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THE EXISTENCE OF ENTROPY SOLUTIONS FOR NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. In this article, we prove the existence of entropy solutions for the Dirichlet problem

$$\begin{cases} -\text{div}[\mathscr{A}(x,\nabla u)\,\omega_1 + \mathscr{B}(x,u,\nabla u)\,\omega_2] = f(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$ and $f \in L^1(\Omega)$. An example is provided to support our result.

Keywords. Nonlinear degenerate elliptic equations; Entropy solutions; Weighted Sobolev spaces.

1. Introduction

The main purpose of this paper is to establish the existence of entropy solutions for the following Dirichlet problem

$$\begin{cases} Lu(x) = f(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where

$$Lu = -\,\text{div}[\mathscr{A}(x,\nabla u)\,\omega_1 + \mathscr{B}(x,u,\nabla u)\,\omega_2], \tag{1.1}$$

$\Omega \subset \mathbb{R}^N$ is a bounded open set, ω_1 and ω_2 are two weight functions (i.e., a locally integrable function on $\mathbb{R}^N$ such that $0 < \omega_j(x) < \infty$ (j=1,2) a.e. $x \in \mathbb{R}^N$) which represent the degeneration (or singularity) in equation (1.1), $1 < q < p < \infty$, $f \in L^1(\Omega)$, the functions $\mathscr{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathscr{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfies the following conditions:

(H1) $x \mapsto \mathscr{A}(x,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^N$,
$\xi \mapsto \mathscr{A}(x,\xi)$ is continuous on $\mathbb{R}^N$ for almost all $x \in \Omega$.

(H2) There exists a constant $\theta_1 > 0$ such that

$$\langle \mathscr{A}(x,\xi) - \mathscr{A}(x,\xi'), (\xi - \xi') \rangle \geq \theta_1\,|\xi - \xi'|^p,$$

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whenever $\xi,\xi'\in\mathbb{R}^N$, $\xi\neq\xi'$, and $\mathscr{A}(x,\xi)=(\mathscr{A}_1(x,\xi),\dots,\mathscr{A}_N(x,\xi))$ (where $\langle .,.\rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^N$).

(H3) $\langle \mathscr{A}(x,\xi),\xi\rangle \geq \lambda_1|\xi|^p$, where λ_1 is a positive constant.

(H4) $|\mathscr{A}(x,\xi)|\leq K_1(x)+h_1(x)|\xi|^{p/p'}$, where K_1 and h_1 are nonnegative functions with $h_1\in L^\infty(\Omega)$ and $K_1\in L^{p'}(\Omega,\omega_1)$ (with $1/p+1/p'=1$).

(H5) $x\mapsto\mathscr{B}(x,s,\xi)$ is measurable on Ω for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$,
$(s,\xi)\mapsto\mathscr{B}(x,s,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost all $x\in\Omega$.

(H6) $\langle\mathscr{B}(x,s,\xi)-\mathscr{B}(x,s',\xi'),\xi-\xi'\rangle>0$, whenever $\xi,\xi'\in\mathbb{R}^N$, $\xi\neq\xi'$.

(H7) $\langle\mathscr{B}(x,s,\xi),\xi\rangle\geq\lambda_2\,|\xi|^q$, with $1<q<p<\infty$, and $\lambda_2>0$.

(H8) $|\mathscr{B}(x,s,\xi)|\leq K_2(x)+g_1(x)\,|s|^{q/q'}+g_2(x)\,|\xi|^{q/q'}$, where K_2, g_1 and g_2 are nonnegative functions with $g_1\in L^\infty(\Omega)$, $g_2\in L^\infty(\Omega)$ and $K_2\in L^{q'}(\Omega,\omega_2)$ (with $1/q+1/q'=1$).

The notion of entropy solutions was introduced in [1] where the authors studied the non-degenerate elliptic equation $-\mathrm{div}(a(x,Du))=f(x)$ with $f\in L^1(\Omega)$. In [2], the author studied the degenerate elliptic equation $Lu=f$, where L is a degenerate elliptic operator in divergence form (i.e., $Lu=-\sum_{i,j=1}^{n}D_j(a_{ij}(x)D_iu)$) and $f\in L^1(\Omega)$. In [3] ,the author studied the case when $\mathscr{A}(x,\xi)\equiv 0$ (i.e., $Lu=-\,\mathrm{div}(\mathscr{B}(x,u,\nabla u)\,\omega)$). Note that, in the proof of our main result, many ideas have been adapted from [1], [2] and [3]. For degenerate partial differential equations, i.e., the equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see, e.g., [4, 5, 6, 7, 8, 9]). A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by Muckenhoupt in the early 1970's (see [10]).

In this paper, we propose to solve problem (P) by approximation with variational solutions. We take $f_n\in C_0^\infty(\Omega)$ such that $f_n\to f$ in $L^1(\Omega)$, and find a solution $u_n\in W_0^{1,p}(\Omega,\omega_1)$ for the problem with right-hand side f_n and G_n.

2. Definitions and basic results

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathscr{W}(\Omega)$, we denote the set of all measurable, a.e., in Ω positive and finite functions $\omega=\omega(x)$, $x\in\Omega$. Elements of $\mathscr{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E)=\int_E\omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

Definition 2.1. Let $1\leq p<\infty$. A weight ω is said to be an A_p-weight if there is a positive constant $C=C(p,\omega)$ such that, for every ball $B\subset\mathbb{R}^N$,

$$\left(\frac{1}{|B|}\int_B\omega\,dx\right)\left(\frac{1}{|B|}\int_B\omega^{1/(1-p)}\,dx\right)^{p-1}\leq C\ \text{ if }\ p>1,$$
$$\left(\frac{1}{|B|}\int_B\omega\,dx\right)\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{\omega}\right)\leq C,\ \text{ if }\ p=1,$$

where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

If $1 < q \leq p$, then $A_q \subset A_p$ (see [7, 8, 11] for more details about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [9], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = e^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [12]).

Remark 2.2. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C\,\frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [8]). Therefore, if $\mu(E) = 0$, then $|E| = 0$. Thus, if $\{u_n\}$ is a sequence of functions defined in B and $u_n \to u$ μ-a.e. then $u_n \to u$ a.e.. The measure μ and the Lebesgue measure $|.|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$ if and only if $|E| = 0$). So, there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e.

Definition 2.3. Let ω be a weight. We denote by $L^p(\Omega, \omega)$ ($1 \leq p < \infty$) the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \omega\, dx\right)^{1/p} < \infty.$$

We denote $[L^{p'}(\Omega, \omega)]^N = L^{p'}(\Omega, \omega) \times \ldots \times L^{p'}(\Omega, \omega)$.

Remark 2.4. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ (see [12, Remark 1.2.4]). It thus makes sense to talk about the weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ a bounded open set, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega\, dx + \sum_{1 \leq |\alpha| \leq k} \int_\Omega |D^\alpha u|^p \omega\, dx\right)^{1/p}.$$

We also define the space $W^{k,p}_0(\Omega, \omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}_0(\Omega,\omega)} = \left(\sum_{1 \leq |\alpha| \leq k} \int_\Omega |D^\alpha u|^p \omega\, dx\right)^{1/p}.$$

The dual space of $W^{1,p}_0(\Omega, \omega)$ is the space $[W^{1,p}_0(\Omega, \omega)]^* = W^{-1,p'}(\Omega, \omega)$,

$$W^{-1,p'}(\Omega, \omega) = \{T = f - \operatorname{div}(G) : G = (g_1, \ldots, g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega, \omega)\}.$$

It is evident that a weight function ω, which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$ (where C_1 and C_2 are constants), gives nothing new (the space $W^{k,p}(\Omega, \omega)$ and is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight function ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basic result.

Theorem 2.6. *(The weighted Sobolev inequality) Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and let ω be an A_p-weight, $1<p<\infty$. Then there exist positive constants C_Ω and δ such that, for all $u\in W_0^{1,p}(\Omega,\omega)$ and $1\leq\eta\leq N/(N-1)+\delta$,*

$$\|u\|_{L^{\eta p}(\Omega,\omega)}\leq C_\Omega\|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}. \tag{2.1}$$

Proof. Its suffices to prove the inequality for functions $u\in C_0^\infty(\Omega)$ (see [6, Theorem 1.3]). To extend the estimates (2.1) to arbitrary $u\in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying estimates (2.1) to differences $u_{m_1}-u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{\eta p}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.1). □

Definition 2.7. Let $\omega\in A_p$, $1<p<\infty$. We say that $u\in\mathscr{T}_0^{1,p}(\Omega,\omega)$ if $T_k(u)\in W_0^{1,p}(\Omega,\omega)$, for all $k>0$, where the function $T_k:\mathbb{R}\to\mathbb{R}$ is defined by

$$T_k(s)=\begin{cases} s, & \text{if } |s|\leq k\\ k\,\text{sign}(s), & \text{if } |s|>k.\end{cases}$$

Remark 2.8. (i) Note that, for given $h>0$ and $k>0$,

$$T_h(u-T_k(u))=\begin{cases} 0, & \text{if } |u|\leq k\\ (|u|-k)\,\text{sign}(u), & \text{if } k<|u|\leq k+h\\ h\,\text{sign}(u), & \text{if } |u|>k+h.\end{cases}$$

And if $\alpha\in\mathbb{R}$, $\alpha\neq 0$, then $T_k(\alpha u)=\alpha\,T_{k/|\alpha|}(u)$.
(ii) If $u\in W^{1,1}_{\text{loc}}(\Omega,\omega)$ then

$$\nabla T_k(u)=\chi_{\{|u|<k\}}\nabla u,$$

where χ_E denotes the characteristic function of a measurable set $E\subset\mathbb{R}^N$.

Definition 2.9. Let $f\in L^1(\Omega)$ and $u\in\mathscr{T}_0^{1,p}(\Omega,\omega)$. We say that u is an entropy solution to problem (P) if

$$\int_\Omega\langle\mathscr{A}(x,\nabla u),\nabla T_k(u-\varphi)\rangle\,\omega_1\,dx+\int_\Omega\langle\mathscr{B}(x,u,\nabla u),\nabla T_k(u-\varphi)\rangle\,\omega_2\,dx$$
$$=\int_\Omega f\,T_k(u-\varphi)\,dx, \tag{2.2}$$

for all $k>0$ and all $\varphi\in W_0^{1,p}(\Omega,\omega_1)\cap L^\infty(\Omega)$.

We recall that the gradient of u which appears in (2.2) is defined as in [2, Remark 2.8], that is, $\nabla u=\nabla T_k(u)$ on the set where $|u|<k$.

Remark 2.10. Note that if $u_1,u_2\in W_0^{1,p}(\Omega,\omega)$ then

$$\varphi=T_k(u_1+u_2)\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$$

and

$$\nabla\varphi=\nabla T_k(u_1+u_2)=\nabla(u_1+u_2)\,\chi_{\{|u_1+u_2|<k\}}.$$

Definition 2.11. Let $1\leq p<\infty$ and let ω be a weight function. We define the weighted Marcinkiewicz space $\mathscr{M}^p(\Omega,\omega)$ as the set of measurable functions $f:\Omega\to\mathbb{R}$ such that the function

$$\Gamma_f(k)=\mu(\{x\in\Omega\ :\ |f(x)|>k\}),\ k>0,$$

satisfies an estimate of the form $\Gamma_f(k)\leq Ck^{-p}$, $0<C<\infty$.

Remark 2.12. (a) If $1<q<p<\infty$ and $\Omega\subset\mathbb{R}^N$ is a bounded set, then

$$L^p(\Omega,\omega)\subset\mathscr{M}^p(\Omega,\omega),\ \text{and}\ \mathscr{M}^p(\Omega,\omega)\subset L^q(\Omega,\omega)$$

(the proof follows the lines of [13, Theorem 2.18.8]).
(b) If $\dfrac{\omega_2}{\omega_1}\in L^r(\Omega,\omega_1)$, where $r=p/(p-q)$ (and $r'=p/q$), then $\mathscr{M}^p(\Omega,\omega_1)\subset\mathscr{M}^q(\Omega,\omega_2)$. In fact, we have for all $A\subset\mathbb{R}^N$ measurable set

$$\begin{aligned}\mu_2(A) &= \int_A \omega_2\,dx\\ &= \int_A \frac{\omega_2}{\omega_1}\,\omega_1\,dx\\ &\leq \left(\int_A \omega_1\,dx\right)^{1/r'}\left(\int_A\left(\frac{\omega_2}{\omega_1}\right)^r\omega_1\,dx\right)^{1/r}\\ &= [\mu_1(A)]^{1/r'}\,\|\omega_2/\omega_1\|_{L^r(\Omega,\omega_1)}.\end{aligned}$$

Hence $\mu_2(A)\leq C_r\,[\mu_1(A)]^{1/r'}$, where $C_r=\|\omega_2/\omega_1\|_{L^r(\Omega,\omega_1)}$. Therefore, if $\Omega_{f,k}=\{x\in\Omega:|f(x)|>k\}$, $\Gamma_f^{(1)}(k)=\mu_1(\Omega_{f,k})$, $\Gamma_f^{(2)}(k)=\mu_2(\Omega_{f,k})$ and $f\in\mathscr{M}^p(\Omega,\omega_1)$ (that is, $\mu_1(\Omega_{f,k})\leq C\,k^{-p}$), then

$$\begin{aligned}\Gamma_f^{(2)}(k) &= \mu_2(\Omega_{f,k})\\ &\leq C_r\,[\mu_1(\Omega_{f,k})]^{1/r'}\\ &\leq C_r(C\,k^{-p})^{1/r'}\\ &= C_rC^{1/r'}\,k^{-q},\end{aligned}$$

that is, $f\in\mathscr{M}^q(\Omega,\omega_2)$
(c) If $\dfrac{\omega_2}{\omega_1}\in L^r(\Omega,\omega_1)$ (where $r=p/(p-q)$, $1<q<p<\infty$), then

$$\|u\|_{L^q(\Omega,\omega_2)}\leq C_{p,q}\,\|u\|_{L^p(\Omega,\omega_1)},$$

where $C_{p,q}=\|\omega_2/\omega_1\|^{1/q}_{L^r(\Omega,\omega_1)}$. In fact, by Hölder's inequality, we obtain

$$\begin{aligned}\|u\|^q_{L^q(\Omega,\omega_2)} &= \int_\Omega |u|^q\,\omega_2\,dx=\int_\Omega|u|^q\,\frac{\omega_2}{\omega_1}\,\omega_1\,dx\\ &\leq \left(\int_\Omega |u|^{q\,p/q}\,\omega_1\,dx\right)^{q/p}\left(\int_\Omega\left(\omega_2/\omega_1\right)^{p/(p-q)}\omega_1\,dx\right)^{(p-q)/p}\\ &= \|u\|^q_{L^p(\Omega,\omega_1)}\,\|\omega_2/\omega_1\|_{L^r(\Omega,\omega_1)}.\end{aligned}$$

Hence,

$$\|u\|_{L^q(\Omega,\omega_2)}\leq C_{p,q}\|u\|_{L^p(\Omega,\omega_1)}.$$

Lemma 2.13. [2, Lemma 3.3] *Let* $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ *and* $\omega \in A_p$, $1 < p < \infty$, *be such that*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \omega \, dx \le M, \tag{2.3}$$

for every $k > 0$. *Then* $u \in \mathcal{M}^{p_1}(\Omega,\omega)$, *where* $p_1 = (p-1)$. *More precisely, there exists* $C > 0$ *such that* $\Gamma_u(k) \le C M k^{-p_1}$.

Lemma 2.14. [2, Lemma 3.4] *Let* $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$, *where* $\omega \in A_p$, $1 < p < \infty$, *be such that*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \omega \, dx \le M,$$

for every $k > 0$. *Then* $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega)$, *where* $p_2 = p\,p_1/(p_1+1)$ *(with* $p_1 = (p-1)$*). More precisely, there exists* $C > 0$ *such that* $\Gamma_{|\nabla u|}(k) \le C M k^{-p_2}$.

Lemma 2.15. *Let* $\omega \in A_p$, $1 < p < \infty$ *and a sequence* $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega,\omega)$ *satisfies*
(i) $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega,\omega)$ *and* μ*-a.e. in* Ω*;*
(ii) $\displaystyle\int_\Omega \langle \mathcal{B}(x,u_n,\nabla u_n) - \mathcal{B}(x,u_n,\nabla u), \nabla(u_n - u)\rangle \, \omega \, dx \to 0$ *with* $n \to \infty$.
Then $u_n \to u$ *in* $W_0^{1,p}(\Omega,\omega)$.

Proof. The proof of this lemma follows the lines of Lemma 5 in [14]. □

3. MAIN RESULT

In this section, we prove the main result of this paper.

Theorem 3.1. *Let* $\omega_1 \in A_p$, $\omega_2 \in \mathcal{W}(\Omega)$, $1 < q < p < \infty$, *with* $\dfrac{\omega_2}{\omega_1} \in L^r(\Omega,\omega_1)$ *(where* $r = p/(p-q)$*) and the conditions (H1)-(H8) be satisfied. Then there exists an entropy solutions* u *of problem (P). Moreover,* $u \in \mathcal{M}^{p_1}(\Omega,\omega_1)$ *and* $|\nabla u| \in \mathcal{M}^{p_2}(\Omega,\omega_1)$, *with* $p_1 = (p-1)$ *and* $p_2 = p_1\,p/(p_1+1)$.

Proof. Considering a sequence $\{f_n\}$, $f_n \in C_0^\infty(\Omega)$,

$$f_n \to f \text{ in } L^1(\Omega) \text{ and } \|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}.$$

For each n, there exists a solution $u_n \in W_0^{1,p}(\Omega,\omega_1)$ of the Dirichlet problem

$$(P_n)\begin{cases} -\text{div}[\mathcal{A}(x,\nabla u_n)\,\omega_1 + \mathcal{B}(x,u_n,\nabla u_n)\,\omega_2] = f_n(x) \text{ in } \Omega, \\ u_n(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

(by Theorem 1.1 in [15]), that is,

$$\int_\Omega \langle \mathcal{A}(x,\nabla u_n), \nabla\varphi\rangle \, \omega_1 \, dx + \int_\Omega \langle \mathcal{B}(x,u_n,\nabla u_n), \nabla\varphi\rangle \, \omega_2 \, dx = \int_\Omega f_n \varphi \, dx, \tag{3.1}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$. For $\varphi = T_k(u_n)$, we obtain from (3.1) that

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x,\nabla u_n), \nabla T_k(u_n)\rangle \, \omega_1 \, dx + \int_\Omega \langle \mathcal{B}(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle \, \omega_2 \, dx \\ &= \int_\Omega f_n T_k(u_n) \, dx. \end{aligned} \tag{3.2}$$

From (H3) and Remark 2.8 (ii), we have

$$\begin{aligned}\int_\Omega \langle \mathscr{A}(x,\nabla u_n),\nabla T_k(u_n)\rangle\,\omega_1\,dx &= \int_\Omega \langle \mathscr{A}(x,\nabla T_k(u_n),\nabla T_k(u_n)\rangle\,\omega_1\,dx\\ &\geq \lambda_1\int_\Omega |\nabla T_k(u_n)|^p\,\omega_1\,dx.\end{aligned}$$

By use of (H7), we have

$$\begin{aligned}&\int_\Omega \langle \mathscr{B}(x,u_n,\nabla u_n),\nabla T_k(u_n)\rangle\,\omega_2\,dx\\ =&\int_\Omega \langle \mathscr{B}(x,u_n,\nabla T_k(u_n)),\nabla T_k(u_n)\rangle\,\omega_2\,dx\\ \geq&\lambda_2\int_\Omega |\nabla T_k(u_n)|^q\omega_2\,dx>0\end{aligned}$$

and we also have

$$\left|\int_\Omega f_n T_k(u_n)\,dx\right|\leq \int_\Omega |f_n||T_k(u_n)|\,dx\leq k\|f_n\|_{L^1(\Omega)}\leq k\|f\|_{L^1(\Omega)}.$$

In view of (3.2), we obtain

$$\lambda_1\int_\Omega |\nabla T_k(u_n)|^p\omega_1\,dx+\lambda_2\int_\Omega |\nabla T_k(u_n)|^q\omega_2\,dx\leq k\,\|f\|_{L^1(\Omega)}.$$

Then, if $C_1=\|f\|_{L^1(\Omega)}/\lambda_1$, then

$$\int_\Omega |\nabla T_k(u_n)|^p\omega_1\,dx\leq\frac{k}{\lambda_1}\|f\|_{L^1(\Omega)}=C_1\,k,\ \text{for all } k>0. \tag{3.3}$$

By use of Lemma 2.13 and Lemma 2.14, we have that the sequence $\{u_n\}$ is bounded in $\mathscr{M}^{p_1}(\Omega,\omega_1)$ (with $p_1=(p-1)$ and $\{|\nabla u_n|\}$ is bounded in $\mathscr{M}^{p_2}(\Omega,\omega_1)$ (with $p_2=p_1\,p/(p_1+1)$). Moreover, $\{u_n\}$ is a Cauchy sequence in μ_1-measure. Consequently, there exist a function u and a subsequence, that we will still denote by $\{u_n\}$, such that

$$u_n\to u\ \text{ a.e. in } \Omega. \tag{3.4}$$

Using (3.3) and (3.4), we have

$$\begin{aligned}T_k(u_n)&\rightharpoonup T_k(u)\text{ weakly in } W_0^{1,p}(\Omega,\omega_1),\\ T_k(u_n)&\to T_k(u)\text{ strongly in } L^p(\Omega,\omega_1)\text{ and a.e. in }\Omega,\end{aligned}\tag{3.5}$$

for all $k>0$. Hence $T_k(u)\in W_0^{1,p}(\Omega,\omega_1)$. Furthermore, from the weak lower semicontinuity of the norm $W_0^{1,p}(\Omega,\omega_1)$, we have that (3.3) still holds for u, that is,

$$\int_\Omega |\nabla T_k(u)|^p\omega_1\,dx\leq C_1\,k.$$

Applying Lemma 2.13 and Lemma 2.14, we have that $u\in\mathscr{M}^{p_1}(\Omega,\omega_1)$ (with $p_1=(p-1)$) and $|\nabla u|\in\mathscr{M}^{p_2}(\Omega,\omega_1)$ (with $p_2=p_1\,p/(p_1+1)$).

- We need to shown that $T_k(u_n)\to T_k(u)$ strongly in $W_0^{1,p}(\Omega,\omega_1)$, for all $k>0$.

Letting $h > k$ and applying (3.1) with function $\varphi_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$, we get

$$\int_\Omega \langle \mathscr{A}(x, \nabla u_n), \nabla \varphi_n \rangle\, \omega_1\, dx + \int_\Omega \langle \mathscr{B}(x, u_n, \nabla u_n), \nabla \varphi_n \rangle\, \omega_2\, dx$$
$$= \int_\Omega f_n \varphi_n\, dx. \tag{3.6}$$

If $M = 4k + h$, then $\nabla \varphi_n = 0$ for $|u_n| > M$. Hence, since condition (H7) implies that $\mathscr{B}(x,s,0) = 0$ and condition (H3) implies that $\mathscr{A}(x,0) = 0$, we can write (3.6) in the form

$$\int_\Omega \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla \varphi_n \rangle\, \omega_1\, dx + \int_\Omega \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla \varphi_n \rangle\, \omega_2\, dx$$
$$= \int_\Omega f_n \varphi_n\, dx. \tag{3.7}$$

In the left-hand side of (3.7), we have

$$\begin{aligned}
&\int_\Omega \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_1\, dx \\
&+ \int_\Omega \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_2\, dx \\
&= \int_{\{|u_n| \le k\}} \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_1\, dx \\
&+ \int_{\{|u_n| \le k\}} \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_2\, dx \\
&+ \int_{\{|u_n| > k\}} \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_1\, dx \\
&+ \int_{\{|u_n| > k\}} \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_2\, dx.
\end{aligned} \tag{3.8}$$

(a) $|u_n| \le k$. Since $h > k$, if $|u_n| \le k < h$, then $T_h(u_n) = T_k(u_n) = u_n$. Hence,

$$u_n - T_h(u_n) + T_k(u_n) - T_k(u) = u_n - T_k(u).$$

We also have $|u_n - u| \le 2k$. Since $\nabla T_M(u_n) = \nabla T_k(u_n)$ (because $|u_n| \le k < M$), we obtain

$$\begin{aligned}
&\int_{\{|u_n| \le k\}} \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_1\, dx \\
&= \int_{\{|u_n| \le k\}} \langle \mathscr{A}(x, \nabla T_k(u_n)), \nabla (T_k(u_n) - T_k(u))\, \omega_1 dx \\
&= \int_\Omega \langle \mathscr{A}(x, \nabla T_k(u_n)), \nabla (T_k(u_n) - T_k(u)) \rangle\, \omega_1 dx.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{|u_n| \le k\}} \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle\, \omega_2\, dx \\
&= \int_{\{|u_n| \le k\}} \langle \mathscr{B}(x, T_k(u_n), \nabla T_k(u_n)), \nabla (T_k(u_n) - T_k(u))\, \omega_2 dx \\
&= \int_\Omega \langle \mathscr{B}(x, T_k(u_n), \nabla T_k(u_n)), \nabla (T_k(u_n) - T_k(u)) \rangle\, \omega_2 dx.
\end{aligned}$$

(b) $|u_n| > k$. Since u_n, $T_k(u_n)$ and $T_k(u)$ are in $W_0^{1,p}(\Omega, \omega_1)$, if $|u_n - T_h(u_n) + T_k(u_n) - T_k(u)| \le 2k$, then

$$\begin{aligned}
&\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \nabla(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \\
&= \nabla u_n - \nabla T_h(u_n) + \nabla T_k(u_n) - \nabla T_k(u) \\
&= \nabla u_n - \nabla T_h(u_n) - \nabla T_k(u)
\end{aligned}$$

(because $\nabla T_k(u_n) = 0$ if $|u_n| > k$). There are two possible cases as follows:
(i) If $k < |u_n| < h$, then $\nabla T_h(u_n) = \nabla u_n$. It follows that

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = -\nabla T_k(u);$$

(ii) If $h < |u_n| \le M$, then $\nabla T_h(u_n) = 0$. It follows that

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \nabla u_n - \nabla T_k(u) = \nabla T_M(u_n) - \nabla T_k(u).$$

Since $\langle \mathscr{A}(x,\xi),\xi\rangle \ge \lambda_1|\xi|^p \ge 0$ and $\langle \mathscr{B}(x,s,\xi),\xi\rangle \ge \lambda_2|\xi|^q \ge 0$, in both cases, we obtain

$$\begin{aligned}
&\langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle \\
&\ge -\langle \mathscr{A}(x, \nabla T_M(u_n), \nabla T_k(u)\rangle \\
&\ge -|\mathscr{A}(x, \nabla T_M(u_n))||\nabla T_k(u)|,
\end{aligned}$$

and

$$\begin{aligned}
&\langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle \\
&\ge -\langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n), \nabla T_k(u)\rangle \\
&\ge -|\mathscr{B}(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|.
\end{aligned}$$

Therefore we obtain from (3.8) that

$$\begin{aligned}
&\int_\Omega \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\, \omega_1\, dx \\
&+ \int_\Omega \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\, \omega_2\, dx \\
&= \int_{\{|u_n| \le k\}} \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\, \omega_1\, dx \\
&+ \int_{\{|u_n| \le k\}} \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\, \omega_2\, dx \\
&+ \int_{\{|u_n| > k\}} \omega\, \langle \mathscr{A}(x, \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\, \omega_1\, dx \\
&+ \int_{\{|u_n| > k\}} \omega\, \langle \mathscr{B}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\rangle\, \omega_2\, dx \\
&\ge \int_\Omega \langle \mathscr{A}(x, \nabla T_k(u_n)), \nabla (T_k(u_n) - T_k(u))\, \omega_1\, dx \\
&+ \int_\Omega \langle \mathscr{B}(x, T_k(u_n), \nabla T_k(u_n)), \nabla (T_k(u_n) - T_k(u))\, \omega_2\, dx \\
&- \int_{\{|u_n| > k\}} |\mathscr{A}(x, \nabla T_M(u_n))||\nabla T_k(u)|\, \omega_1\, dx \\
&- \int_{\{|u_n| > k\}} |\mathscr{B}(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\, \omega_2\, dx.
\end{aligned}$$

By use of (3.7), we obtain

$$\begin{aligned}
&\int_\Omega \langle \mathscr{A}(x,\nabla T_k(u_n)) - \mathscr{A}(x,\nabla T_k(u)), \nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx \\
&+\int_\Omega \langle \mathscr{B}(x,T_k(u_n),\nabla T_k(u_n)) - \mathscr{B}(x,T_k(u_n),\nabla T_k(u)), \nabla(T_k(u_n)-T_k(u))\rangle\,\omega_2\,dx \\
&\le \int_{\{|u_n|>k\}} |\mathscr{A}(x,\nabla T_M(u_n))||\nabla T_k(u)|\,\omega_1 dx \\
&+\int_{\{|u_n|>k\}} |\mathscr{B}(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,\omega_2 dx \\
&+\int_\Omega f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx \\
&-\int_\Omega \langle \mathscr{A}(x,\nabla T_k(u)), \nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx \\
&-\int_\Omega \langle \mathscr{B}(x,T_k(u_n),\nabla T_k(u)), \nabla(T_k(u_n)-T_k(u))\rangle\,\omega_2\,dx. \qquad (3.9)
\end{aligned}$$

Considering the test function $\psi_n = T_{2k}(u_n - T_h(u_n))$ in (3.1), we have

$$\int_\Omega \langle \mathscr{A}(x,\nabla u_n), \nabla\psi_n\rangle\,\omega_1\,dx + \int_\Omega \langle \mathscr{B}(x,u_n,\nabla u_n), \nabla\psi_n\rangle\,\omega_2\,dx = \int_\Omega f_n\psi_n\,dx,$$

and using (3.3), we obtain

$$\int_\Omega |\nabla T_{2k}(u_n - T_h(u_n))|^p\,\omega_1\,dx \le C_1(2k+1), \text{ for all } k\ge 1.$$

Now using the fact that $T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u))$ weakly in $W_0^{1,p}(\Omega,\omega_1)$ (by (3.5) and Remark 2.8 (i)), we have

$$\int_\Omega |\nabla T_{2k}(u - T_h(u))|^p\,\omega_1\,dx \le C_1(2k+1). \qquad (3.10)$$

Letting $\eta = 1$ in Theorem 2.6, we find that

$$\begin{aligned}
\int_\Omega |T_{2k}(u - T_h(u))|^p\,\omega_1\,dx &\le C_\Omega \int_\Omega |\nabla T_{2k}(u - T_h(u))|^p\,\omega_1\,dx \\
&\le C_\Omega C_1(2k+1).
\end{aligned}$$

Moreover, from Lebesgue's theorem, we obtain

$$\lim_{h\to\infty}\int_\Omega f\,T_{2k}(u - T_h(u))\,dx = 0.$$

We can fix a positive real number h_ε sufficiently large to have

$$\int_\Omega f T_{2k}(u - T_{h_\varepsilon}(u))\,dx \le \varepsilon. \qquad (3.11)$$

Letting $h = h_\varepsilon$ in (3.9) (and $M = M_\varepsilon = 4k + h_\varepsilon$), we have the following.

(i) By use of (H4) and (3.3), we have

$$\begin{aligned}
&\int_\Omega |\mathscr{A}(x,\nabla T_M(u_n))|^{p'}\omega_1\,dx\\
\le&\int_\Omega \left(K_1(x)+h_1(x)|\nabla T_M(u_n)|^{p/p'}\right)^{p'}\omega_1\,dx\\
\le& C\left[\int_\Omega K_1^{p'}(x)\,\omega_1\,dx+\int_\Omega h_1^{p'}(x)|\nabla T_M(u_n)|^p\omega_1\,dx\right]\\
\le& C\left(\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega|\nabla T_M(u_n)|^p\omega_1\,dx\right)\\
\le& C\left(\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)}+\|h_1\|^{p'}_{L^\infty(\Omega)}MC_1\right),
\end{aligned}$$

that is, $|\mathscr{A}(x,\nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega,\omega_1)$.
(ii) By use of (H8), Theorem 2.6 (with $\eta=1$), Remark 2.12 (c) and (3.3), we have

$$\begin{aligned}
&\int_\Omega |\mathscr{B}(x,T_M(u_n),\nabla T_M(u_n))|^{q'}\omega_2\,dx\\
\le&\int_\Omega\left(K_2(x)+g_1(x)|T_M(u_n)|^{q/q'}+g_2(x)|\nabla T_M(u_n)|^{q/q'}\right)^{q'}\omega_2\,dx\\
\le& C\left[\int_\Omega K_2^{q'}(x)\,\omega_2\,dx+\int_\Omega g_1^{q'}(x)|T_M(u_n)|^q\omega_2\,dx\right.\\
&\left.+\int_\Omega g_2^{q'}(x)|\nabla T_M(u_n)|^q\omega_2\,dx\right]\\
\le& C\left(\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)}+\|g_1\|^{q'}_{L^\infty(\Omega)}\int_\Omega|T_M(u_n)|^q\omega_2\,dx\right.\\
&\left.+\|g_2\|^{q'}_{L^\infty(\Omega)}\int_\Omega|\nabla T_M(u_n)|^q\omega_2\,dx\right)\\
\le& C\left(\|K_2\|^{q'}_{L^q(\Omega,\omega_2)}+\|g_1\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|T_M(u_n)\|^q_{L^p(\Omega,\omega_1)}\right.\\
&\left.+\|g_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\,|\nabla T_M(u_n)|\,\|^q_{L^p(\Omega,\omega_1)}\right)\\
\le& C\left(\|K_2\|^{q'}_{L^q(\Omega,\omega_2)}+\|g_1\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}C^q_\Omega\|\,|\nabla T_M(u_n)|\,\|^q_{L^p(\Omega,\omega_1)}\right.\\
&\left.+\|g_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\,|\nabla T_M(u_n)|\,\|^q_{L^p(\Omega,\omega_1)}\right)\\
\le& C\left(\|K_2\|^{q'}_{L^q(\Omega,\omega_2)}+\|g_1\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}C^q_\Omega(MC_1)^{q/p}+\|g_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}(MC_1)^{q/p}\right),
\end{aligned}$$

that is, $|\mathscr{B}(x,T_M(u_n),\nabla T_M(u_n))|$ is bounded in $L^{q'}(\Omega,\omega_2)$. Moreover, $\chi_{\{|u_n|>k\}}|\nabla T_k(u)|\to 0$ in $L^p(\Omega,\omega_1)$ as $n\to\infty$. We also have $\chi_{\{|u_n|>k\}}|\nabla T_k(u)|\to 0$ in $L^q(\Omega,\omega_2)$ as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}}|\mathscr{A}(x,\nabla T_M(u_n))||\nabla T_k(u)|\,\omega_1\,dx=0, \tag{3.12}$$

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}}|\mathscr{B}(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,\omega_2\,dx=0. \tag{3.13}$$

Furthermore,

$$T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\rightharpoonup T_{2k}(u-T_h(u)),$$

weakly in $W_0^{1,p}(\Omega,\omega_1)$, as $n\to\infty$. Hence, passing to the limit in (3.9) and using (3.5), (3.11), (3.12) and (3.13), we have

$$\begin{aligned}&\lim_{n\to\infty}\Bigg[\int_\Omega\langle\mathscr{A}(x,\nabla T_k(u_n))-\mathscr{A}(x,\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx\\&+\int_\Omega\langle\mathscr{B}(x,T_k(u_n),\nabla T_k(u_n))-\mathscr{B}(x,T_k(u_n),\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_2\,dx\Bigg]\\&\le\int_\Omega fT_{2k}(u-T_{h_\varepsilon}(u))\,dx\\&\le\varepsilon,\end{aligned}$$

for all $\varepsilon>0$, that is,

$$\begin{aligned}&\int_\Omega\langle\mathscr{A}(x,\nabla T_k(u_n))-\mathscr{A}(x,\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx\\&+\int_\Omega\langle\mathscr{B}(x,T_k(u_n),\nabla T_k(u_n))-\mathscr{B}(x,T_k(u_n),\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_2\,dx\\&\to 0,\ \text{as } n\to\infty.\end{aligned}$$

By use of (H2), $\langle\mathscr{A}(x,\xi)-\mathscr{A}(x,\xi'),(\xi-\xi')\rangle\ge\theta_1\,|\xi-\xi'|^p\ge 0$ and (H6), we obtain

$$\begin{aligned}0&\le\int_\Omega\langle\mathscr{A}(x,\nabla T_k(u_n))-\mathscr{A}(x,\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx\\&\le\int_\Omega\langle\mathscr{A}(x,\nabla T_k(u_n))-\mathscr{A}(x,\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx\\&+\int_\Omega\langle\mathscr{B}(x,T_k(u_n),\nabla T_k(u_n))-\mathscr{B}(x,T_k(u_n),\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_2\,dx.\end{aligned}$$

Hence,

$$\int_\Omega\langle\mathscr{A}(x,\nabla T_k(u_n))-\mathscr{A}(x,\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_1\,dx\to 0$$

as $n\to\infty$. Analogously, we obtain

$$\int_\Omega\langle\mathscr{B}(x,T_k(u_n),\nabla T_k(u_n))-\mathscr{B}(x,T_k(u_n),\nabla T_k(u)),\nabla(T_k(u_n)-T_k(u))\rangle\,\omega_2\,dx\to 0$$

as $n\to\infty$. Applying Lemma 2.15, we get

$$T_k(u_n)\to T_k(u) \tag{3.14}$$

strongly in $W_0^{1,p}(\Omega,\omega_1)$ for every $k>0$. Moreover (by Remark 2.12 (c)), we also have that

$$T_k(u_n)\to T_k(u) \tag{3.15}$$

strongly in $W_0^{1,q}(\Omega,\omega_2)$ for every $k>0$. This convergence implies that, for every fixed $k>0$,

$$\mathscr{A}(x,\nabla T_k(u_n))\to\mathscr{A}(x,\nabla T_k(u)), \tag{3.16}$$

in $(L^{p'}(\Omega,\omega_1))^N=L^{p'}(\Omega,\omega_1)\times\ldots\times L^{p'}(\Omega,\omega_1)$ and

$$\mathscr{B}(x,T_k(u_n),\nabla T_k(u_n))\to\mathscr{B}(x,T_k(u),\nabla T_k(u)) \tag{3.17}$$

in $(L^{q'}(\Omega,\omega_2))^N = L^{q'}(\Omega,\omega_2)\times\ldots\times L^{q'}(\Omega,\omega_2)$.

• Finally, we need to shown that u is an entropy solution to Dirichlet problem (P). Let us take $\psi_n = T_k(u_n-\varphi)$ as test function in (3.1), with $\varphi\in W_0^{1,p}(\Omega,\omega_1)\cap L^\infty(\Omega)$. We obtain

$$\int_\Omega \langle \mathscr{A}(x,\nabla u_n),\nabla\psi_n\rangle\,\omega_1\,dx + \int_\Omega \langle \mathscr{B}(x,u_n,\nabla u_n),\nabla\psi_n\rangle\,\omega_2\,dx = \int_\Omega f_n\psi_n\,dx. \tag{3.18}$$

If $M = k+\|\varphi\|_{L^\infty(\Omega)}$ and $n > M$, then

$$\begin{aligned}&\int_\Omega \mathscr{A}(x,\nabla u_n),\nabla T_k(u_n-\varphi)\rangle\,\omega_1 dx + \int_\Omega \mathscr{B}(x,u_n,\nabla u_n),\nabla T_k(u_n-\varphi)\rangle\,\omega_2 dx\\ &= \int_\Omega \langle \mathscr{A}(x,\nabla T_M(u_n)),\nabla T_k(u_n-\varphi)\rangle\,\omega_1\,dx\\ &+ \int_\Omega \langle \mathscr{B}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_k(u_n-\varphi)\rangle\,\omega_2\,dx.\end{aligned}$$

It follows from (3.18) that

$$\begin{aligned}&\int_\Omega \langle \mathscr{A}(x,\nabla T_M(u_n)),\nabla T_k(u_n-\varphi)\rangle\,\omega_1\,dx\\ &+ \int_\Omega \langle \mathscr{B}(x,T_M(u_n),\nabla T_M(u_n)),\nabla T_k(u_n-\varphi)\rangle\,\omega_2\,dx\\ &= \int_\Omega f_n T_k(u_n-\varphi)\,dx.\end{aligned} \tag{3.19}$$

Therefore, passing to the limit as $n\to\infty$ in (3.19), and using (3.5), (3.16) and (3.17), we obtain

$$\int_\Omega \langle \mathscr{A}(x,\nabla u),\nabla T_k(u-\varphi)\rangle\,\omega_1 dx + \int_\Omega \langle \mathscr{B}(x,u,\nabla u),\nabla T_k(u-\varphi)\rangle\,\omega_2 dx = \int_\Omega f\,T_k(u-\varphi)\,dx,$$

for all $\varphi\in W_0^{1,p}(\Omega,\omega_1)\cap L^\infty(\Omega)$ and for each $k>0$. Therefore u is an entropy solutions of problem (P). This completes the proof. □

Example 3.2. Let $\Omega = \{(x,y)\in\mathbb{R}^2 \ : \ x^2+y^2<1\}$. Consider the weight functions $\omega_1(x,y) = (x^2+y^2)^{-1/2}$, $\omega_2(x,y) = (x^2+y^2)^{-1/3}$ ($\omega_1\in A_4$, $\omega_2\in A_3$, $p=4$ and $q=3$), $f(x,y) = \dfrac{\cos(xy)}{(x^2+y^2)^{1/3}}$ and

$$\begin{aligned}&\mathscr{A} : \Omega\times\mathbb{R}^2\to\mathbb{R}^2,\\ &\mathscr{A}((x,y),\xi) = h(x,y)\,|\xi|^2\,\xi,\end{aligned}$$

where $h(x,y) = 2\,e^{(x^2+y^2)}$, and

$$\begin{aligned}&\mathscr{B} : \Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2,\\ &\mathscr{B}((x,y),\eta,\xi) = g_2(x,y)\,|\xi|\,\xi,\end{aligned}$$

where $g_2(x,y) = 2+\cos(x^2+y^2)$. from Theorem 3.1, the problem

$$(P)\begin{cases} -\operatorname{div}[\mathscr{A}((x,y),\nabla u)\,\omega_1(x,y) + \mathscr{B}((x,y),u,\nabla u)\,\omega_2(x,y)] = \dfrac{\cos(xy)}{(x^2+y^2)^{1/3}} \text{ in } \Omega\\ u(x,y) = 0 \ \text{ in } \ \partial\Omega\end{cases}$$

has an entropy solution.

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