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DYNAMIC ANALYSIS OF COMPETING PREDATOR-PREY SYSTEMS WITH PURE DELAYS

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Abstract. Two classes of nonautonomous three-species Lotka-Volterra type one predator-two competitive prey systems with pure discrete time delays are investigated. Some new sufficient conditions on the boundedness, the permanence, the extinction and the global attractivity of the systems are established by using the comparison method and the construction of suitable Lyapunov functional. Finally, the theoretical results are illustrated by one example with numerical simulations.

Keywords. Predator-prey-competitive system; Time delay; Lyapunov functional; Global attractivity.

1. Introduction

It is well known that the mathematical population dynamical system is one of the important discipline in modern applied mathematics, where population dynamical competitive systems, population dynamical cooperative systems, population dynamical predator-prey systems become the most popular topics recently. There has been a lot of studies related to the population dynamical systems; see, e.g., [1]-[18] and the references cited therein. Most of these studies concerned with the extinction, the permanence, the global attractivity and the existence of periodic solutions and so on. For example, in [2], Lin and Lu considered the following two species autonomous Lotka-Volterra systems with delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_1x_1(t) + a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})], \\ \dot{x}_2(t) &= x_2(t)[r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22})]. \end{aligned} \tag{1.1}$$

They obtained some sufficient conditions for the permanence of system (1.1) for competitive case and cooperative case respectively. In [4], Lu, Lu and Lian considered the following two species autonomous Lotka-Volterra cooperative systems with delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_1x_1(t) - a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})], \\ \dot{x}_2(t) &= x_2(t)[r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) - a_{22}x_2(t-\tau_{22})]. \end{aligned} \tag{1.2}$$

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They obtained some sufficient conditions for the permanence of system (1.2). In [7], Nakata and Muroya studied the following two species non-autonomous Lotka-Volterra system with delays

$$\begin{aligned} \dot{x}_1(t) = {} & x_1(t)[r_1(t) - a_{11}^1(t)x_1(t-\tau) \\ & -a_{11}^2(t)x_1(t-2\tau) + a_{12}^1(t)x_2(t-\tau)], \\ \dot{x}_2(t) = {} & x_2(t)[r_2(t) + a_{21}^0(t)x_1(t) + a_{21}^1(t)x_1(t-\tau) \\ & -a_{22}^0(t)x_2(t) - a_{22}^1(t)x_2(t-\tau)]. \end{aligned} \tag{1.3}$$

They established some sufficient conditions which ensured the system to be permanent. In [10], Lv, Yan and Lu considered the following competitor-competitor-mutualist Lotka-Volterra systems with pure delays

$$\begin{aligned} \dot{x}_1(t) = {} & x_1(t)\big(r_1(t) - a_{11}(t)x_1(t-\tau_{11}(t)) \\ & -a_{12}(t)x_2(t-\tau_{12}(t)) + a_{13}(t)x_3(t-\tau_{13}(t))\big), \\ \dot{x}_2(t) = {} & x_2(t)\big(r_2(t) - a_{21}(t)x_1(t-\tau_{21}(t)) \\ & -a_{22}(t)x_2(t-\tau_{22}(t)) + a_{23}(t)x_3(t-\tau_{23}(t))\big), \\ \dot{x}_3(t) = {} & x_3(t)\big(r_3(t) + a_{31}(t)x_1(t-\tau_{31}(t)) \\ & +a_{32}(t)x_2(t-\tau_{32}(t)) - a_{33}(t)x_3(t-\tau_{33}(t))\big). \end{aligned} \tag{1.4}$$

They e obtained sufficient conditions for the existence of periodic solutions by Krasnosselsii's fixed point theorem. For the case of $\tau_{ii}(t) \equiv 0$, they also obtained the global attractivity of positive periodic solution of system (1.4) by means of the construction of Liapunov functions. Base on the above works, Muhammadhaji, Teng and Zhang [12] studied the following three species non-autonomous Lotka-Volterra competitive - cooperative systems with delays

$$\begin{aligned} \dot{x}_1(t) = {} & x_1(t)\big[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) \\ & -a_{12}(t)x_2(t-2\tau) + a_{13}(t)x_3(t-\tau)\big], \\ \dot{x}_2(t) = {} & x_2(t)\big[r_2(t) - a_{21}(t)x_1(t-2\tau) - a_{22}^1(t)x_2(t-\tau) \\ & -a_{22}^2(t)x_2(t-2\tau) + a_{23}(t)x_3(t-\tau)\big], \\ \dot{x}_3(t) = {} & x_3(t)\big[r_3(t) + a_{31}(t)x_1(t-\tau) + a_{32}(t)x_2(t-\tau) \\ & -a_{33}^1(t)x_3(t) - a_{33}^2(t)x_3(t-\tau)\big]. \end{aligned} \tag{1.5}$$

They obtained some sufficient conditions on the permanence of species and the global attractivity of the system by construction of Liapunov functional and the method given in [7]. However, the systems of (1.1), (1.2), (1.3) and (1.5) are not pure delay systems. In [10], Lv, Yan and Lu did not consider the global attractivity of the systems for pure delay case. For example, systems ((1.1) and (1.2) include two non-delayed terms $a_1x_1(t)$, and $a_2x_2(t)$, system (1.3) includes two non-delayed terms $a_{21}^0(t)x_1(t)$ and $a_{22}^0(t)x_2(t)$ and system (1.5) includes one non-delayed term $a_{33}^1(t)x_3(t)$. For that reason and based on the above works, in this, paper, we consider the following two classes three species non-autonomous Lotka-Volterra type one predator-two competitive prey systems with pure discrete time delays

$$\begin{aligned} \dot{x}_1(t) = {} & x_1(t)\big[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) \\ & -a_{12}(t)x_2(t-2\tau) - a_{13}(t)x_3(t-2\tau)\big], \\ \dot{x}_2(t) = {} & x_2(t)\big[r_2(t) - a_{21}(t)x_1(t-2\tau) - a_{22}^1(t)x_2(t-\tau) \\ & -a_{22}^2(t)x_2(t-2\tau) - a_{23}(t)x_3(t-2\tau)\big], \\ \dot{x}_3(t) = {} & x_3(t)\big[-r_3(t) + a_{31}(t)x_1(t-2\tau) + a_{32}(t)x_2(t-2\tau) \\ & -a_{33}^1(t)x_3(t-\tau) - a_{33}^2(t)x_3(t-2\tau)\big] \end{aligned} \tag{1.6}$$

and

$$\begin{aligned}\dot{x}_1(t) = {} & x_1(t)\big[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) \\ & -a_{12}(t)x_2(t-2\tau) - a_{13}(t)x_3(t-2\tau)\big], \\ \dot{x}_2(t) = {} & x_2(t)\big[r_2(t) - a_{21}(t)x_1(t-2\tau) - a_{22}^1(t)x_2(t-\tau) \\ & -a_{22}^2(t)x_2(t-2\tau) - a_{23}(t)x_3(t-2\tau)\big], \\ \dot{x}_3(t) = {} & x_3(t)\big[r_3(t) + a_{31}(t)x_1(t-2\tau) + a_{32}(t)x_2(t-2\tau) \\ & -a_{33}^1(t)x_3(t-\tau) - a_{33}^2(t)x_3(t-2\tau)\big].\end{aligned} \tag{1.7}$$

As far as we know, the dynamic relationship between predators and preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [6]. In addition, an important problem in the predator-prey theory and the related topics in mathematical ecological dynamical systems concern the permanence, the extinction and the global attractivity of considered dynamical system. Hence, in this paper, our main purpose is to establish some sufficient conditions on the boundedness, the permanence, the extinction and the global attractivity for system (1.6) and system (1.7). The method used in this paper is motivated by the work in [7] and the work in [12].

2. Preliminaries

In system (1.6) and system (1.7), $x_1(t)$ and $x_2(t)$ denote the density of the two competitive prey species at time t, respectively, and $x_3(t)$ denotes the density of the predator species at time t. Throughout this paper, we always assume that system (1.6) and system (1.7) satisfy the following assumption

(H_1) τ is a positive constant, $r_i(t)$, $(i=1,2,3)$, $a_{11}^l(t)$, $a_{22}^l(t)$, $a_{33}^l(t)(l=1,2)$, $a_{12}(t)$, $a_{13}(t)$, $a_{21}(t)$, $a_{23}(t)$, $a_{31}(t)$ and $a_{32}(t)$ are continuous, bounded and strictly positive functions on $[0,\infty)$.

Throughout this paper, for system (1.6) and system (1.7), we consider the solution with the following initial condition

$$x_i(t) = \phi_i(t) \quad \text{for all} \quad t \in [-2\tau, 0),\ i = 1,2,3,$$

where $\phi_i(t)$ $(i=1,2,3)$ are nonnegative continuous functions defined on $[-2\tau,0)$ satisfying $\phi_i(0) > 0$ $(i=1,2,3)$. For a continuous and bounded function $f(t)$ defined on $[0,\infty)$, we define $f^L = \inf_{t\in[0,\infty)}\{f(t)\}$ and $f^M = \sup_{t\in[0,\infty)}\{f(t)\}$. On the global attractivity of system (1.6) and system (1.7), we have the following definition.

Definition 2.1. System (1.6) or system (1.7) is said to be global attractive if for any two positive solutions $(x_1(t),x_2(t),x_3(t))$ and $(y_1(t),y_2(t),y_3(t))$ of (1.6) or system (1.7), one has

$$\lim_{t\to\infty}(x_i(t) - y_i(t)) = 0, \quad i = 1,2,3.$$

Now, we present some useful lemmas.

The following three lemmas will be used in the main results on the boundedness of system (1.6) and system (1.7).

Lemma 2.2. [7] *Assume that function $y(t) \geq 0$ defined on $[-m\tau,\infty)$, satisfies that*

$$\dot{y}(t) \leq y(t)(\lambda - \sum_{l=1}^{m} \mu^l y(t-l\tau)) + D,$$

where

$$\lambda > 0,\ \mu^l \geq 0 (l = 0,1,2,\cdots m),\ \mu = \sum_{l=0}^{m} \mu^l > 0,\ D \geq 0,$$

are constants. Then there exists a positive constant M_y such that

$$\limsup_{t\to\infty} y(t) \leq M_y = -\frac{D}{\lambda} + \big(\frac{D}{\lambda} + y^*\big)\exp(\lambda m\tau), \tag{2.1}$$

where $y = y^*$ is the unique positive solution of equation

$$y(\lambda - \mu y) + D = 0.$$

Lemma 2.3. [7] *Assume that function $y(t) \geq 0$ defined on $[-m\tau,\infty)$ satisfies that*

$$\dot{y}(t) \geq y(t)(\lambda - \sum_{l=1}^{m} \mu^l y(t - l\tau)) + D,$$

where

$$\lambda > 0,\ \mu^l \geq 0 (l = 0,1,2,\cdots m),\ \mu = \sum_{l=0}^{m} \mu^l > 0 \quad \text{and } D \geq 0,$$

are constants. If (2.1) holds, then there exists a positive constant m_y such that

$$\liminf_{t\to\infty} y(t) \geq m_y = \frac{\lambda}{\mu}\exp\{(\lambda - \mu M_y)m\tau\}.$$

Lemma 2.4. [18] *Consider the following equation $\dot{u}(t) = u(t)(d_1 - d_2 u(t))$, where $d_2 > 0$, we have (1) if $d_1 > 0$, then $\lim_{t\to+\infty} u(t) = d_1/d_2$, and (2) if $d_1 < 0$, then $\lim_{t\to+\infty} u(t) = 0$.*

3. BOUNDEDNESS, PERMANENCE AND EXTINCTION

In this section, we will obtain some sufficient conditions for the boundedness of system (1.6) and system (1.7), permanence of species $x_3(t)$ in system (1.7) and extinction of species $x_3(t)$ in system (1.6).

Theorem 3.1. *If (H_1) holds, then system (1.6) is ultimately bounded.*

Proof. First, we show that $x_1(t)$ is ultimately bounded. From the first equation of system (1.6), we have

$$\dot{x_1}(t) \leq x_1(t)\big(r_1^M - a_{11}^{1L}x_1(t-\tau) - a_{11}^{2L}x_1(t-2\tau)\big),$$

By Lemma 2.2, we get

$$\limsup_{t\to\infty} x_1(t) \leq M_1 \triangleq \frac{r_1^M}{a_{11}^{1L} + a_{11}^{2L}}\exp\big(r_1^M 2\tau\big).$$

Similar to the above discussion, for $x_2(t)$, we can obtain

$$\limsup_{t\to\infty} x_2(t) \leq M_2 \triangleq \frac{r_2^M}{a_{22}^{1L} + a_{22}^{2L}}\exp\big(r_2^M 2\tau\big).$$

Finally, from the above discussion, for any positive constant $\varepsilon_0 > 0$, there exists a positive constant T_0 such that

$$x_1(t) \leq M_1 + \varepsilon_0,\ \ x_2(t) \leq M_2 + \varepsilon_0 \ \text{ for all } \ t \geq T_0.$$

Using the third equation of system (1.6), we have

$$\dot{x_3}(t) \le x_3(t)\big(a_{31}^M(M_1+\varepsilon_0)+a_{32}^M(M_2+\varepsilon_0)-a_{33}^{1L}x_3(t-\tau)-a_{33}^{2L}x_3(t-2\tau)\big), \quad t \ge T_0.$$

Since ε_0 is arbitrary, we get

$$\limsup_{t\to\infty} x_3(t) \le M_3 \triangleq \frac{M_0}{a_{33}^{1L}+a_{33}^{2L}}\exp\{M_0 2\tau\},$$

where $M_0 = a_{31}^M M_1 + a_{32}^M M_2$. This completes the proof. □

The following theorem is on the boundedness of system (1.7).

Theorem 3.2. *If (H_1) holds, then system (1.7) is ultimately bounded.*

Proof. By use of the similar method with Theorem 3.1, we can get the desired conclusion immediately. □

Remark 3.3. From the proof of Theorem 3.1 and Theorem 3.2, we can see that if species $x_1(t)$ and $x_2(t)$ have positive intrinsic growth rates, then system (1.6) and system (1.7) are must be ultimately bounded.

The following theorem is about the permanence of species $x_3(t)$ in system (1.7).

Theorem 3.4. *If (H_1) holds, then species $x_3(t)$ in system (1.7) is permanent.*

Proof. From third equation of system (1.7), we get

$$\dot{x_3}(t) \ge x_3(t)\big(r_3^L - a_{33}^{1M}x_3(t-\tau)-a_{33}^{2M}x_3(t-2\tau)\big).$$

By Lemma 2.3, we can obtain that

$$\liminf_{t\to\infty} x_3(t) \ge m_3 \triangleq \frac{r_3^L}{a_{33}^{1M}+a_{33}^{2M}}\exp\{(r_3^L-(a_{33}^{1M}+a_{33}^{2M})M_3)\tau\}.$$

This completes the proof. □

Remark 3.5. We can see that if species $x_3(t)$ has a positive intrinsic growth rate, then species $x_3(t)$ in system (1.7) must be permanent.

We also have the following result.

Corollary 3.6. *Assume that (H_1) holds and $r_3^L - a_{31}^M M_1 - a_{32}^M M_2 > 0$. Then the predator species in system (1.6) goes to extinction.*

4. Global attractivity

In this section, we will obtain the sufficient conditions for the global attractivity of system (1.6) and system (1.7). The following theorem is about the global attractivity of system (1.6).

Theorem 4.1. *Suppose that (H_1) holds. Further suppose that the following (H_2) holds.*

(H_2) There exist constants $\mu_i > 0$ $(i = 1,2,3)$ such that

$$\liminf_{t\to\infty} A_i(t) > 0, \quad i = 1,2,3,$$

where

$$\begin{aligned}A_1(t)= \ & \mu_1(a_{11}^1(t)+a_{11}^2(t))-\mu_1\textstyle\sum_{l=1}^2\int_{t-l\tau}^t a_{11}^l(u+l\tau)du\big[r_1(t)+(a_{11}^1(t)\\ &+a_{11}^2(t))M_1+a_{12}(t)M_2+a_{13}(t)M_3\big]-\mu_1M_1\textstyle\sum_{l=1}^2\int_t^{t+l\tau}a_{11}^l(u+l\tau)du\\ &\times a_{11}^l(t+l\tau)-\mu_2(1+\tau M_2(a_{22}^{1M}+2a_{22}^{2M}))a_{21}(t+2\tau)\\ &-\mu_3(1+\tau M_3(a_{33}^{1M}+2a_{33}^{2M}))a_{31}(t+2\tau),\end{aligned}$$

$$\begin{aligned}A_2(t)= \ & \mu_2(a_{22}^1(t)+a_{22}^2(t))-\mu_2\textstyle\sum_{l=1}^2\int_{t-l\tau}^t a_{22}^l(u+l\tau)du\big[r_2(t)+(a_{22}^1(t)\\ &+a_{22}^2(t))M_2+a_{21}(t)M_1+a_{23}(t)M_3\big]-\mu_2M_2\textstyle\sum_{l=1}^2\int_t^{t+l\tau}a_{22}^l(u+l\tau)du\\ &\times a_{22}^l(t+l\tau)-\mu_1(1+\tau M_1(a_{11}^{1M}+2a_{11}^{2M}))a_{12}(t+2\tau)\\ &-\mu_3(1+\tau M_3(a_{33}^{1M}+2a_{33}^{2M}))a_{32}(t+2\tau),\end{aligned}$$

and

$$\begin{aligned}A_3(t)= \ & \mu_3(a_{33}^1(t)+a_{33}^2(t))-\mu_3\textstyle\sum_{l=1}^2\int_{t-l\tau}^t a_{33}^l(u+l\tau)du\big[r_3(t)+(a_{33}^1(t)\\ &+a_{33}^2(t))M_3+a_{31}(t)M_1+a_{32}(t)M_2\big]-\mu_3M_3\textstyle\sum_{l=1}^2\int_t^{t+l\tau}a_{33}^l(u+l\tau)du\\ &\times a_{33}^l(t+l\tau)-\mu_1(1+\tau M_1(a_{11}^{1M}+2a_{11}^{2M}))a_{13}(t+2\tau)\\ &-\mu_2(1+\tau M_2(a_{22}^{1M}+2a_{22}^{2M}))a_{23}(t+2\tau).\end{aligned}$$

Then system (1.6) is globally attractive.

Proof. Suppose that $(x_1(t),x_2(t),x_3(t))$ and $(y_1(t),y_2(t),y_3(t))$ are any two positive solutions of system (1.6). From Theorem 3.1, there exist positive constants T_0 and M_i $(i=1,2,3)$ such that $0<x_i(t)$, $y_i(t)\le M_i$, $(i=1,2,3)$, for all $t\ge T_0$. Let

$$W_1(t)=\mu_1V_{11}(t)+\mu_2V_{21}(t)+\mu_3V_{31}(t),$$

where

$$V_{i1}=|\ln x_i(t)-\ln y_i(t)|,\quad i=1,2,3.$$

Calculating the right-upper derivative of $W_1(t)$ along system (1.6), we have

$$\begin{aligned}D^+W_1(t)= \ & \mu_1\mathrm{sign}(x_1(t)-y_1(t))\big[-a_{11}^1(t)(x_1(t-\tau)-y_1(t-\tau))\\ &-a_{11}^2(t)(x_1(t-2\tau)-y_1(t-2\tau))-a_{12}(t)(x_2(t-2\tau)\\ &-y_2(t-2\tau))-a_{13}(t)(x_3(t-2\tau)-y_3(t-2\tau))\big]\\ &+\mu_2\mathrm{sign}(x_2(t)-y_2(t))\big[-a_{21}(t)(x_1(t-2\tau)-y_1(t-2\tau))\\ &-a_{22}^1(t)(x_2(t-\tau)-y_2(t-\tau))-a_{22}^2(t)(x_2(t-2\tau)-y_2(t-2\tau))\\ &-a_{23}(t)(x_3(t-2\tau)-y_3(t-2\tau))\big]\\ &+\mu_3\mathrm{sign}(x_3(t)-y_3(t))\big[a_{31}(t)(x_1(t-2\tau)-y_1(t-2\tau))\\ &+a_{32}(t)(x_2(t-2\tau)-y_2(t-2\tau))-a_{33}^1(t)(x_3(t-\tau)-y_3(t-\tau))\\ &-a_{33}^2(t)(x_3(t-2\tau)-y_3(t-2\tau))\big]\end{aligned}$$

$$
\begin{aligned}
= \quad & \mu_1 \mathrm{sign}(x_1(t)-y_1(t))\big[-(a_{11}^1(t)+a_{11}^2(t))(x_1(t)-y_1(t))-a_{12}(t)(x_2(t-2\tau) \\
& -y_2(t-2\tau))-a_{13}(t)(x_3(t-2\tau)-y_3(t-2\tau))+\sum_{l=1}^{2} a_{11}^l(t)\int_{t-l\tau}^{t}\big((x_1(u)-y_1(u)) \\
& \times\big[r_1(u)-a_{11}^1(u)y_1(u-\tau)-a_{11}^2(u)y_1(u-2\tau)-a_{12}(u)y_2(u-2\tau) \\
& -a_{13}(u)y_3(u-2\tau)\big]+x_1(u)\big[-a_{11}^1(u)(x_1(u-\tau)-y_1(u-\tau)) \\
& -a_{11}^2(u)(x_1(u-2\tau)-y_1(u-2\tau))-a_{12}(u)(x_2(u-2\tau)-y_2(u-2\tau)) \\
& -a_{13}(u)(x_3(u-2\tau)-y_3(u-2\tau))\big]\big)du\big] \\
& +\mu_2 \mathrm{sign}(x_2(t)-y_2(t))\big[-(a_{22}^1(t)+a_{22}^2(t))(x_2(t)-y_2(t)) \\
& -a_{21}(t)(x_1(t-2\tau)-y_1(t-2\tau))-a_{23}(t)(x_3(t-2\tau)-y_3(t-2\tau)) \\
& +\sum_{l=1}^{2} a_{22}^l(t)\int_{t-l\tau}^{t}\big((x_2(u)-y_2(u))\big[r_2(u)-a_{22}^1(u)y_2(u-\tau) \\
& -a_{22}^2(u)y_2(u-2\tau)-a_{21}(u)y_1(u-2\tau)-a_{23}(u)y_3(u-2\tau)\big] \\
& +x_2(u)\big[-a_{22}^1(u)(x_2(u-\tau)-y_2(u-\tau))-a_{22}^2(u)(x_2(u-2\tau)-y_2(u-2\tau)) \\
& -a_{21}(u)(x_1(u-2\tau)-y_1(u-2\tau))-a_{23}(u)(x_3(u-2\tau)-y_3(u-2\tau))\big]\big)du\big] \\
& +\mu_3 \mathrm{sign}(x_3(t)-y_3(t))\big[-(a_{33}^1(t)+a_{33}^2(t))(x_3(t)-y_3(t)) \\
& +a_{31}(t)(x_1(t-2\tau)-y_1(t-2\tau))+a_{32}(t)(x_2(t-2\tau)-y_2(t-2\tau)) \\
& +\sum_{l=1}^{2} a_{33}^l(t)\int_{t-l\tau}^{t}\big((x_3(u)-y_3(u))\big[-r_3(u)-a_{33}^1(u)y_3(u-\tau) \\
& -a_{33}^2(u)y_3(u-2\tau)+a_{31}(u)y_1(u-2\tau)+a_{32}(u)y_2(u-2\tau)\big] \\
& +x_3(u)\big[-a_{33}^1(u)(x_3(u-\tau)-y_3(u-\tau))-a_{33}^2(u)(x_3(u-2\tau)-y_3(u-2\tau)) \\
& +a_{31}(u)(x_1(u-2\tau)-y_1(u-2\tau))+a_{32}(u)(x_2(u-2\tau)-y_2(u-2\tau))\big]\big)du\big] \\
\leq \quad & -\sum_{i=1}^{3}\mu_i(a_{ii}^1(t)+a_{ii}^2(t))|x_i(t)-y_i(t)|+(\mu_2 a_{21}(t)+\mu_3 a_{31}(t))|x_1(t-2\tau) \\
& -y_1(t-2\tau)|+(\mu_1 a_{12}(t)+\mu_3 a_{32}(t))|x_2(t-2\tau)-y_2(t-2\tau)| \\
& +(\mu_1 a_{13}(t)+\mu_2 a_{23}(t))|x_3(t-2\tau)-y_3(t-2\tau)| \\
& +\mu_1\sum_{l=1}^{2} a_{11}^l(t)\int_{t-l\tau}^{t}\big(|x_1(u)-y_1(u)|\big[r_1(u)+a_{11}^1(u)y_1(u-\tau) \\
& +a_{11}^2(u)y_1(u-2\tau)+a_{12}(u)y_2(u-2\tau)+a_{13}(u)y_3(u-2\tau)\big] \\
& +x_1(u)\big[a_{11}^1(u)|x_1(u-\tau)-y_1(u-\tau)|+a_{11}^2(u)|x_1(u-2\tau)-y_1(u-2\tau)|
\end{aligned}
$$

$$
\begin{aligned}
&+a_{12}(u)|x_2(u-2\tau)-y_2(u-2\tau)|+a_{13}(u)|x_3(u-2\tau)-y_3(u-2\tau)|\big]\big)du\\
&+\mu_2\sum_{l=1}^{2}a_{22}^{l}(t)\int_{t-l\tau}^{t}\big(|x_2(u)-y_2(u)|\big[r_2(u)+a_{22}^{1}(u)y_2(u-\tau)\\
&+a_{22}^{2}(u)y_2(u-2\tau)+a_{21}(u)y_1(u-2\tau)+a_{23}(u)y_3(u-2\tau)\big]\\
&+x_2(u)\big[a_{22}^{1}(u)|x_2(u-\tau)-y_2(u-\tau)|+a_{22}^{2}(u)|x_2(u-2\tau)-y_2(u-2\tau)|\\
&+a_{21}(u)|x_1(u-2\tau)-y_1(u-2\tau)|+a_{23}(u)|x_3(u-2\tau)-y_3(u-2\tau)|\big]\big)du \qquad (4.1)\\
&+\mu_3\sum_{l=1}^{2}a_{33}^{l}(t)\int_{t-l\tau}^{t}\big(|x_3(u)-y_3(u)|\big[r_3(u)+a_{33}^{1}(u)y_3(u-\tau)\\
&+a_{33}^{2}(u)y_3(u-2\tau)+a_{31}(u)y_1(u-2\tau)+a_{32}(u)y_2(u-2\tau)\big]\\
&+x_3(u)\big[a_{33}^{1}(u)|x_3(u-\tau)-y_3(u-\tau)|+a_{33}^{2}(u)|x_3(u-2\tau)-y_3(u-2\tau)|\\
&+a_{31}(u)|x_1(u-2\tau)-y_1(u-2\tau)|+a_{32}(u)|x_2(u-2\tau)-y_2(u-2\tau)|\big]\big)du.
\end{aligned}
$$

Define $W_2(t)=\mu_1V_{12}(t)+\mu_2V_{22}(t)+\mu_3V_{32}(t)$, where

$$
\begin{aligned}
V_{12}(t)=\;&\sum_{l=1}^{2}\int_{t-l\tau}^{t}\int_{u}^{t}a_{11}^{l}(u+l\tau)\big(\big[r_1(s)+a_{11}^{1}(s)y_1(s-\tau)+a_{11}^{2}(s)y_1(s-2\tau)\\
&+a_{12}(s)y_2(s-2\tau)+a_{13}(s)y_3(s-2\tau)\big]|x_1(s)-y_1(s)|\\
&+x_1(s)\big[a_{11}^{1}(s)|x_1(s-\tau)-y_1(s-\tau)|+a_{11}^{2}(s)|x_1(s-2\tau)-y_1(s-2\tau)|\\
&+a_{12}(s)|x_2(s-2\tau)-y_2(s-2\tau)|+a_{13}(s)|x_3(s-2\tau)-y_3(s-2\tau)|\big]\big)dsdu,\\
V_{22}(t)=\;&\sum_{l=1}^{2}\int_{t-l\tau}^{t}\int_{u}^{t}a_{22}^{l}(u+l\tau)\big(\big[r_2(s)+a_{22}^{1}(s)y_2(s-\tau)+a_{22}^{2}(s)y_2(s-2\tau)\\
&+a_{21}(s)y_1(s-2\tau)+a_{23}(s)y_3(s-2\tau)\big]|x_2(s)-y_2(s)|\\
&+x_2(s)\big[a_{22}^{1}(s)|x_2(s-\tau)-y_2(s-\tau)|+a_{22}^{2}(s)|x_2(s-2\tau)-y_2(s-2\tau)|\\
&+a_{21}(s)|x_1(s-2\tau)-y_1(s-2\tau)|+a_{23}(s)|x_3(s-2\tau)-y_3(s-2\tau)|\big]\big)dsdu,\\
V_{32}(t)=\;&\sum_{l=1}^{2}\int_{t-l\tau}^{t}\int_{u}^{t}a_{33}^{l}(u+l\tau)\big(\big[r_3(s)+a_{33}^{1}(s)y_3(s-\tau)+a_{33}^{2}(s)y_3(s-2\tau)\\
&+a_{31}(s)y_1(s-2\tau)+a_{32}(s)y_2(s-2\tau)\big]|x_3(s)-y_3(s)|\\
&+x_3(s)\big[a_{33}^{1}(s)|x_3(s-\tau)-y_3(s-\tau)|+a_{33}^{2}(s)|x_3(s-2\tau)-y_3(s-2\tau)|\\
&+a_{31}(s)|x_1(s-2\tau)-y_1(s-2\tau)|+a_{32}(s)|x_2(s-2\tau)-y_2(s-2\tau)|\big]\big)dsdu.
\end{aligned}
$$

Calculating the right-upper derivative of $W_2(t)$ and from (4.1), we have

$$\begin{aligned}
\sum_{i=1}^{2} D^+W_i(t) \leq & -\sum_{i=1}^{3}\mu_i(a_{ii}^1(t)+a_{ii}^2(t))|x_i(t)-y_i(t)|+(\mu_2 a_{21}(t)+\mu_3 a_{31}(t))|x_1(t-2\tau) \\
& -y_1(t-2\tau)|+(\mu_1 a_{12}(t)+\mu_3 a_{32}(t))|x_2(t-2\tau)-y_2(t-2\tau)| \\
& +(\mu_1 a_{13}(t)+\mu_2 a_{23}(t))|x_3(t-2\tau)-y_3(t-2\tau)| \\
& +\mu_1\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{11}^l(u+l\tau)du\big[r_1(t)+(a_{11}^1(t)+a_{11}^2(t))M_1+a_{12}(t)M_2 \\
& +a_{13}(t)M_3\big]|x_1(t)-y_1(t)|+\mu_1 M_1\textstyle\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{11}^l(u+l\tau)du \\
& \times a_{11}^l(t)|x_1(t-l\tau)-y_1(t-l\tau)|+\mu_1 M_1\tau(a_{11}^{1M}+2a_{11}^{2M}) \\
& \times[a_{12}(t)|x_2(t-2\tau)-y_2(t-2\tau)|+a_{13}(t)|x_3(t-2\tau)-y_3(t-2\tau)|] \\
& +\mu_2\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{22}^l(u+l\tau)du\big[r_2(t)+(a_{22}^1(t)+a_{22}^2(t))M_2+a_{21}(t)M_1 \\
& +a_{23}(t)M_3\big]|x_2(t)-y_2(t)|+\mu_2 M_2\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{22}^l(u+l\tau)du \\
& \times a_{22}^l(t)|x_2(t-l\tau)-y_2(t-l\tau)|+\mu_2 M_2\tau(a_{22}^{1M}+2a_{22}^{2M}) \\
& \times\big[a_{21}(t)|x_1(t-2\tau)-y_1(t-2\tau)|+a_{23}(t)|x_3(t-2\tau)-y_3(t-2\tau)|\,\big], \\
& +\mu_3\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{33}^l(u+l\tau)du\big[r_3(t)+(a_{33}^1(t)+a_{33}^2(t))M_3+a_{31}(t)M_1 \\
& +a_{32}(t)M_2\big]|x_3(t)-y_3(t)|+\mu_3 M_3\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{33}^l(u+l\tau)du \\
& \times a_{33}^l(t)|x_3(t-l\tau)-y_3(t-l\tau)|+\mu_3 M_3\tau(a_{33}^{1M}+2a_{33}^{2M}) \\
& \times\big[a_{31}(t)|x_1(t-2\tau)-y_1(t-2\tau)|+a_{32}(t)|x_2(t-2\tau)-y_2(t-2\tau)|\,\big],
\end{aligned} \tag{4.2}$$

Define $W_3(t)=\mu_1 V_{13}(t)+\mu_2 V_{23}(t)+\mu_3 V_{33}(t)$, where

$$\begin{aligned}
V_{13}(t) = & M_1\sum_{l=1}^{2}\int_{t-l\tau}^{t}\int_{s}^{s+l\tau} a_{11}^l(u+l\tau)a_{11}^l(s+l\tau)|x_1(s)-y_1(s)|duds \\
& +(1+M_1\tau(a_{11}^{1M}+2a_{11}^{2M}))\int_{t-2\tau}^{t} a_{12}(u+2\tau)|x_2(u)-y_2(u)|du \\
& +(1+M_1\tau(a_{11}^{1M}+2a_{11}^{2M}))\int_{t-2\tau}^{t} a_{13}(u+2\tau)|x_3(u)-y_3(u)|du,
\end{aligned}$$

$$V_{23}(t) = M_2\sum_{l=1}^{2}\int_{t-l\tau}^{t}\int_{s}^{s+l\tau} a_{22}^{l}(u+l\tau)a_{22}^{l}(s+l\tau)|x_2(s)-y_2(s)|duds$$
$$+(1+M_2\tau(a_{22}^{1M}+2a_{22}^{2M}))\int_{t-2\tau}^{t} a_{21}(u+2\tau)|x_1(u)-y_1(u)|du$$
$$+(1+M_2\tau(a_{22}^{1M}+2a_{22}^{2M}))\int_{t-2\tau}^{t} a_{23}(u+2\tau)|x_3(u)-y_3(u)|du,$$
$$V_{33}(t) = M_3\sum_{l=1}^{2}\int_{t-l\tau}^{t}\int_{s}^{s+l\tau} a_{33}^{l}(u+l\tau)a_{33}^{l}(s+l\tau)|x_3(s)-y_3(s)|duds$$
$$+(1+M_3\tau(a_{33}^{1M}+2a_{33}^{2M}))\int_{t-2\tau}^{t} a_{31}(u+2\tau)|x_1(u)-y_1(u)|du$$
$$+(1+M_3\tau(a_{33}^{1M}+2a_{33}^{2M}))\int_{t-2\tau}^{t} a_{32}(u+2\tau)|x_2(u)-y_2(u)|du.$$

Calculating the right-upper derivative of $W_3(t)$ and from (4.2), we have

$$\sum_{i=1}^{3} D^+W_i(t) \le -\Big(\mu_1(a_{11}^1(t)+a_{11}^2(t)) - \mu_1\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{11}^{l}(u+l\tau)du\big[r_1(t)+(a_{11}^1(t)$$
$$+a_{11}^2(t))M_1+a_{12}(t)M_2+a_{13}(t)M_3\big] - \mu_1M_1\sum_{l=1}^{2}\int_{t}^{t+l\tau} a_{11}^{l}(u+l\tau)du$$
$$\times a_{11}^{l}(t+l\tau) - \mu_2(1+\tau M_2(a_{22}^{1M}+2a_{22}^{2M}))a_{21}(t+2\tau)$$
$$-\mu_3(1+\tau M_3(a_{33}^{1M}+2a_{33}^{2M}))a_{31}(t+2\tau)\Big)|x_1(t)-y_1(t)|$$
$$-\Big(\mu_2(a_{22}^1(t)+a_{22}^2(t)) - \mu_2\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{22}^{l}(u+l\tau)du\big[r_2(t)+(a_{22}^1(t)$$
$$+a_{22}^2(t))M_2+a_{21}(t)M_1+a_{23}(t)M_3\big] - \mu_2M_2\sum_{l=1}^{2}\int_{t}^{t+l\tau} a_{22}^{l}(u+l\tau)du$$
$$\times a_{22}^{l}(t+l\tau) - \mu_1(1+\tau M_1(a_{11}^{1M}+2a_{11}^{2M}))a_{12}(t+2\tau)$$
$$-\mu_3(1+\tau M_3(a_{33}^{1M}+2a_{33}^{2M}))a_{32}(t+2\tau)\Big)|x_2(t)-y_2(t)|$$
$$-\Big(\mu_3(a_{33}^1(t)+a_{33}^2(t)) - \mu_3\sum_{l=1}^{2}\int_{t-l\tau}^{t} a_{33}^{l}(u+l\tau)du\big[r_3(t)+(a_{33}^1(t)$$
$$+a_{33}^2(t))M_3+a_{31}(t)M_1+a_{32}(t)M_2\big] - \mu_3M_3\sum_{l=1}^{2}\int_{t}^{t+l\tau} a_{33}^{l}(u+l\tau)du$$

$$\begin{aligned}&\times a_{33}^{l}(t+l\tau)-\mu_1(1+\tau M_1(a_{11}^{1M}+2a_{11}^{2M}))a_{13}(t+2\tau)\\&-\mu_2(1+\tau M_2(a_{22}^{1M}+2a_{22}^{2M}))a_{23}(t+2\tau)\Big)|x_3(t)-y_3(t)|.\end{aligned}\tag{4.3}$$

Further, we define a Lyapunov function as follows

$$V(t)=\sum_{i=1}^{3}W_i(t).$$

Calculating the right-upper derivative of $V(t)$, we obtain from (4.3) that, for all $t\geq T_0$,

$$D^{+}V(t)\ \leq -\sum_{i=1}^{3}A_i(t)|x_i(t)-y_i(t)|.\tag{4.4}$$

From assumption (H_2), there exists a constant $\alpha>0$ and $T^*\geq T_0$ such that, for all $t\geq T^*$,

$$A_i(t)\geq\alpha>0,\quad i=1,2,3.\tag{4.5}$$

Integrating from T^* to t on both sides of (4.4) and by (4.5) produces

$$V(t)+\alpha\int_{T^*}^{t}\Big(\sum_{i=1}^{3}|x_i(s)-y_i(s)|\Big)ds\leq V(T^*).$$

Hence, $V(t)$ is bounded on $[T^*,\infty)$ and we have

$$\int_{T^*}^{t}\Big(\sum_{i=1}^{3}|x_i(s)-y_i(s)|\Big)ds<\infty.$$

From Theorem 3.1, we can obtain that $(x_i(t)-y_i(t))(i=1,2,3)$ and their derivatives remain bounded on $[T^*,\infty)$. As a consequence, $|x_i(t)-y_i(t)|$ $(i=1,2,3)$ is uniformly continuous on $[T^*,\infty)$. By use of Barbalat's lemma, it follows that

$$\lim_{t\to\infty}\sum_{i=1}^{3}|x_i(t)-y_i(t)|=0.$$

Hence,

$$\lim_{t\to\infty}(x_i(t)-y_i(t))=0,\quad i=1,2,3.$$

This completes the proof. □

The following theorem is about the globally attractivity of system (1.7).

Theorem 4.2. *If all conditions of Theorem 4.1 hold, then system (1.7) is globally attractive.*

Proof. By using the similar method with Theorem 4.1, we can conclude the desired result. □

Remark 4.3. The aim of the construction of the multiple Lyapunov functional is to produce non-delayed terms in the right-upper derivative calculation of Lyapunov functional. Thus, we can offset the delayed terms by non-delayed terms. From the Lyapunov functional $W_2(t)$ and $W_3(t)$ in the proof, we can see that the upper bounds $M_i(i=1,2,3)$ are very useful to construction of the Lyapunov functional.

5. The example

In this section, we will give an example to illustrate the results obtained in this paper.

Example 5.1. We consider the following system

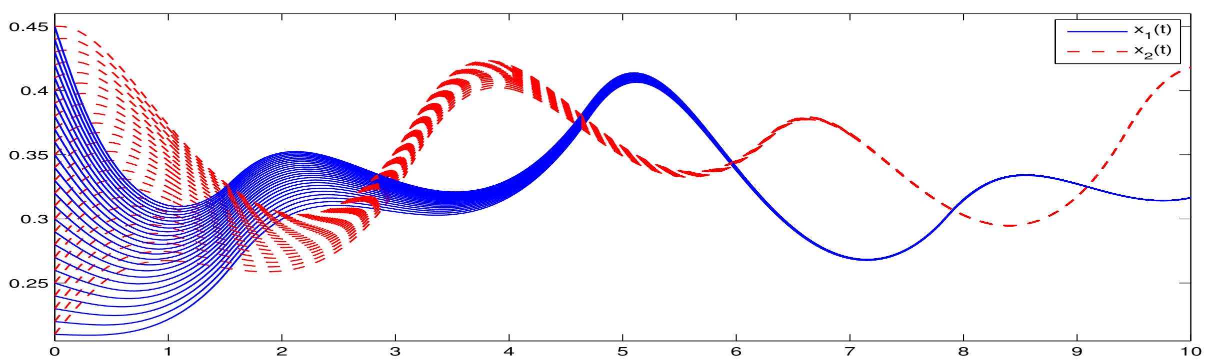
$$
\begin{aligned}
\dot{x}_1(t) = \;& x_1(t)\Big(\frac{2-|cos(t)|}{2} - (1.8+0.35cos(t))x_1(t-\tau) - \frac{1.1+sin(t)}{4}x_1(t-2\tau) \\
& - \frac{4+4|cos(t)|}{1000}x_2(t-2\tau) - \frac{7+7|sin(t)|}{1000}x_3(t-2\tau)\Big), \\
\dot{x}_2(t) = \;& x_2(t)\Big(\frac{2-|sin(t)|}{2} - \frac{4+4|sin(t)|}{1000}x_1(t-2\tau) - \frac{3.3+0.3sin(t)}{2}x_2(t-\tau) \\
& - \frac{1.1+cos(t)}{4}x_2(t-2\tau) - \frac{7+7|sin(t)|}{1000}x_3(t-2\tau)\Big), \\
\dot{x}_3(t) = \;& x_3(t)\Big(-\frac{2-|cos(t)|}{1000} + \frac{8+|sin(t)|}{100}x_1(t-2\tau) + \frac{8+|cos(t)|}{100}x_2(t-2\tau) \\
& - \frac{5+sin(t)}{2}x_3(t-\tau) - \frac{5+cos(t)}{6}x_3(t-2\tau)\Big).
\end{aligned}
\tag{5.1}
$$

In system (5.1), if $\tau = 0.15$, then

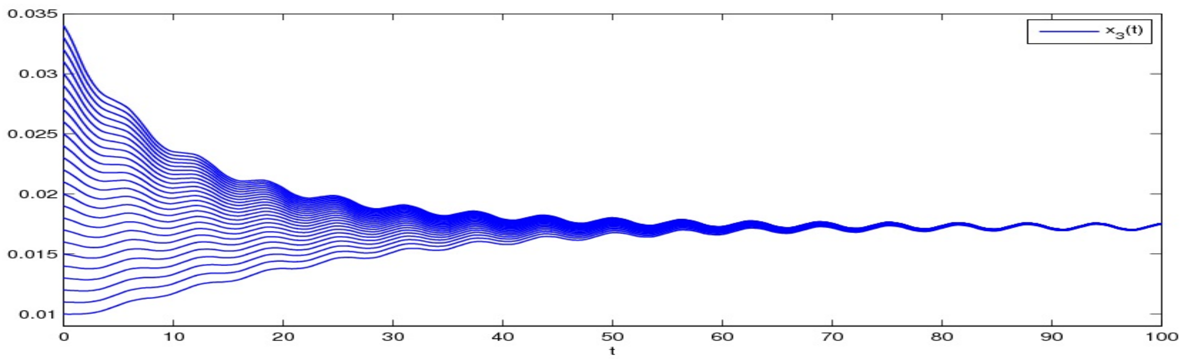
$$M_1 \approx 0.9152, \quad M_2 \approx 0.8852, \quad M_3 \approx 0.5062,$$

$$\liminf_{t\to\infty} A_1(t) \approx 0.4551, \quad \liminf_{t\to\infty} A_2(t) \approx 0.5696, \quad \liminf_{t\to\infty} A_3(t) \approx 1.4764.$$

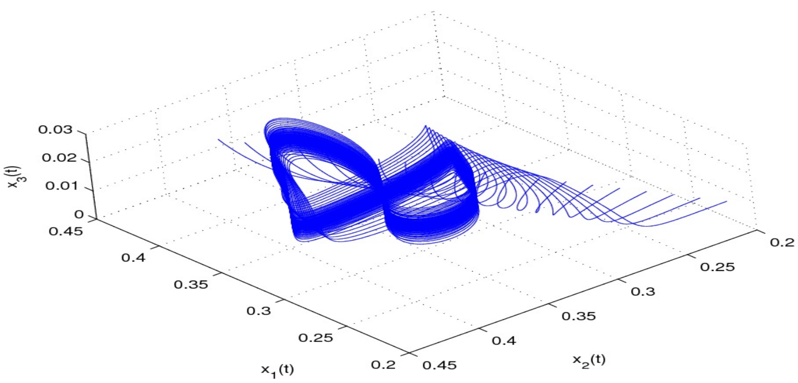
It is clear that the conditions of Theorem 4.1 hold.



(a)



(b)



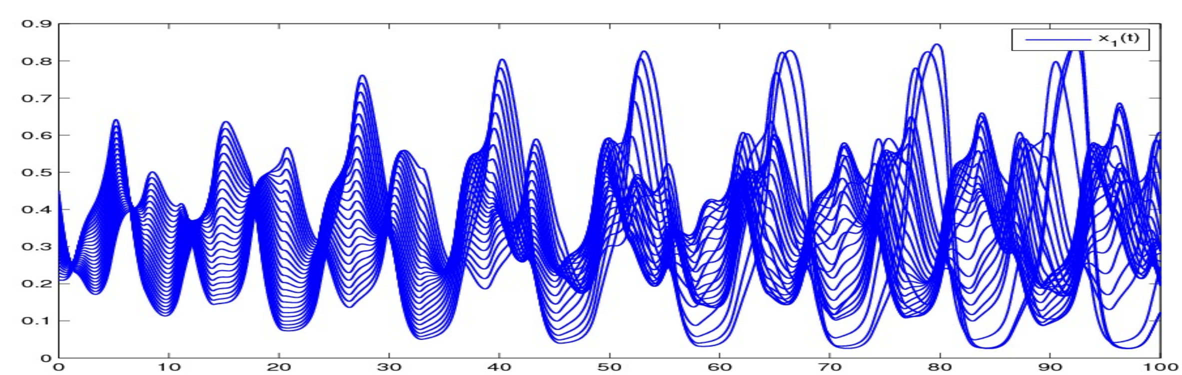
(c)

Fig. 1. Global attractivity of system (5.1)

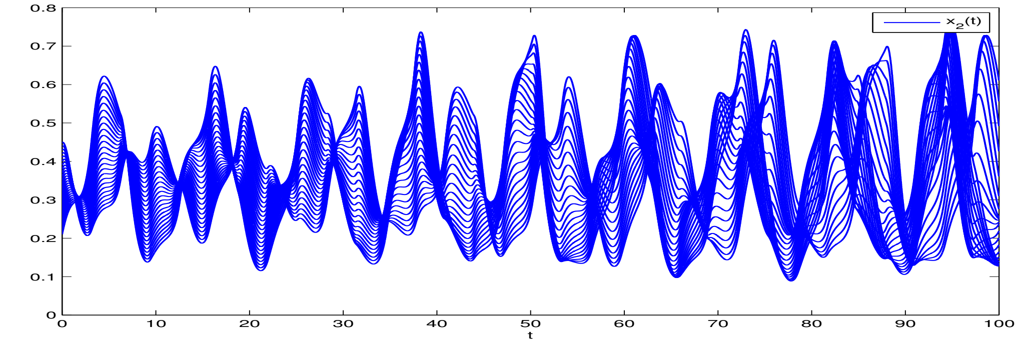
From Fig. 1. we can see, system (5.1) is globally attractive.
Next, if $\tau = 2.5$, then

$$M_1 \approx 100.6191, \quad M_2 \approx 97.3201, \quad M_3 \approx 55.6549, \qquad \liminf_{t\to\infty} A_1(t) \approx -2232.2 < 0$$

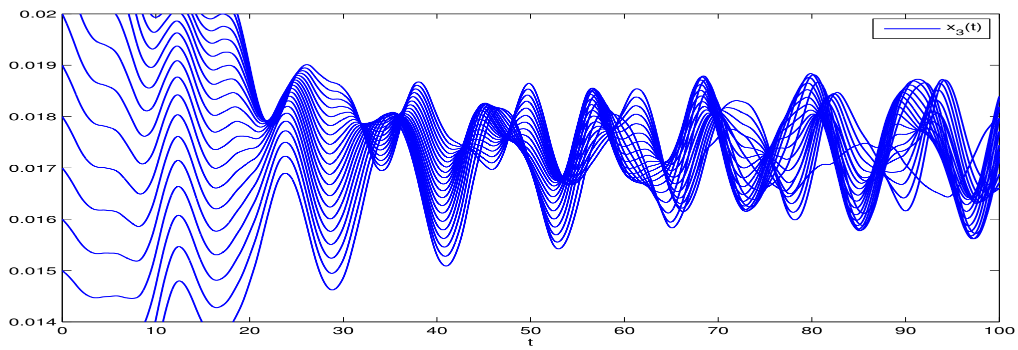
It is clear that the conditions of Theorem 4.1 do not hold.



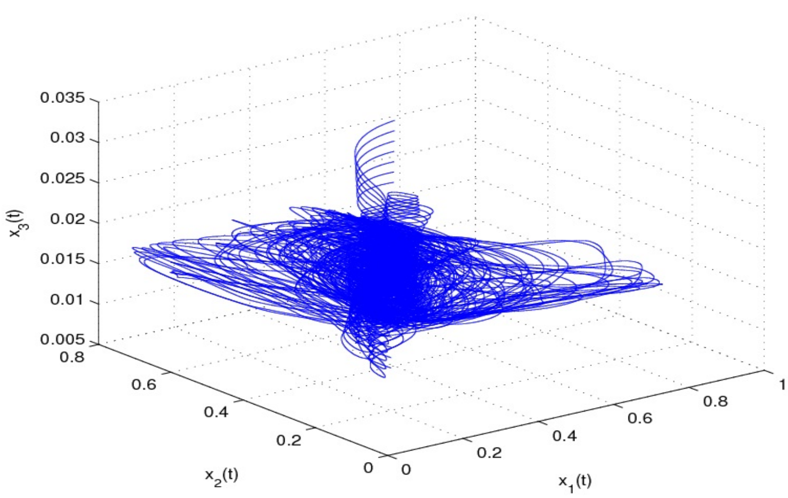
(a)



(b)



(c)



(d)

Fig. 2. Non-global attractivity of system (5.1)

From the Fig. 2. we can see, system (5.1) is not globally attractive.

Remark 5.1. From the above example, we can see that the effect of time delays on the global attractivity of the system. The results of this paper suggest the biological implication that the length of time delays can change the global attractivity of the system, in some cases, small time delays are harmless for the global attractivity of the system. However, the global attractivity of the system may be destroyed by some other larger time delays.

6. Conclusion

In this paper, two classes of nonautonomous three-species Lotka-Volterra type one predator-two competitive prey systems with pure discrete time delays are proposed and analyzed to study the boundedness of the solution and global attractivity of the systems. Based on the comparison method and the construction of the multiple Lyapunov functional, some new sufficient conditions on boundedness, permanence, extinction and global attractivity of the systems are obtained. In addition, numerical simulation results shows the feasibility of our results. Moreover, the models and results present in this paper can been seen as the improvement and extension of the models and results obtained in [2, 4, 7, 9, 10, 12].

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