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THREE SOLUTIONS FOR A FRACTIONAL $(p(x,.),q(x,.))$-KIRCHHOFF TYPE ELLIPTIC SYSTEM

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Abstract. This paper is concerned with the existence of weak solutions for a nonlocal fractional elliptic system of $(p(x,.),q(x,.))$-Kirchhoff type with homogeneous Dirichlet boundary conditions. The approach is based on the three critical points theorem introduced by Recceri and on the theory of fractional Sobolev spaces with variable exponents.

Keywords. Fractional $p(.)$-laplacian; Fractional Sobolev spaces with variable exponents; Nonlocal problem; Three critical points theorem.

1. Introduction

In this paper, we aim to investigate the existence and the multiplicity of weak solutions for the following fractional elliptic system of $(p(x,.),q(x,.))-$ Kirchhoff type

$$\begin{cases} M(I_{s,p(x,y)}(u))((-\Delta)^s_{p(x,.)}u+|u|^{\bar{p}(x)-2}u)=\lambda F_u(x,u,v)+\mu G_u(x,u,v), & \text{in } \Omega,\\ M(I_{s,q(x,y)}(v))((-\Delta)^s_{q(x,.)}v+|v|^{\bar{q}(x)-2}v)=\lambda F_v(x,u,v)+\mu G_v(x,u,v), & \text{in } \Omega,\\ u=v=0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \qquad (P_1)$$

$$I_{s,r(x,y)}(w)=\int_{\Omega\times\Omega}\frac{1}{r(x,y)}\frac{|w(x)-w(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}dxdy+\int_{\Omega}\frac{|w(x)|^{\bar{r}(x)}}{\bar{r}(x)}dx,$$

where Ω is an open bounded subset in $\mathbb{R}^N$, $N\geqslant 2$, with Lipschitz boundary $\partial\Omega$, $M:[0,+\infty)\longrightarrow(0,+\infty)$ is a strictly nondecreasing continuous function and $p,q:\mathbb{R}^N\times\mathbb{R}^N\longrightarrow]1,+\infty[$ are symmetric continuous functions such that

$$1<p^-\leqslant p(x,y)\leqslant p^+<+\infty, \qquad 1<q^-\leqslant q(x,y)\leqslant q^+<+\infty,$$

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where

$$p^- = \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y), \quad p^+ = \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y),$$
$$q^- = \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} q(x,y), \quad q^+ = \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} q(x,y).$$

The critical fractional Sobolev exponent is given by

$$m_s^*(x) = \begin{cases} \frac{Nm(x,x)}{N-sm(x,x)}, & N > sm(x,x), \\ +\infty, & N \leqslant sm(x,x). \end{cases}$$

$F,G : \Omega\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are functions such that $F(.,t,\xi)$, $G(.,t,\xi)$ are measurable in Ω for all $t,\xi \in \mathbb{R}$, and $F(x,.,.)$ and $G(x,.,.)$ are continuously differentiable in $\mathbb{R}^2$ for a.e. $x\in\Omega$. $F_u(x,u,v)$, $F_v(x,u,v)$, $G_u(x,u,v)$ and $G_v(x,u,v)$ are the partial derivatives of F and G with respect to u and v, respectively. λ and μ are two real parameters and $(-\Delta)^s_{p(x,.)}$ is a nonlocal integro-differential operator of elliptic type defined by

$$(-\Delta)^s_{p(x,.)}u(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy,$$

which is a pseudo-differential operator allowing in particular the inclusion of the non-integer derivative order. This operator is a fractional version of the so-called $p(.)-$laplacian operator, which is given by $(-\Delta)_{p(.)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$.

The problem (P_1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

presented by Kirchhoff [1] in 1883, is an extension of the classical d'Alembert's wave equation by considering the changes in the length of the string during vibrations. In (1.1), L is the length of string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example, in [2]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, for example, the population density. We refer the reader to [3, 4, 5] for some related works.

In the local case, $s = 1$, many publications [6, 7, 8, 9, 10, 11] have appeared concerning the quasilinear elliptic systems of Kirchhoff type. Existence and multiplicity results have been investigated. For example, when the exponents $p(.)$ and $q(.)$ are reduced to be constants, Li and Tang considered in [9] the following boundary problem involving (p,q)-Kirchhoff type

$$\begin{cases} [M(\int_\Omega|\nabla u|^p dx)]^{p-1}(-\Delta)_p u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega \\ [M(\int_\Omega|\nabla v|^q dx)]^{q-1}(-\Delta)_q v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_2}$$

where Ω is a bounded smooth domain, $\lambda,\mu\in[0,+\infty)$, $p > N$, $q > N$, and $(-\Delta)_p$ is the p-Laplacian operator $(-\Delta)_p(u) = -div(|\nabla u|^{p-2}\nabla u)$. Based on the three critical points theorem, they established the existence of three weak solutions.

In [10], using the same approach, Liu and Shi proved the existence of three solutions for a $(p(.),q(.)$-Laplacian system of the form

$$\begin{cases} (-\Delta)_{p(x)}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega \\ (-\Delta)_{q(x)}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega \end{cases} \tag{P_3}$$

In [7], Dai proved the existence and multiplicity of solotions for the following $p(.)$-Kirchhoff type system with Dirichlet boundary conditions

$$\begin{cases} -M_1\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) div(|\nabla u|^{p(x)-2}\nabla u) = F_u(x,u,v), & \text{in } \Omega \\ -M_1\left(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx\right) div(|\nabla v|^{q(x)-2}\nabla u) = F_v(x,u,v), & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_4}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p(.),q(.) \in C_+(\overline{\Omega})$ and $M_1(.)$, $M_2(.)$ are continuous functions. The function $F : \Omega \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be continuous in $x \in \overline{\Omega}$ and of class C^1 in $u,v \in \mathbb{R}$. Under growth conditions on the reaction terms, the author established the existence results using a direct variational approach.

The study of the fractional Lebesgue and Sobolev spaces and their generalizations to variable exponents has received more considerable attention (see, for example, [12, 13, 14]) recently. We refer to Di Nezza, Palatucci and Valdinoci [14] for a comprehensive introduction to the study of nonlocal fractional problems. Because of non homogeneous materials (such that electrorheological fluids and smart fluids), the use of Lebesgue and Sobolev spaces L^p and $W^{s,p}$ are not adequate. This leads to the study of variable exponent Lebesgue and Sobolev spaces $L^{p(.)}$ and $W^{s,p(.)}$. On the other hand, studying the problems which involve the fractional $p(.)$-Laplacian and corresponding nonlocal elliptic equations becomes a new domain for research. This attracts the attention of many mathematicians; see, for example, [13, 15, 16, 17, 18, 19]. This type of problems arises in many physical phenomena such as conservation laws, ultra-materials and water waves, optimization, population dynamics, soft thin films, mathematical finance, phases transitions, stratified materials, anomalous diffusion, crystal dislocation, semipermeable membranes, flames propagation, ultra-relativistic limits of quantum mechanics (see [14, 20]).

In [21], Azroul et al. considered the following fractional p-Kirchhoff type system

$$\begin{cases} M_1\left([u]^p_{s,p}\right)(-\Delta)^s_p u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega, \\ M_2\left([v]^p_{s,p}\right)(-\Delta)^s_p v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_5}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, M_i, $i = 1,2$, are nondecreasing continuous functions, $1 < p < \infty$ and $\lambda,\ \mu$ are two real parameters, and $(-\Delta)^s_p$ denotes the nonlocal fractional p-Laplacian operator of elliptic type given by

$$(-\Delta)^s_p u(x) = 2\lim_{\varepsilon \searrow 0}\int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy.$$

Motivated by [7, 9, 10, 21], our main goal is to prove the existence of three positive solutions for problem (P_1). Our approach is based on the three critical points theorem introduced by Ricceri. This paper is organized as follows. In Section 2, we recall some properties of fractional

Lebesgue and Sobolev spaces with variable exponents. In Section 3, using the three critical points theorem, we prove our main existence result.

2. Preliminaries and basic assumptions

In this section, we recall some necessary properties of variable exponent spaces. For more details, we refer to [22, 23, 24], and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}) : p(x) > 1, \forall x \in \overline{\Omega}\right\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ as

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}.$$

This space equipped with the *Luxemburg* norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

is a separable reflexive Banach space. Let $\hat{p} \in C_+(\overline{\Omega})$ be the conjugate exponent of p, i.e., $\frac{1}{p(x)} + \frac{1}{\hat{p}(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.1. *(Hölder inequality). If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{\hat{p}(\cdot)}(\Omega)$. Then,*

$$\left|\int_\Omega uv dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{\hat{p}^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{\hat{p}(\cdot)}(\Omega)} \leqslant 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{\hat{p}(\cdot)}(\Omega)}.$$

The modular of $L^{p(\cdot)}(\Omega)$ is defined by

$$\begin{aligned} \rho_{p(.)} : L^{p(\cdot)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longrightarrow \rho_{p(.)}(u) = \textstyle\int_\Omega |u(x)|^{p(x)} dx. \end{aligned}$$

Proposition 2.2. [23, 25] *Let $u \in L^{p(\cdot)}(\Omega)$, then we have,*

(1) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ (resp $= 1, > 1$) $\Leftrightarrow \rho_{p(.)}(u) < 1$ (resp $= 1, > 1$),
(2) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leqslant \rho_{p(.)}(u) \leqslant \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$,
(3) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leqslant \rho_{p(.)}(u) \leqslant \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

Proposition 2.3. *If $u, u_k \in L^{p(\cdot)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

(1) $\lim\limits_{k \to +\infty} \|u_k - u\|_{L^{p(\cdot)}(\Omega)} = 0$,
(2) $\lim\limits_{k \to +\infty} \rho_{p(.)}(u_k - u) = 0$,
(3) $u_k \longrightarrow u$ in measure in Ω and $\lim\limits_{k \to +\infty} \rho_{p(.)}(u_k) = \rho_{p(.)}(u)$.

Proposition 2.4. [25] *Let Ω be an open bounded subset of $\mathbb{R}^N$, $p(.) \in C_+(\bar{\Omega})$. Then $(L^{p(\cdot)}(\Omega), ||u||_{L^{p(\cdot)}(\Omega)})$ is a reflexive uniformly convex and separable Banach space.*

Now, let introduce our fundamental space. Let Ω be an open bounded set in $\mathbb{R}^N$, $s \in (0,1)$, and let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous variable exponent. We define the usual fractional Sobolev space with variable exponent as

$$W^{s,p(x,y)}(\Omega) = \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy < \infty \right\},$$

which endowed with the luxemberg norm

$$||u||_{W^{s,p(x,y)}(\Omega)} = ||u||_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $\bar{p}(x) = p(x,x)$ and $[u]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent defined by

$$[u]_{s,p(x,y)} = \inf\left\{ \lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy \leqslant 1 \right\}.$$

For any $u \in W^{s,p(.,.)}(\Omega)$, we define the modular $\rho_{p(.,.)} : W^{s,p(.,.)}(\Omega) \longrightarrow \mathbb{R}$, by

$$\rho_{p(x,y)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy + \int_{\Omega} |u(x)|^{\bar{p}(x)} dx,$$

and

$$\|u\|_{s,p(.,.)} = \|u\|_{\rho_{p(.,.)}} = \inf\left\{ \lambda > 0 : \rho_{p(.,.)}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

Proposition 2.5. *Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous variable exponent and $s \in (0,1)$. For any $u \in W^{s,p(x,y)}(\Omega)$, we have*

(1) $1 \leqslant \|u\|_{s,p(.,.)} \Rightarrow \|u\|^{p^-}_{s,p(.,.)} \leqslant \rho_{p(.,.)}(u) \leqslant \|u\|^{p^+}_{s,p(.,.)}$,
(2) $\|u\|_{s,p(.,.)} \leqslant 1 \Rightarrow \|u\|^{p^+}_{s,p(.,.)} \leqslant \rho_{p(.,.)}(u) \leqslant \|u\|^{p^-}_{s,p(.,.)}$.

The proof is similar to Proposition 2.2.

Proposition 2.6. *If $u, u_k \in W^{s,p(.,.)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

(1) $\lim_{k\to+\infty} \|u_k - u\|_{s,p(.,.)} = 0$,
(2) $\lim_{k\to+\infty} \rho_{p(.,.)}(u_k - u) = 0$,
(3) $u_k \longrightarrow u$ *in measure in* Ω *and* $\lim_{k\to+\infty} \rho_{p(.,.)}(u_k) = \rho_{p(.,.)}(u)$.

The proof is similar to Proposition 2.3.

Let denote by $W_0^{s,p(x,y)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{s,p(x,y)}(\Omega)$. Let $E = W_0^{s,p(x,y)}(\Omega) \times W_0^{s,q(x,y)}(\Omega)$ be the Cartesian product of two Banach space, which is a reflexive Banach space endowed with the norm

$$||(u,v)||_E = \|u\|_{s,p(x,y)} + \|v\|_{s,q(x,y)}. \tag{2.1}$$

Theorem 2.7. [26] *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $s \in (0,1)$. Let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous variable exponent with $sp^+ < N$. Let $r : \overline{\Omega} \longrightarrow (1,+\infty)$ be a continuous variable exponent such that,*

$$p_s^*(x) = \frac{Np(x,x)}{N - sp(x,x)} > r(x) \geqslant r^- > 1, \ \forall x \in \overline{\Omega}.$$

Then the space $W^{s,p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact. In other words, there exists a constant $C = C(N,s,p,q,\Omega) > 0$ such that, for every $u \in W^{s,p(x,y)}(\Omega)$,

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_{s,p(x,y)}.$$

Finally, the proof of our main result is based on the following three critical points theorem.

Theorem 2.8. [27] *Let X be a reflexive real Banach space, $I \subset \mathbb{R}$ an interval, $\Psi : X \longrightarrow \mathbb{R}$ a sequentially weakly lower semi-continuous C^1 functional whose derivative admits a continuous inverse on X^*, and $J : X \longrightarrow \mathbb{R}$ a C^1 functional with compact derivative. In addition, Ψ is bounded on each bounded subset of X. Assume that*

$$\lim_{\|x\|_X \to +\infty} (\Psi(x) - \lambda J(x)) = +\infty, \tag{2.2}$$

for all $\lambda \in I$, and that there exists $\delta \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\psi(x) - \lambda (J(x) + \delta)) < \inf_{x \in X} \sup_{\lambda \in I} (\psi(x) - \lambda (J(x) + \delta)). \tag{2.3}$$

Then, there exist a nonempty open set $\Lambda \subset I$and a positive real number r with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $\Gamma : X \longrightarrow \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that, for each $\mu \in [0, \mu^*]$, the equation

$$\Psi'(x) - \lambda J'(x) + \mu \Gamma'(x) = 0,$$

has at least three solutions whose norms are less than r.

Proposition 2.9. [28] *Let X be a non-empty set and Ψ, J two real functionals on X. Assume that there are $\gamma > 0$, u_0, $u_1 \in X$, such that*

$$\begin{gathered} \Psi(u_0) = J(u_0) = 0, \qquad \Psi(u_1) > \gamma, \\ \sup_{u \in \Psi^{-1}((-\infty,\gamma))} J(u) < \gamma \frac{J(u_1)}{\Psi(u_1)}. \end{gathered} \tag{2.4}$$

Then, for each r satisfying

$$\sup_{u \in \Psi^{-1}((-\infty,\gamma))} J(u) < r < \gamma \frac{J(u_1)}{\Psi(u_1)},$$

one has

$$\sup_{\lambda \in I} \inf_{x \in X} (\psi(x) - \lambda (J(x) + \delta)) < \inf_{x \in X} \sup_{\lambda \in I} (\psi(x) - \lambda (J(x) + \delta)).$$

3. Mains Results

In this section, we prove the existence of three weak solutions for problem (P_1) by applying Theorem 2.8. For this, we suppose that the Kirchhoff function $M : [0, +\infty) \longrightarrow (0, +\infty)$ satisfies the following condition: there exists a constant $m_0 > 0$ such that

$$M(t) \geqslant m_0, \quad \forall t \geqslant 0 \tag{M_0}$$

and put

$$\widehat{M}(t) = \int_0^t M(s)ds, \quad \forall t \geqslant 0.$$

On the other hand, we suppose that F satisfies the following conditions:

(F1) $F(x,0,0)=0$, for a.e. $x\in\Omega$,

(F2) there exists $d>0$ such that $F(x,u,v)>0$, for a.e. $x\in\Omega$ and for all $(s,t)\in[0,d]\times[0,d]$,

(F3) There exist $p_1,\ q_1\in C(\bar{\Omega})$, $p^+<p_1^-<p_1^+<p_s^*(x)$ and $q^+<q_1^-<q_1^+<q_s^*(x)$ such that

$$\limsup_{(t,\xi)\to(0,0)}\frac{\sup\limits_{x\in\Omega}F(x,t,\xi)}{|t|^{p_1(x)}+|\xi|^{q_1(x)}}<+\infty,$$

(F4) There are functions $\alpha(.),\beta(.)\in C(\bar{\Omega})$, $1<\alpha^-<\alpha^+<p^-$ and $1<\beta^-<\beta^+<q^-$ such that

$$F(x,t,\xi)\leqslant c(1+|t|^{\alpha(x)}+|\xi|^{\beta(x)})\text{ for a.e. }x\in\Omega\text{ and all }(s,t)\in\mathbb{R}^2.$$

Definition 3.1. For $sp^+,sq^+<N$, we denote by $\mathscr{A}$ the class of functions $F:\Omega\times\mathbb{R}^2\longrightarrow\mathbb{R}$ such that $F_u=\frac{\partial F}{\partial u}$ and $F_v=\frac{\partial F}{\partial v}$ are two Caratheodory functions and

$$\sup_{(x,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}}\frac{|F_u(x,t,\xi)|}{1+|t|^{r_1(x)-1}+|\xi|^{r_2(x)-1}}<+\infty$$

$$\text{and}\quad\sup_{(x,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}}\frac{|F_v(x,t,\xi)|}{1+|t|^{r_1(x)-1}+|\xi|^{r_2(x)-1}}<+\infty,$$

for all $r_1(.)\in(p^+,p_s^*(x))$ and $r_2(.)\in(q^+,q_s^*(x))$.

Definition 3.2. We say that $(u,v)\in E$ is a weak solution of problem (P_1) if

$$\begin{aligned}
&M\left(I_{s,p(x,y)}(u)\right)\times\left[\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy\right.\\
&\quad\left.+\int_\Omega|u(x)|^{\bar{p}(x)-2}u(x)\varphi(x)dx\right]\\
&+M\left(I_{s,q(x,y)}(v)\right)\times\left[\int_\Omega\int_\Omega\frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}}dxdy\right.\\
&\quad\left.+\int_\Omega|v(x)|^{\bar{q}(x)-2}v(x)\psi(x)dx\right]\\
&=\lambda\int_\Omega F_u(x,u,v)\varphi dx+\mu\int_\Omega G_u(x,u,v)\varphi dx+\lambda\int_\Omega F_v(x,u,v)\psi dx+\mu\int_\Omega G_v(x,u,v)\psi dx,
\end{aligned}$$

for all $(\varphi,\psi)\in E$.

Theorem 3.3. *For $sp^+,sq^+<N$ and $F\in\mathscr{A}$, suppose that (M_0) and (F2)-(F4) are fulfilled. Then, there exist a nonempy open set $\Lambda\subset I$ and a number $\delta>0$ such that: for every $\lambda\in\Lambda$ and every $G\in\mathscr{A}$ there exists $\mu^*>0$ such that, for each $\mu\in[0,\mu^*]$, the problem (P_1) has at least three weak solutions whose norms are less than δ.*

Following the development of Kirchhoff type problems, our results can generalize many physical and biological problems, such as, the movement of elastic threads, the description of the population of a bacterium, the deformation of beams in flexion, the deformation of plates or tensioned ropes, etc. As an example, we can take the case when $M(t)=a+bt$, $a,b>0$.

Consider the functions $J,\Psi : E \longrightarrow \mathbb{R}$ defined by

$$J(u,v)=\int_\Omega F(x,u,v)dx,$$

and

$$\Psi(u,v)=\widehat{M}\left(I_{s,p(x,y)}(u)\right)+\widehat{M}\left(I_{s,q(x,y)}(v)\right).$$

Lemma 3.4. *Let $F\in\mathscr{A}$. Then the function $J\in C^1(E,\mathbb{R})$ and its derivative is given by*

$$\left\langle J'(u,v),(\varphi,\psi)\right\rangle=\int_\Omega F_u(x,u,v)\varphi+\int_\Omega F_v(x,u,v)\psi dx, \tag{3.1}$$

for all $(\varphi,\psi)\in E$. Moreover $J' : E\longrightarrow E^*$ is compact.

Proof. From the definition of $\mathscr{A}$ and the embedding theorem, we can see that J is well-defined on E. Usual arguments show that J is Gâteaux differentiable on E with the derivative is given by (3.1) . Assume that $sp^+,sq^+<N$. We claim that the operator J' is continuous. Let $\{(u_n,v_n)\}\subset E$ be a sequence converging strongly to $(u,v)\in E$. Due to the compact embedding of $W_0^{s,p(x,y)}(\Omega)$ and $W_0^{s,q(x,y)}(\Omega)$ in $L^{r(x)}(\Omega)$, where $\max\{p^+,q^+\}\leqslant r(x)\leqslant\min\{p_s^*(x),q_s^*(x)\}$ for a.e $x\in\overline{\Omega}$, we have that $\{(u_n,v_n)\}$ converges strongly to (u,v) in $L^{r(x)}(\Omega)$. So, for a subsequence, still denoted by $\{(u_n,v_n)\}$, there exist functions $\overline{u}\in L^{r(x)}(\Omega)$ and $\overline{v}\in L^{r(x)}(\Omega)$ such that $u_n\to u$ a.e. in Ω, $v_n\to v$ a.e. in Ω, and $|u_n|\leqslant|\overline{u}|$, $|v_n|\leqslant|\overline{v}|$ for all $n\in\mathbb{N}$ and almost everywhere in Ω. Since $F\in\mathscr{A}$, for all measurable functions u, $v:\Omega\longrightarrow\mathbb{R}$, the operator defined by $(u,v)\longmapsto F_u(.,u(.),v(.))+F_v(.,u(.),v(.))$ maps $L^{r(x)}(\Omega)\times L^{r(x)}(\Omega)$ into $L^{\hat{r}(x)}(\Omega)$. Fix $(\tilde{u},\tilde{v})\in E$ with $||(\tilde{u},\tilde{v})||_E\leqslant 1$. Using Hölder inequality, we get

$$\begin{aligned}
&|\left\langle J'(u_n,v_n)-J'(u,v),(\tilde{u},\tilde{v})\right\rangle|\\
&=\left|\int_\Omega\left(F_u(x,u_n(x),v_n(x))-F_u(x,u(x),v(x))\right)\tilde{u}(x)dx\right|\\
&\quad+\left|\int_\Omega\left(F_v(x,u_n(x),v_n(x))-F_v(x,u(x),v(x))\right)\tilde{v}(x)dx\right|\\
&\leqslant C_1\,||F_u(x,u_n(x),v_n(x))-F_u(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)}||\tilde{u}||_{L^{r(x)}(\Omega)}\\
&\quad+C_1\,||F_u(x,u_n(x),v_n(x))-F_u(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)}||\tilde{v}||_{L^{r(x)}(\Omega)}\\
&\leqslant C_2\,||F_u(x,u_n(x),v_n(x))-F_u(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)}||\tilde{u}||_{W^{s,p(x,y)}(\Omega)}\\
&\quad+C_2\,||F_u(x,u_n(x),v_n(x))-F_u(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)}||\tilde{v}||_{W^{s,q(x,y)}(\Omega)},
\end{aligned}$$

for some constants $C_1,C_2>0$. Thus, passing to the sup for $||(\tilde{u},\tilde{v})||\leqslant 1$, we get

$$\begin{aligned}
||J'(u_n,v_n)-J'(u,v)||_{E^*}&\leqslant C_1\,||F_u(x,u_n(x),v_n(x))-F_u(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)}\\
&\quad+C_1\,||F_v(x,u_n(x),v_n(x))-F_v(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)}.
\end{aligned}$$

From the definition of $\mathscr{A}$, for a.e. $x\in\Omega$, we have

$$\begin{aligned}
F_u(x,u_n(x),v_n(x)-F_u(x,u(x),v(x)\longrightarrow 0 \text{ as } n\to\infty,\\
F_v(x,u_n(x),v_n(x)-F_v(x,u(x),v(x)\longrightarrow 0 \text{ as } n\to\infty,
\end{aligned}$$

and

$$|F_u(x,u_n(x),v_n(x))|\leqslant C_2(1+|\overline{u}|^{r(x)-1}+|\overline{v}|^{r(x)-1})\quad\text{a.e. } x\in\Omega,$$

$$|F_v(x,u_n(x),v_n(x))|\leqslant C_2(1+|\overline{u}|^{r(x)-1}+|\overline{v}|^{r(x)-1})\quad\text{a.e. } x\in\Omega.$$

Note that the right hand side in the previous relation belongs to $L^{\hat{r}(x)}(\Omega)$. Hence, by applying the dominate convergence theorem, we get

$$||F_u(x,u_n(x),v_n(x)) - F_u(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)} \to 0 \ \ as \ \ n \to \infty,$$
$$||F_v(x,u_n(x),v_n(x)) - F_v(x,u(x),v(x))||_{L^{\hat{r}(x)}(\Omega)} \to 0 \ \ as \ \ n \to \infty.$$

This proves that J' is continuous.

Now, in order to verify the compactness of J', we take $\{(u_n,v_n)\}$ a bounded sequence in E. Then there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, converging weakly in E. So, it converges strongly in $L^{r(x)}(\Omega)\times L^{r(x)}(\Omega)$. Arguing in the same way as before, we show that $\{J'(u_n,v_n)\}$ converges strongly and then the operator J' is compact. □

Lemma 3.5. Ψ' *satisfies the property:*

(S) : [*Let* $\{(u_n,v_n)\}$ *be a sequence that converges weakly to* (u,v) *in* E *and*

$$\lim_{n\to\infty} \Psi'(u_n,v_n) = \Psi'(u,v) \quad \textit{in } E^*.$$

Then

$$(u_n,v_n) \to (u,v) \textit{ strongly in } E.\Big]$$

Proof. Put

$$\begin{aligned}
&\left\langle \mathscr{L}_{p(x,y)}(u),\varphi\right\rangle \\
&= M(I_{s,p(x,y)}(u))\Bigg[\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&\qquad\qquad + \int_\Omega |u(x)|^{\bar{p}(x)-2}u(x)\varphi(x)dx\Bigg].
\end{aligned}$$

Let $(u_n,v_n) \rightharpoonup (u,v)$ in E and $\Psi'(u_n,v_n) \longrightarrow \Psi'(u,v)$. The compact embedding in Theorem 2.7 implies that

$$\begin{aligned}
u_n(x) &\longrightarrow u(x) \text{ a.e. } x\in\Omega, \\
\text{and} \quad v_n(x) &\longrightarrow v(x) \text{ a.e. } x\in\Omega.
\end{aligned} \tag{3.2}$$

This together with Fatou's lemma yields

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n(x)|^{\bar{p}(x)}dx \\
&\qquad \geqslant \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u(x)|^{\bar{p}(x)}dx, \\
&\text{and} \quad \liminf_{n\to\infty}\int_{\Omega\times\Omega}\frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v_n(x)|^{\bar{q}(x)}dx \\
&\qquad \geqslant \int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v(x)|^{\bar{q}(x)}dx.
\end{aligned} \tag{3.3}$$

Now using Young's inequality, we have

$$
\begin{aligned}
&\left\langle \mathscr{L}_{p(x,y)}(u_n), u_n - u\right\rangle + \left\langle \mathscr{L}_{q(x,y)}(v_n), v_n - v\right\rangle \\
&= M(I_{s,p(x,y)}(u_n))\bigg(\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n(x)|^{\bar{p}(x)}dx \\
&\quad - \int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad - \int_\Omega |u_n(x)|^{\bar{p}(x)-2}u_n(x)u(x)dx\bigg) + M(I_{s,q(x,y)}(v_n))\bigg(\int_{\Omega\times\Omega} \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy \\
&\quad + \int_\Omega |v_n(x)|^{\bar{q}(x)}dx - \int_{\Omega\times\Omega} \frac{|v_n(x)-v_n(y)|^{q(x,y)-2}(v_n(x)-v_n(y))(v(x)-v(y))}{|x-y|^{N+sq(x,y)}}dxdy \\
&\quad - \int_\Omega |v_n(x)|^{\bar{q}(x)-2}v_n(x)v(x)dx\bigg) \\
&\geqslant M(I_{s,p(x,y)}(u_n))\bigg(\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n(x)|^{\bar{p}(x)}dx \\
&\quad - \int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)-1}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dxdy - \int_\Omega |u_n(x)|^{\bar{p}(x)-1}u(x)dx\bigg) \\
&\quad + M(I_{s,q(x,y)}(v_n))\bigg(\int_{\Omega\times\Omega} \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v_n(x)|^{\bar{q}(x)}dx \\
&\quad - \int_{\Omega\times\Omega} \frac{|v_n(x)-v_n(y)|^{q(x,y)-1}(v(x)-v(y))}{|x-y|^{N+sq(x,y)}}dxdy - \int_\Omega |v_n(x)|^{\bar{q}(x)-1}v(x)dx\bigg) \\
&\geqslant M(I_{s,p(x,y)}(u_n))\bigg(\frac{p^- + 1 - p^+}{p^-}\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad + \frac{p^- + 1 - p^+}{p^-}\int_\Omega |u_n(x)|^{\bar{p}(x)}dx - \frac{1}{p^-}\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad - \frac{1}{p^-}\int_\Omega |u(x)|^{\bar{p}(x)}dx\bigg) + M(I_{s,q(x,y)}(v_n))\bigg(\frac{q^- + 1 - q^+}{q^-}\int_{\Omega\times\Omega} c_1\frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy \\
&\quad + \frac{q^- + 1 - q^+}{q^-}\int_\Omega |v_n(x)|^{\bar{q}(x)}dx - \int_{\Omega\times\Omega} \frac{1}{q^-}\frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy - \frac{1}{q^-}\int_\Omega |v(x)|^{\bar{q}(x)}dx\bigg).
\end{aligned}
$$

By passing to lim inf, we have

$$
\begin{aligned}
0 \geqslant m_0\bigg(\liminf_{n\to\infty}\bigg[\int_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n(x)|^{\bar{p}(x)}dx\bigg] \\
- \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \int_\Omega |u(x)|^{\bar{p}(x)}dx\bigg)
\end{aligned}
$$

$$+m_0\Big(\liminf_{n\to\infty}\ \Big[\int_{\Omega\times\Omega}\frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega|v_n(x)|^{\bar{q}(x)}dx\Big]$$
$$-\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy-\int_\Omega|v(x)|^{\bar{q}(x)}dx\Big).$$

Combining the above inequality with (3.3), we get

$$\liminf_{n\to\infty}\Big[\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega|u_n(x)|^{\bar{p}(x)}dx\Big]$$
$$=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega|u(x)|^{\bar{p}(x)}dx,$$

$$\text{and}\quad\liminf_{n\to\infty}\Big[\int_{\Omega\times\Omega}\frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega|v_n(x)|^{\bar{q}(x)}dx\Big]$$
$$=\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega|v(x)|^{\bar{q}(x)}dx.$$

For a subsequence, we have

$$\lim_{n\to\infty}\Big[\int_{\Omega\times\Omega}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega|u_n(x)|^{\bar{p}(x)}dx\Big]$$
$$=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega|u(x)|^{\bar{p}(x)}dx,$$

$$\text{and}\quad\lim_{n\to\infty}\Big[\int_{\Omega\times\Omega}\frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega|v_n(x)|^{\bar{q}(x)}dx\Big]$$
$$=\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega|v(x)|^{\bar{q}(x)}dx.$$

That is, $\lim_{n\to\infty}\rho_{s,p(x,y)}(u_n)=\rho_{s,p(x,y)}(u)$, and $\lim_{n\to\infty}\rho_{s,q(x,y)}(v_n)=\rho_{s,q(x,y)}(v)$. By virtue of Proposition 2.6, we conclude that $(u_n,v_n)\longrightarrow(u,v)$ in E. □

Lemma 3.6. *The following properties hold true*

(i) the functional Ψ is sequentially weakly lower semi-continuous and bounded on each bounded subset of E;

(ii) $\Psi':E\longrightarrow E^*$ is a homeomorphism.

Proof. (i) From [18, Lemma 2.4], the operators $u\longmapsto I'_{s,p(x,y)}(u)$ and $v\longmapsto I'_{s,q(x,y)}(v)$ are strictly monotone. From [29, Proposition 25.10], $I_{s,p(x,y)}$ and $I_{s,q(x,y)}$ are strictly convex. Let $\{u_n,v_n\}\subset E$ be a sequence such that $u_n\rightharpoonup u$ in $W_0^{s,p(x,y)}(\Omega)$ and $v_n\rightharpoonup v$ in $W_0^{s,q(x,y)}(\Omega)$. Then, by the convexity of $\rho_{s,p(x,y)}$, we have

$$I_{s,p(x,y)}(u_n)-I_{s,p(x,y)}(u)\geqslant\Big\langle I'_{s,p(x,y)}(u),u_n-u\Big\rangle.$$

Hence we obtain

$$I_{s,p(x,y)}(u) \leqslant \liminf_{n\to\infty} I_{s,p(x,y)}(u_n).$$

So the map $u \longmapsto I_{s,p(x,y)}(u)$ is weakly lower semi-continuous. Similarly, the map $v \longmapsto I_{s,q(x,y)}(v)$ is weakly lower semi-continuous. On the other hand, since the function $t \longmapsto \widehat{M}(t)$ is monotone and continuous, we get

$$\begin{aligned}
\liminf_{n\to\infty} \Psi(u_n, v_n) &= \liminf_{n\to\infty} \widehat{M}\left(I_{s,p(x,y)}(u_n)\right) + \liminf_{n\to\infty} \widehat{M}\left(I_{s,q(x,y)}(v_n)\right) \\
&\geqslant \widehat{M}\left(\liminf_{n\to\infty} I_{s,p(x,y)}(u_n)\right) + \widehat{M}\left(\liminf_{n\to\infty} I_{s,q(x,y)}(v_n)\right) \\
&\geqslant \widehat{M}\left(I_{s,p(x,y)}(u)\right) + \widehat{M}\left(I_{s,q(x,y)}(v)\right) = \Psi(u,v).
\end{aligned}$$

Thus, the functional Ψ is sequentially weakly lower semi-continuous. Now, we prove that Ψ is bounded. Let $\{(u_n, v_n)\}$ be a bounded sequence in E. By Proposition 2.5, there exists positive constants C_1, C_2 such that $I_{s,p(x,y)}(u_n) \leqslant C_1$ and $I_{s,q(x,y)}(v_n) \leqslant C_2$. Since $\widehat{M}$ is monotone, by the definition of Ψ, we can write

$$\Psi(u_n, v_n) = \widehat{M}\left(I_{s,p(x,y)}(u_n)\right) + \widehat{M}\left(I_{s,q(x,y)}(v_n)\right) \leqslant \widehat{M}\left(C_1\right) + \widehat{M}\left(C_2\right).$$

Hence, Ψ is bounded on each bounded subset of E.

(ii) It is easy to see that the functional $\Psi \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\begin{aligned}
&\left\langle \Psi'(u,v), (\varphi, \psi) \right\rangle \\
&= M\left(I_{s,p(x,y)}(u)\right)\left[\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}} dxdy \right.\\
&\quad \left. + \int_\Omega |u(x)|^{\bar{p}(x)-2} u(x)\varphi(x)dx\right] \\
&\quad + M\left(I_{s,q(x,y)}(v)\right)\left[\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}} dxdy \right.\\
&\quad \left. + \int_\Omega |v(x)|^{\bar{q}(x)-2} v(x)\psi(x)dx\right],
\end{aligned}$$

for all $(u,v), (\varphi, \psi) \in E$. Moreover, for each $(u,v) \in E$, $\Psi'(u,v) \in E^*$.

Ψ' is strictly monotone Let us first show that $\widehat{M}$ is strictly convex. Note that $\rho_{s,p(x,y)}$ is strictly convex and $\widehat{M}$ is increasing. If $(\varphi, \psi) \in E$ with $\varphi \neq \psi$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$, we have

$$\begin{aligned}
\widehat{M}(I_{s,p(x,y)}(\lambda\varphi + \mu\psi)) &< \widehat{M}\left(\lambda I_{s,p(x,y)}(\varphi) + \mu I_{s,p(x,y)}(\psi)\right) \\
&\leqslant \lambda \widehat{M}\left(I_{s,p(x,y)}(\varphi)\right) + \mu \widehat{M}\left(I_{s,p(x,y)}(\psi)\right).
\end{aligned} \tag{3.4}$$

Now, let $(u_1,v_1),(u_2,v_2)\in E$ with $(u_1,v_1)\neq(u_2,v_2)$ and $\lambda,\mu\in[0,1]$ with $\lambda+\mu=1$. From the strict convexity of $\hat{M}$, we have

$$\begin{aligned}\Psi(\lambda(u_1,v_1)+\mu(u_2,v_2)) &= \Psi(\lambda u_1+\mu u_2,\lambda v_1+\mu v_2)\\ &= \widehat{M}\left(I_{s,p(x,y)}(\lambda u_1+\mu u_2)\right)+\widehat{M}\left(I_{s,q(x,y)}(\lambda v_1+\mu v_2)\right)\\ &< \lambda\widehat{M}\left(I_{s,p(x,y)}(u_1)\right)+\mu\widehat{M}\left(I_{s,p(x,y)}(u_2)\right)\\ &\quad+\lambda\widehat{M}\left(I_{s,q(x,y)}(v_1)\right)+\mu\widehat{M}\left(I_{s,q(x,y)}(v_2)\right)\\ &= \lambda\left[\widehat{M}\left(I_{s,p(x,y)}(u_1)\right)+\widehat{M}\left(I_{s,q(x,y)}(v_1)\right)\right]\\ &\quad+\mu\left[\widehat{M}\left(I_{s,p(x,y)}(u_2)\right)+\widehat{M}\left(I_{s,q(x,y)}(v_2)\right)\right]\\ &= \lambda\Psi(u_1,v_1)+\mu\Psi(u_2,v_2).\end{aligned}$$

Thus Ψ is strictly convex. From [29, Proposition 25.10], we have that Ψ' is strictly monotone.

Ψ' is coercive Let $(u,v)\in E$ be such that $\|(u,v)\|_E>1$. By use of (M_0), we have

$$\begin{aligned}\left\langle\Psi'(u,v),(u,v)\right\rangle &= M(I_{s,p(x,y)}(u))\rho_{s,p(x,y)}(u)+M(I_{s,q(x,y)}(v))\rho_{s,q(x,y)}(v)\\ &\geqslant m_0\left(\rho_{s,p(x,y)}(u)+\rho_{s,q(x,y)}(v)\right)\\ &\geqslant m_0\left(||u||^{p^\pm}_{s,p(x,y)}+||v||^{q^\pm}_{s,q(x,y)}\right)\\ &\geqslant m_0\left(||u||_{s,p(x,y)}+||v||_{s,q(x,y)}\right)^{\min\{p^\pm,q^\pm\}}\\ &= m_0||(u,v)||_E^{\min\{p^\pm,q^\pm\}},\end{aligned}$$

where $p^\pm=\begin{cases} p^- & if \quad \|u\|_{s,p(x,y)}>1\\ p^+ & if \quad \|u\|_{s,p(x,y)}<1\end{cases}$, and $q^\pm=\begin{cases} q^- & if \quad \|v\|_{s,q(x,y)}>1\\ q^+ & if \quad \|v\|_{s,q(x,y)}<1\end{cases}$.

Thus

$$\lim_{||(u,v)||_E\to+\infty}\frac{\langle\Psi'(u,v),(u,v)\rangle}{||(u,v)||_E}=+\infty.$$

Hence Ψ' is coercive. Thanks to the Munty-Browder theorem see [29], the operator Ψ' is surjective. Because of its monotonicity, Ψ' is an injection. So $(\Psi')^{-1}$ exists. It remains to show that $(\Psi')^{-1}$ is continuous. Let $(\tilde{u}_n,\tilde{v}_n)$ be a sequence that converges strongly to $(\tilde{u},\tilde{v})$ in E^* and let $u_n=\mathscr{L}^{-1}_{p(x,y)}(\tilde{u}_n)$, $v_n=\mathscr{L}^{-1}_{q(x,y)}(\tilde{v}_n)$, $u=\mathscr{L}^{-1}_{p(x,y)}(\tilde{u})$ and $v=\mathscr{L}^{-1}_{q(x,y)}(\tilde{v})$. Since Ψ' is coercive, we have that $\{(u_n,v_n)\}$ is bounded in E. We may assume that it converges weakly to a certain $(u_0,v_0)\in E$. Since $(\tilde{u}_n,\tilde{v}_n)$ converges strongly to $(\tilde{u},\tilde{v})$, we have

$$\lim_{n\to\infty}\left\langle\Psi'(u_n,v_n)-\Psi'(u_0,v_0),(u_n,v_n)-(u_0,v_0)\right\rangle=\lim_{n\to\infty}\langle(\tilde{u}_n,\tilde{v}_n),(u_n,v_n)-(u_0,v_0)\rangle=0.$$

The fact that ψ' is of type (S) (see Lemma3.5) implies that $(u_n, v_n) \longrightarrow (u_0, v_0)$. By the continuity of Ψ' we have

$$\Psi'(u_n, v_n) \longrightarrow \Psi'(u_0, v_0). \tag{3.5}$$

On the other hand, since

$$\tilde{u}_n = \mathscr{L}_{p(x,y)}(u_n) \longrightarrow \tilde{u} = \mathscr{L}_{p(x,y)}(u),$$

$$\text{and} \quad \tilde{v}_n = \mathscr{L}_{q(x,y)}(v_n) \longrightarrow \tilde{v} = \mathscr{L}_{q(x,y)}(v),$$

it follows that

$$\Psi'(u_n, v_n) \longrightarrow \Psi'(u, v). \tag{3.6}$$

From (3.5), (3.6) and the fact that ψ' is an injection, we conclude that $(u_0, v_0) = (u, v)$. This achieves the proof. □

Proof of Theorem 3.3. In order to apply Theorem 2.8, we take $X = E$, Ψ and J as before. By Lemma 3.4, the functional J is of class C^1 with compact derivative. By Lemma 3.6, Ψ is a sequentially weakly lower semi-continuous, of class C^1 and belongs to $\mathscr{W}_E$ and the operator Ψ' admits a continuous inverse $\Psi' : E^* \longrightarrow E$. Moreover

$$\lim_{||(u,v)|| \to \infty} (\Psi(u,v) - \lambda J(u,v)) = +\infty.$$

for all $\lambda > 0$. Indeed, from Proposition 2.5, if $||u||_{s,p(x,y)} \geqslant 1$, then

$$\frac{1}{p^+} ||u||^{p^-}_{s,p(x,y)} \leqslant I_{s,p(x,y)}(u) \leqslant \frac{1}{p^-} ||u||^{p^+}_{s,p(x,y)},$$

and if $||u||_{s,p(x,y)} < 1$, then

$$\frac{1}{p^+} ||u||^{p^+}_{s,p(x,y)} \leqslant I_{s,p(x,y)}(u) \leqslant \frac{1}{p^-} ||u||^{p^-}_{s,p(x,y)}.$$

In fact, when $||u||_{s,p(x,y)} < 1$, we can set $C_0 \geqslant \frac{1}{p^+} ||u||^{p^-}_{s,p(x,y)} - \frac{1}{p^+} ||u||^{p^+}_{s,p(x,y)} \geqslant 0$. It follows that

$$I_{s,p(x,y)}(u) \geqslant \frac{1}{p^+} ||u||^{p^-}_{s,p(x,y)} - C_0.$$

Hence,

$$I_{s,p(x,y)}(u) \geqslant \frac{1}{p^+} ||u||^{p^-}_{s,p(x,y)} - C_0, \quad \forall u \in W^{s,p(x,y)}(\Omega).$$

So, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \Psi(u,v) &= \widehat{M}\left(I_{s,p(x,y)}(u)\right) + \widehat{M}\left(I_{s,q(x,y)}(v)\right) \\ &\geqslant \frac{m_0}{p^+} \rho_{s,p(x,y)}(u) + \frac{m_0}{q^+} \rho_{s,q(x,y)}(v) \\ &\geqslant m_0 \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \left(||u||^{p^-}_{s,p(x,y)} + ||v||^{q^-}_{s,q(x,y)} - C_1 \right), \end{aligned} \tag{3.7}$$

for all $(u, v) \in E$.

On the other hand, for all $\lambda \geqslant 0$, according to the condition (F4), there exists a constant $C_2 > 0$ such that

$$-\lambda J(u,v) = -\lambda \int_\Omega F(x,u,v)dx \geqslant -\lambda C_2(1+||u||^{\alpha^+}_{s,p(x,y)} + ||v||^{\beta^+}_{s,p(x,y)}). \tag{3.8}$$

By (3.7) and (3.8), it follows that

$$\Psi(u,v) - \lambda J(u,v) \geqslant m_0 \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \left(||u||^{p^-}_{s,p(x,y)} + ||v||^{q^-}_{s,q(x,y)} - C_1\right) - \lambda C_2 \left(1+||u||^{\alpha^+}_{s,p(x,y)} + ||v||^{\beta^+}_{s,p(x,y)}\right).$$

Since $\alpha^+ < p^-$ and $\beta^+ < q^-$, we conclude that

$$\lim_{||(u,v)||\to\infty} (\Psi(u,v) - \lambda J(u,v)) = +\infty.$$

Let $x_0 \in \Omega$, $r_2 > r_1 > 0$ and put

$$w(x) = \begin{cases} 0, & x \in \bar{\Omega} \setminus B(x_0, r2), \\ \frac{d}{r_2 - r_1}(r_2 - |x - x_0|), & x \in B(x_0, r_2) \setminus B(x_0, r_1), \\ d, & x \in B(x_0, r_1). \end{cases}$$

From (F2), we have $J(w,w) = \int_\Omega F(x,w,w)dx > 0$. From (F3), we have that there exist $\eta \in [0,1]$ and C_0 such that

$$F(x,t,\xi) < C_0(|t|^{p_1(x)} + |\xi|^{q_1(x)}) < C_0(|t|^{p_1^-} + |\xi|^{q_1^-}), \tag{3.9}$$

for all $(t,\xi) \in [-\eta,\eta] \times [-\eta,\eta]$ a.e $x \in \Omega$. From (F3) and (F4), there are positive numbers $C_i (i = 1,2,...,9)$ according to $|t|$ and $|\xi|$ larger or smaller than η and 1. For example, for $|t| < \eta$ and $|\xi| < \eta$, we take

$$C_1 = \sup_{|t|<\eta, |\xi|<\eta} \frac{C\left(1+|t|^{\alpha^+} + |\xi|^{\beta^+}\right)}{|t|^{p_1^-} + |\xi|^{q_1^-}}.$$

For $|t| < \eta$ and $\eta < |\xi| < 1$, we take

$$C_2 = \sup_{|t|<\eta, \eta<|\xi|<1} \frac{C\left(1+|t|^{\alpha^+} + |\xi|^{\beta^+}\right)}{|t|^{p_1^-} + |\xi|^{q_1^-}}.$$

Let $C = \max\{C_0,\ C_1, ...,\ C_9\}$. Then $F(x,t,\xi) < C(|t|^{p_1^-} + |\xi|^{q_1^-})$, $\forall (t,\xi) \in \mathbb{R}^2$ a.e $x \in \Omega$. Fix γ such that $0 < \gamma < 1$. If

$$m_0 \min\{\frac{1}{p^+}, \frac{1}{q^+}\} \left(||u||^{p^+}_{s,p(x,y)} + ||v||^{q^+}_{s,q(x,y)}\right) < \lambda < 1,$$

due to the continuous embedding of $W_0^{s,p(x,y)}$ in $L^{p_1^-}(\Omega)$ and of $W_0^{s,q(x,y)}$ in $L^{q_1^-}(\Omega)$, we have

$$\begin{aligned} J(u,v) = \int_\Omega F(x,u,v)dx &\leqslant C_0 \int_\Omega (|t|^{p_1^-} + |\xi|^{q_1^-})dx \\ &\leqslant C_{10} ||u||^{p_1^-}_{s,p(x,y)} + ||v||^{q_1^-}_{s,q(x,y)} \\ &\leqslant C_{11} \left(\gamma^{\frac{p_1^-}{p^+}} + \gamma^{\frac{q_1^-}{q^+}}\right), \end{aligned}$$

for some positive constants $C_{10},\ C_{11}$. Since $p_1^- > p^+$ and $q_1^- > q^+$, we get

$$\lim_{\gamma\to 0}\frac{\sup_{m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||u||^{p^+}_{s,p(x,y)}+||v||^{q^+}_{s,q(x,y)}\right)<\gamma} J(u,v)}{\gamma}=0. \tag{3.10}$$

Choose w as before and fix $\gamma_0>0$ such that

$$\begin{aligned}&0<\gamma<\gamma_0\\ &<m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\min\{||w||^{p^-}_{s,p(x,y)}+||w||^{q^-}_{s,q(x,y)},||w||^{p^+}_{s,q(x,y)}+||w||^{q^+}_{s,q(x,y)},1\}.\end{aligned}$$

Then we divide the proof into two cases.

- If $||w||_E<1$, then

$$\begin{aligned}\Psi(w,w)\ &=\widehat{M}\left(I_{s,p(x,y)}(w)\right)+\widehat{M}\left(I_{s,q(x,y)}(w)\right)\\ &\geqslant m_0\left(I_{s,p(x,y)}(w)+I_{s,q(x,y)}(w)\right)\\ &\geqslant m_0\left(\int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega\frac{|w(x)|^{\bar{p}(x)}}{\bar{p}(x)}dx\right)\\ &\quad+m_0\left(\int_{\Omega\times\Omega}\frac{1}{q(x,y)}\frac{|w(x)-w(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega\frac{|w(x)|^{\bar{q}(x)}}{\bar{q}(x)}dx\right)\\ &\geqslant m_0\min\{\frac{1}{p+},\frac{1}{q^+}\}\left(\rho_{s,p(x,y)}(w)+\rho_{s,q(x,y)}(w)\right)\\ &\geqslant m_0\min\{\frac{1}{p+},\frac{1}{q^+}\}\left(||w||^{p^+}_{s,p(x,y)}+||w||^{q^+}_{s,q(x,y)}\right)\\ &\geqslant\gamma_0>\gamma.\end{aligned}$$

  From (3.10), we have

$$\begin{aligned}&\sup_{m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||u||^{p^+}_{s,p(x,y)}+||v||^{q^+}_{s,q(x,y)}\right)<\gamma} J(u,v)\\ &\leqslant\frac{\gamma}{2}\frac{J(w,w)}{m_0\max\{\frac{1}{p^-},\frac{1}{q^-}\}\left(||w||^{p^-}_{s,p(x,y)}+||w||^{q^-}_{s,q(x,y)}\right)}\\ &\leqslant\frac{\gamma}{2}\frac{J(w,w)}{\Psi(w,w)}\leqslant\gamma\frac{J(w,w)}{\Psi(w,w)}.\end{aligned}$$

- If$||w||_E\geqslant 1$, then

$$\begin{aligned}\Psi(w,w)\ &=\widehat{M}\left(I_{s,p(x,y)}(w)\right)+\widehat{M}\left(I_{s,q(x,y)}(w)\right)\\ &\geqslant m_0\left(I_{s,p(x,y)}(w)+I_{s,q(x,y)}(w)\right)\\ &\geqslant m_0\left(\int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|w(x)-w(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega\frac{|w(x)|^{\bar{p}(x)}}{\bar{p}(x)}dx\right)\\ &\quad+m_0\left(\int_{\Omega\times\Omega}\frac{1}{q(x,y)}\frac{|w(x)-w(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy+\int_\Omega\frac{|w(x)|^{\bar{q}(x)}}{\bar{q}(x)}dx\right)\\ &\geqslant m_0\min\{\frac{1}{p+},\frac{1}{q^+}\}\left(\rho_{s,p(x,y)}(w)+\rho_{s,q(x,y)}(w)\right)\\ &\geqslant m_0\min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||w||^{p^-}_{s,p(x,y)}+||w||^{q^-}_{s,q(x,y)}\right)\\ &\geqslant\gamma_0>\gamma.\end{aligned}$$

From (3.10), we have

$$\sup_{m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||u||^{p^+}_{s,p(x,y)}+||v||^{q^+}_{s,q(x,y)}\right)<\gamma} J(u,v)$$

$$\leqslant \frac{\gamma}{2}\frac{J(w,w)}{m_0 \max\{\frac{1}{p^-},\frac{1}{q^-}\}\left(||w||^{p^+}_{s,p(x,y)}+||w||^{q^+}_{s,q(x,y)}\right)}$$

$$\leqslant \frac{\gamma}{2}\frac{J(w,w)}{\Psi(w,w)} \leqslant \gamma\frac{J(w,w)}{\Psi(w,w)}.$$

For any $(u,v) \in \Psi^{-1}(-\infty,\gamma)$, we obtain

$$\Psi(u,v) = \widehat{M}\left(I_{s,p(x,y)}(u)\right) + \widehat{M}\left(I_{s,q(x,y)}(v)\right) \leqslant \gamma.$$

So, we have

$$m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(\rho_{s,p(x,y)}(u)+\rho_{s,q(x,y)}(v)\right) \leqslant \gamma.$$

This implies that

$$\rho_{s,p(x,y)}(u)+\rho_{s,q(x,y)}(v) \leqslant \frac{\gamma}{m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}} < \frac{\gamma_0}{m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}} < 1.$$

Then

$$\rho_{s,p(x,y)}(u) < 1, \qquad \rho_{s,q(x,y)}(v) < 1.$$

It follows that $||u||_{s,p(x,y)} < 1$, and $||v||_{s,q(x,y)} < 1$. Therefore, we have

$$m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||u||^{p^+}_{s,p(x,y)}+||v||^{q^+}_{s,q(x,y)}\right) \leqslant \gamma.$$

Hence, we conclude that

$$\Psi^{-1}(-\infty,\gamma) \subset \left\{(u,v)\in E/\, m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||u||^{p^+}_{s,p(x,y)}+||v||^{q^+}_{s,q(x,y)}\right) \leqslant \gamma\right\}.$$

Then

$$\sup_{(u,v)\in\Psi^{-1}(-\infty,\gamma)} J(u,v) \leqslant \sup_{m_0 \min\{\frac{1}{p^+},\frac{1}{q^+}\}\left(||u||^{p^+}_{s,p(x,y)}+||v||^{q^+}_{s,q(x,y)}\right)<\gamma} J(u,v) \leqslant \gamma\frac{J(w,w)}{\Psi(w,w)}.$$

Hence, we can find $\gamma > 0$, $u_1 = v_1 = w$ and $\Psi(w,w) < \gamma$ satisfying (2.4). Also we can find δ satisfying

$$\sup_{(u,v)\in\Psi^{-1}(-\infty,\gamma)} J(u,v) \leqslant \delta \leqslant \gamma\frac{J(w,w)}{\Psi(w,w)}.$$

Hence, all assumptions of Theorem 2.8 are satisfied. So, there exist a nonempy open set $\Lambda \subset I$ and a positive real number r with the property described in the conclusion of Theorem 2.8. Fix $\lambda \in \Lambda$ and $G \in \mathscr{A}$. Put

$$\Gamma(u,v) = \int_\Omega G(x,u,v)dx, \quad \left\langle \Gamma'(u,v),(\varphi,\psi)\right\rangle = \int_\Omega G_u(x,u,v)\varphi + \int_\Omega G_v(x,u,v)\psi dx,$$

for all $(u,v),(\varphi,\psi)\in E$. Then, Γ is a C^1 functional on E with compact derivative. Hence, there exists $\mu^*>0$ such that, for each $\mu\in[0,\mu^*]$, the equation

$$\Psi'(u)-\lambda J'(u)+\mu\Gamma'(u)=0$$

has at least three solutions whose norms are less than δ. Moreover, theses solutions are exactly the weak solutions of problem (P_1). Thus, the proof of Theorem 3.3 is complete.

REFERENCES

[1] G. Kirchhoff, Mechanik. Teubner, Leipzig, 1883.
[2] M. Chipot, B. Lovat, Some remarks on non local elliptic and parabolic problems, Nonlinear Anal. 30 (1997), 4619-4627.
[3] C.O. Alves, F.S.J.A. Corrêa, T.F. Ma, Positive solutions for a quasilinear elliptic equations of Kirchhoff type, Comput. Math. Appl. 49 (2005), 85-93.
[4] J. R. Graef, S. Heidarkhani, L. Kong, A variational approach to a Kirchhoff-type problem involving two parameters, Results. Math. 63 (2013), 877-889.
[5] B. Ricceri, On an elliptic Kirchhoff-type problem depending on two parameters, J. Global Optim. 46 (2010), 543-549.
[6] G.S. Chen, H.Y. Tang, D.Q. Zhang, Y.X. Jiao, H.X. Wang, Existence of three solutions for a nonlocal elliptic system of (p, q)-Kirchhoff type, Boundary Value Probl. 2013 (2013), 175.
[7] G. Dai, Existence of solutions for nonlocal elliptic systems with nonstandard growth conditions, Electron. J. Differential Equ. 137 (2011), 1-13.
[8] M. L. Fang, S. Yue, C. Li, H.X. Wang, Existence of three solutions for nonlocal elliptic system of (p1,..., pn)-Kirchhoff type, Italian J. Pure Appl. Math. 33 (2014), 489-500.
[9] C. Li, C.L. Tang, Three solutions for a class of quasilinear elliptic systems involving the (p, q)-Laplacian, Nonlinear Anal. 69 (2008), 3322-3329.
[10] J. Liu, X. Shi, Existence of three solutions for a class of quasilinear elliptic systems involving the (p (x), q (x))-Laplacian, Nonlinear Anal. 71 (2009), 550-557.
[11] A. Taghavi, G.A. Afrouzi, H. Ghorbani, Existence of three solutions for a class of quasilinear elliptic systems involving the p(x)-Laplace operator, Sahand Commun. Math. Anal. 10 (2018), 47-60.
[12] C. Bucur, E. Valdinoci, Non local diffusion and application,Springer, Lecture Notes of the Unione Matematica Italiana, 2016.
[13] U. Kaufmann, J. D. Rossi, R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, Electron. J. Qual. Theory Diff. Equ. 76 (2017), 1-10.
[14] E. Di Nezza, G. Palatucci, E. Valdinoci, Hithiker's guide to the frctional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521-573.
[15] E. Azroul, A. Benkirane, A. Boumazourh, M. Shimi, Existence results for fractional $p(x,.)$-Laplacian problem via the Nehari manifold approach, Appl. Math. Optim. (2020) DOI: 10.1007/s00245-020-09686-z.
[16] E. Azroul, A. Boumazourh, On a class of fractional systems with nonstandard growth conditions, J. Pseudo-Differ. Oper. Appl. 11 (2020), 805-820.
[17] A. Bahrouni, Comparison and sub-supersolution principles for the fractional $p(x)$-Laplacian, J. Math. Anal. Appl. 458 (2018), 1363-1372.
[18] A. Bahrouni, V. Radulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, Discrete Contin. Dyn. Syst. 11 (2018), 379-389.
[19] L.M. Del Pezzo, J.D. Rossi, Traces for fractional Sobolev spaces with variable exponents, Adv. Oper. Theory 2 (2017), 435-446.
[20] L.A. Cafferelli, Non-local Diffusions, Drifts and Games. In: Holden H., Karlsen K. (eds) Nonlinear Partial Differential Equations. Abel Symposia, vol 7. Springer, Berlin, Heidelberg, 2012.
[21] E. Azroul, A. Benkirane, A. Boumazourh, M. Srati, Three solutions for a nonlocal fractional p-Kirchhoff type elliptic system, Appl. Anal. (2019) DOI: 10.1080/00036811.2019.1670347.
[22] X.L. Fan, D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2001), 424-446.
[23] O. Kováčik, J.Rákosník, On Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, Czechoslovak Math. J. 41 (1991), 592-618.

[24] V.D. Rădulescu, D. Repovš, Partial Differential Equations with Variable Exponents, Variational Methods and Qualitative Analysis, Monographs and Research Notes in Mathematics, CRC Press, Taylor and Francis Group, Boca Raton, FL, 2015.

[25] X.L. Fan, D. Zhao, On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$, J. Gansu Edu. College. 12 (1998), 1-6.

[26] E. Azroul, A. Benkirane, M. Shimi, Eigenvalue problems involving the fractional $p(x)$-Laplacian operator, Adv. Oper. Theory 4 (2019), 539-555.

[27] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 70 (2009), 3084-3089.

[28] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problem, Math. Comput. Modelling, 32 (2000), 1485-1494.

[29] E. Zeidler, Nonlinear Functional Analysis and Applications, In Nonlinear monotone operators, Vol. II/B, Springer-Verlag, New York, 1990.