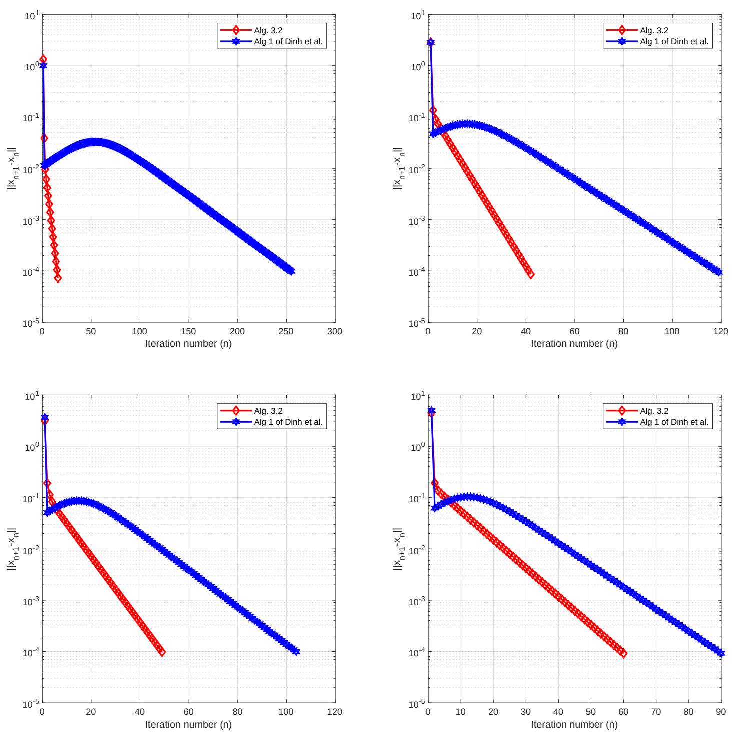

FIGURE 1. Example 5.1, Top Left: Case (i); Top Right: Case (ii); Bottom Left: Case (iii); Bottom right: Case (iv).

$S : Q \to Q$ be defined by $T(x) = \frac{2x}{3}$ and $S(y) = y$, respectively, for all $x \in C$ and $y \in Q$. Then, $F(T) = F(S) = \{0\}$ and $\Gamma \neq \emptyset$. We choose $\alpha_k = \frac{1}{k}$, $\beta_k = \frac{2}{5} + \frac{2k-1}{4k+13}$, $\gamma_k = \frac{2k-1}{4k^2+1}$, $\lambda_k = \frac{1}{2k+3}$ and $u = rand(m,1)$ and use $\frac{\|x_{k+1}-x_k\|}{\|x_2-x_1\|} < 10^{-5}$ as the stopping criterion. We compare the performance of our Algorithm 3.2 with Algorithm 3 [18] by varying the values of m as follows: [Case (i) $m = 10$, Case (ii) $m = 50$, Case (iii) $m = 70$, Case (iv) $m = 100$. We then plot the graph of "$\|x_{k+1} - x_k\|$" against the number of iterations in each cases. The numerical results can be seen in Table 1 and Figure 1.

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